Introduction

In this chapter we will mainly disscuss second order *linear* differential equations of form

$$y'' + p(t)y' + q(t) = g(t)$$

for all $t \in I$. Here I is some open interval, whereas $p(t)$, $q(t)$ and $g(t)$ are assumed continuous on the whole interval I.

Definition: A second order linear differential equation is said to be *homogeneous* if $g(t) \equiv 0$, for all $t \in I$. Otherwise, the equation is said to be *non-homogeneous*.

We shall begin with the following theorem of (*global*) existence and uniqueness.

Theorem: If $p(t)$, $q(t)$, and $g(t)$ are continuous on some open interval I, then there exists a *unique* solution, throughout whole interval I, to the following initial value problem

$$y'' + p(t)y' + q(t)y = g(t), \qquad y(t_0) = y_0, \quad y'(t_0) = y_0'.$$

Here $t_0 \in I$, y_0 and y_0' are two arbitrary constants.

A proof of the proof is presented in the next section.

However, here we want to point out that a second order linear equation can be transformed into a two-dimensional system of first order linear differential equations. Actually, let us difine

$$z_1(t) \triangleq y(t), \quad z_2(t) \triangleq y'(t).$$

The initial value problem can be written as the following two-dimensional linear system

$$\begin{aligned} z_1' &= z_2 \\ z_2' &= -q(t)z_1 - p(t)z_2 + g(t) \end{aligned}$$

with initial condition

$$z_1(t_0) = y_0, \quad z_2(t_0) = y_0'.$$

We can write this in a more compact way using matrix. Define

$$Z(t) \triangleq \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} y(t) \\ y'(t) \end{bmatrix}, \quad A(t) \triangleq \begin{bmatrix} 0 & 1 \\ -p(t) & -q(t) \end{bmatrix}, \quad B(t) \triangleq \begin{bmatrix} 0 \\ g(t) \end{bmatrix}, \quad Z_0 \triangleq \begin{bmatrix} y_0 \\ y_0' \end{bmatrix}.$$

We have a linear system of first order differential equations that looks much familiar.

$$Z' = A(t) \cdot Z + B(t), \qquad Z(t_0) = Z_0.$$

Remark: The above methodology works for linear differential equations of *any* order. That is, linear differential equations of order n can be transformed into an n-dimensional system of first order linear equations.

Remark: It is not difficult to see that nonlinear differential equations of any order can be transformed into an n-dimensional (non-linear) system of first order differential equations.

Example: Change the following second-order equations to a first-order system.

$$y'' - 5y' + ty = 3t^2, \qquad y(0) = 0, \quad y'(0) = 1.$$

Solution: Let us define

$$Z(t) \triangleq \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} y(t) \\ y'(t) \end{bmatrix}.$$

We have

$$Z' = \begin{bmatrix} 0 & 1 \\ 5 & -t \end{bmatrix} \cdot Z + \begin{bmatrix} 0 \\ 3t^2 \end{bmatrix}, \qquad Z(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Example (Reverse): Consider the following system of first-order linear equations.

$$Z' = \begin{bmatrix} 3 & 2 \\ 1 & -1 \end{bmatrix} \cdot Z, \qquad \text{here } Z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

Find the second-order linear differential equation that z_1 satisfies.

Solution: The system is

$$\begin{aligned} z_1' &= 3z_1 + 2z_2 \\ z_2' &= z_1 - z_2 \end{aligned}$$

It follows that

$$z_1' + 2z_2' = 5z_1 \Rightarrow w' = z_1, \quad \text{where } w \triangleq \frac{z_1 + 2z_2}{5}$$

This implies that

$$\begin{aligned} w' &= z_1 \\ z_1' &= 3z_1 + 2z_2 = 3z_1 + (5w - z_1) = 5w + 2z_1. \end{aligned}$$

Therefore, we have

$$z_1'' = 5w' + 2z_1' = 5z_1 + 2z_1' \Rightarrow z_1'' - 2z_1' - 5z_1 = 0.$$

Example: Consider the following system of first order linear equations.

$$\begin{aligned} z_1' &= z_1 + z_2 \\ z_2' &= z_1 - 2z_2 \end{aligned}$$

Write out the differential equation z_1 satisfies.

Solution: To obtain the equation of z_1, we have

$$2z_1' + z_2' = 3z_1 \quad \Rightarrow \quad z_1 = \left(\frac{2z_1 + z_2}{3}\right)' := w' \quad \text{where } w = \frac{2z_1 + z_2}{3}.$$

It follows that

$$z_1' = z_1 + z_2 = z_1 + 3w - 2z_1 = 3w - z_1$$

which implies that

$$z_1'' = 3w' - z_1' = 3z_1 - z_1' \quad \text{or} \quad z_1'' + z_1' - 3z_1 = 0$$

Exercise: Redo the preceding example for z_2.

Solution: We have

$$z_1' - z_2' = 3z_2 \quad \Rightarrow \quad z_2 = \left(\frac{z_1 - z_2}{3}\right)' := w' \quad \text{where } w = \frac{z_1 - z_2}{3}.$$

It follows that

$$z_2' = z_1 - 2z_2 = 3w + z_2 - 2z_2 = 3w - z_2$$

which implies that

$$z_2'' = 3w' - z_2' = 3z_2 - z_2' \quad \text{or} \quad z_2'' + z_2' - 3z_2 = 0$$

Remark: Note that the equations for z_1 and z_2 are the *same*. It is actually a general phenomenon.

Exercise: Change the following third-order linear differential equation into a 3-dimensional system of first-order linear equations.

$$y''' - e^t y' + 2y = e^{2t}$$

Solution: Let $z_1 \triangleq y$, $z_2 \triangleq y'$, and $z_3 \triangleq y''$. We have

$$\begin{aligned} z_1' &= z_2 \\ z_2' &= z_3 \\ z_3' &= -2z_1 + e^t z_2 + e^{2t} \end{aligned}$$

or

$$Z' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & e^t & 0 \end{bmatrix} \cdot Z + \begin{bmatrix} 0 \\ 0 \\ e^{2t} \end{bmatrix}$$

1 Proof of Theorem

To prove the existence and uniqueness, we shall use again the method of successive approximation. Consider the equilavent two-dimenasional linear system of first order differential equations

$$Z' = A(t) \cdot Z + B(t), \qquad Z(t_0) = Z_0,$$

as constructed in Introduction. It is not very difficult to see that there exists a positive number K such that

$$\|A(t)Z\| \leq K\|Z\|$$

for all $t \in [a, b]$ and $Z \in \mathbb{R}^2$. Here $\|Z\| \stackrel{\triangle}{=} \sqrt{z_1^2 + z_2^2}$. Let

$$\phi_0(t) \equiv Z_0 = \begin{bmatrix} y_0 \\ y_0' \end{bmatrix}$$

and succesively, we define

$$\phi_{n+1}(t) = Z_0 + \int_{t_0}^{t} \big[A(s) \cdot \phi_n(s) + B(s)\big]\, ds$$

It follows that

$$\|\phi_{n+1}(t) - \phi_n(t)\| \leq \int_{t_0}^{t} \|A(s) \cdot \big(\phi_n(s) - \phi_{n-1}(s)\big)\|\, dt \leq K \int_{t_0}^{t} \|\phi_n(s) - \phi_{n-1}(s)\|\, ds.$$

However, we have

$$\|\phi_1(t) - \phi_0(t)\| \leq \int_{t_0}^{t} \|A(s) \cdot Z_0 + B(s)\|\, ds \leq M(t - t_0),$$

here

$$M \stackrel{\triangle}{=} K\|Z_0\| + \max_{s \in [a,b]} \|B(s)\|.$$

An easy induction yields that

$$\|\phi_{n+1}(t) - \phi_n(t)\| \leq MK^n \frac{(t - t_0)^n}{(n+1)!} \leq MK^n \frac{(b - a)^n}{(n+1)!}$$

Since

$$\sum_{n=1}^{\infty} MK^n \frac{(b - a)^n}{(n+1)!} < \infty,$$

iterative sequence $\{\phi_n(t); n = 0, 1, 2, \cdots\}$ converges uniformly to some function $\phi(t)$. It follows that $\phi(t)$ is our solution.

The uniqueness follows from the Gronwall Inequality. Suppose that $\phi(t)$ and $\varphi(t)$ are two solutions to the equation, it follows that

$$\|\phi(t) - \varphi(t)\| \leq \int_{t_0}^{t} K\|\phi(s) - \varphi(s)\|\, ds$$

Letting $v(t) \stackrel{\triangle}{=} \|\phi(t) - \varphi(t)\|$, we have

$$v(t) \leq \int_{t_0}^{t} Kv(s)\, ds; \qquad v(t_0) = 0$$

Hence $v(t) \equiv 0$ by Gronwall inequality. This completes the proof. $\square$

2 Linear Independence; Fundamental Solutions

The theorem of existence and uniqueness provides an excellent way to obtain *all* solutions to second order linear differential equations. In this section we exclusively consider homogeneous case, that is, $g(t) \equiv 0$. Our equation, therefore, is

$$y'' + p(t)y' + q(t)y = 0.$$

An immediate result is the so-called *Principle of Superposition*.

Proposition (Principle of Superposition): If y_1 and y_2 are two solutions of the homogeneous differential equation

$$y'' + p(t)y' + q(t)y = 0,$$

then any linear combination of y_1 and y_2, say $c_1y_1 + c_2y_2$, is also a solution. Here c_1 and c_2 are arbitrary contants.

Proof: Use formulae $(\phi_1 + \phi_2)' = \phi_1' + \phi_2'$, $(c\phi)' = c\phi'$. $\square$

Before we present the main theorem, let us read several examples to see how it works.

Example: Solve the following initial value problem

$$y'' - 3y' + 2y = 0, \qquad y(0) = 1, \quad y'(0) = 1$$

Solution: We guess a solution to the differential equation

$$y'' - 3y' + 2y = 0$$

will take form $y(t) = e^{rt}$. It follows that

$$\left(e^{rt}\right)'' - 3\left(e^{rt}\right)' + 2e^{rt} = e^{rt} \cdot (r^2 - 3r + 2) = 0$$

If we choose r as a root of the following algebric equation (*characteristic equation*)

$$r^2 - 3r + 2 = 0 \quad \Rightarrow \quad r_1 = 1, \quad r_2 = 2,$$

then $y = e^{rt}$ will solve the differential equation. Moreover, any linear combination of e^{r_1t} and e^{r_2t} will also solve the differential equation.

Now we guess that the solution to the initial value problem will take form

$$y(t) = c_1e^{r_1t} + c_2e^{r_2t}.$$

This function $y(t)$ will satisfy $y'' - 3y' + 2y = 0$ regardless of what values c_1 and c_2 take. The remaining question is whether we can find a suitable pair (c_1, c_2) such that the initial conditions are satisfied. But this is equivalent to the following equation

$$y(0) = c_1 + c_2 = 1, \qquad y'(0) = c_1r_1 + c_2r_2 = c_1 + 2c_2 = 1$$

This implies $c_1 = 1, c_2 = 0$. Hence $y(t) = e^t$ is a solution to the initial value problem. Furthermore, it follows from previous theorem that it is actually the *only* solution to the initial value problem.

Exercise: In the above initial value problem, if we change the initial conditions to the more general $y(0) = y_0$, $y'(0) = y_0'$, can we still find the unique solution?

Solution: The only difference would be that (c_1, c_2) shall satisfy

$$c_1 + c_2 = y_0, \qquad c_1 + 2c_2 = y_0'.$$

For any values of y_0 and y_0', the above linear equation system is solvable with solution

$$c_1 = 2y_0 - y_0', \qquad c_2 = y_0' - y_0.$$

The solution to the initial value problem is

$$y(t) = (2y_0 - y_0')e^t + (y_0' - y_0)e^{2t}$$

We can conclude from the example one way to solve a general second-order linear second-order differential equation.

Step 1a. Find two solutions, say $y_1(t)$ and $y_2(t)$, of differential equation

$$y'' + p(t)y + q(t) = 0.$$

Step 1b. The two solutions, $y_1(t)$ and $y_2(t)$, shall have the following property: for any initial condition $y(t_0) = y_0$, $y'(t_0) = y_0'$, we can find constants (c_1, c_2) such that

$$y(t) \triangleq c_1 y_1(t) + c_2 y_2(t)$$

satisfies the initial condition.

Step 2. By the principle of superposition, we know $y(t)$ is a solution of the initial value problem. By the theorem of existence and uniqueness, we know $y(t)$ is actually the unique solution to the initial value problem.

It is not difficult to see that step 1b is equivalent to solving the following linear equations

$$\begin{aligned} y_0 &= c_1 y_1(t_0) + c_2 y_2(t_0) \\ y_0' &= c_1 y_1'(t_0) + c_2 y_2'(t_0) \end{aligned}$$

However, this system of equations would be solvable for all y_0 and y_0' *if and only if* the determinant

$$\det \begin{bmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{bmatrix} = y_1(t_0)y_2'(t_0) - y_1'(t_0)y_2(t_0) \neq 0.$$

The determinant is called *Wronskian determinant*, or simply *Wronskian*, of the solutions y_1, y_2. It will be denoted by $W(y_1, y_2)(t_0)$, or simply $W(t_0)$ when no confusion is incurred.

Example(continued): In the previous example, the Wronskian is

$$W(t) = y_1(t)y_2'(t) - y_1'(t)y_2(t) = e^t \cdot 2e^{2t} - e^t \cdot e^{2t} = e^{3t} \neq 0.$$

It follows from above discussion that step 1a+1b is equivalent to

Step 1: Find two solutions, say $y_1(t)$ and $y_2(t)$, whose Wronskian is non-zero. This shall allow us to determine (c_1, c_2) for any initial condition.

Exercise: Show that

1. $W(y, y) = 0$
2. $W(y_1, y_2) = -W(y_2, y_1)$
3. $W(y_1 + y_2, y) = W(y_1, y) + W(y_2, y)$
4. $W(ay_1, y_2) = aW(y_1, y_2)$

for any differentiable function y_1, y_2, y and constant a.

Exercise: Show that for any constants a, b, c, d, any functions f, g, we have

$$W(af + bg, cf + dg) = (ad - bc)W(f, g).$$

Proof: We have

$$\begin{aligned} W(af + bg, cf + dg) &= aW(f, cf + dg) + bW(g, cf + dg) \\ &= a\big[cW(f, f) + dW(f, g)\big] + b\big[cW(g, f) + dW(g, g)\big] \\ &= adW(f, g) + bcW(g, f) = (ad - bc)W(f, g) \end{aligned}$$

E.g. $W(2f + g, 3f - 2g) = (2 \cdot (-2) - 1 \cdot 3)W(f, g) = -7 \cdot W(f, g)$.

2.1 Fundamental Solutions

The following two theorems will give an affirmative answer to the existence of such a pair of solutions, and characterize *all* solutions to a given linear differential equation.

Theorem: Consider differential equation

$$y'' + p(t)y' + q(t)y = 0,$$

where p, q are continuous functions on open interval I. There exist a pair of solutions y_1 and y_2 such that *every* solution of the above differential equation, say ϕ, can be represented as the linear combination of y_1 and y_2. That is, for every solution ϕ, there exist constants c_1 and c_2 such that

$$\phi = c_1 y_1 + c_2 y_2.$$

Remark: Such a pair of solution (y_1, y_2) is called a *fundamental set of solutions*, while the expression $y = c_1 y_1 + c_2 y_2$ with arbitary coefficients (c_1, c_2) are called *general solutions.*

Theorem: Let y_1, y_2 be any two solutions to the differential equation. The following statements are equivalent.

1. A pair of solution (y_1, y_2) is a fundamental set of solution.

2. The Wronskian $W(t) \triangleq W(y_1, y_2)(t)$ is not zero for all $t \in I$.
3. The Wronskian $W(t) \triangleq W(y_1, y_2)(t)$ is not zero for some $t_0 \in I$.

Proof of the first theorem: Fix any $t_0 \in I$. Let y_1 be the solution of the differential equation with initial condition $y_1(t_0) = 1$, $y_1'(t_0) = 0$. Similarly, let y_2 be the solution with initail condition $y_2(t_0) = 0$, $y_2'(t_0) = 1$. Note y_1 and y_2 always exist. Now, let ϕ be an arbitary solution of differential equation. Define $a \triangleq \phi(t_0)$, $b \triangleq \phi'(t_0)$. Consider the following initial value problem

$$y'' + p(t)y' + q(t)y = 0, \qquad y(t_0) = a, \;\; y'(t_0) = b.$$

We know ϕ is a solution to this initial value problem. However, note that

$$\tilde{\phi} \triangleq ay_1 + by_2$$

is also a solution to the initial value problem (check!!). By uniqueness, we know

$$\phi = \tilde{\phi} = ay_1 + by_2.$$

This completes the proof. □

Proof of the second theorem: We shall prove "$2 \Rightarrow 3 \Rightarrow 1 \Rightarrow 2$". But "$2 \Rightarrow 3$" is trivial.

"$3 \Rightarrow 1$". The proof is very similar to that of the first theorem. The only difference is

$$\tilde{\phi} \triangleq Ay_1 + By_2$$

with A and B determined by equations

$$\begin{aligned} Ay_1(t_0) + By_2(t_0) &= a \\ Ay_1'(t_0) + By_2'(t_0) &= b \end{aligned}$$

Note such (A, B) always exist since $W(t_0) \neq 0$ (check!!).

"$1 \Rightarrow 2$". Fix an arbitrary $s \in I$, let ϕ be the solution to the differential equation with initial condition $\phi(s) = 1$, $\phi'(s) = 0$. By definition, there exist constants (c_1, c_2) such that $\phi = c_1y_1 + c_2y_2$. In particular

$$1 = c_1y_1(s) + c_2y_2(s), \qquad 0 = c_1y_1'(s) + c_2y_2'(s)$$

Similarly, let $\tilde{\phi}$ be the solution with initial condition $\tilde{\phi}(s) = 0$, $\tilde{\phi}(s) = 1$. There exist constants $(\tilde{c}_1, \tilde{c}_2)$ such that $\tilde{\phi} = \tilde{c}_1y_1 + \tilde{c}_2y_2$. In particular

$$0 = \tilde{c}_1y_1(s) + \tilde{c}_2y_2(s), \qquad 1 = \tilde{c}_1y_1'(s) + \tilde{c}_2y_2'(s).$$

However, this implies that,

$$[c_1y_1(s) + c_2y_2(s)] \cdot [\tilde{c}_1y_1'(s) + \tilde{c}_2y_2'(s)] - [c_1y_1'(s) + c_2y_2'(s)] \cdot [\tilde{c}_1y_1(s) + \tilde{c}_2y_2(s)] = 1 \cdot 1 - 0 \cdot 0 = 1$$

But the left hand side, by direct calculation, is

$$W(s) \cdot (c_1\tilde{c}_2 - \tilde{c}_1c_2) = 1 \;\Rightarrow\; W(s) \neq 0$$

Since s is arbitrary, it completes the proof. □

Example: For differential equation

$$y'' + y = 0$$

Show that $(y_1, y_2) = (\sin t, \cos t)$ is a pair of fundamental solutions.

Proof: It is easy to verify that $y_1 = \sin t, y - 2 = \cos t$ are two solutions to the differential equation. Moreover, $W(y_1, y_2) = 1$. Therefore, they are a set of fundamental solutions. $\square$

Example: Consider linear equation

$$y'' - \frac{2}{t}y' + \frac{2}{t^2}y = 0; \qquad t > 0$$

Show that $(y_1, y_2) = (t, t^2)$ is a set of fundamental solutions.

Proof: It is easy to verify that $y_1 = t$, $y_2 = t^2$ are indeed a pair of solutions to the linear equation. Moreover,

$$W(y_1, y_2) = t^2 \neq 0$$

on interval $t > 0$. Therefore, this is a pair of fundamental solutions.

Remark: One way to solve this kind of equations is to guess that $y = t^\alpha$ for some yet-to-be-determined constants α. It follows that

$$\alpha(\alpha - 1) - 2\alpha + 2 = 0 \quad \Rightarrow \quad \alpha^2 - 3\lambda + 2 = (\alpha - 1)(\alpha - 2) = 0 \quad \Rightarrow \quad \alpha = 1, 2.$$

or $y_1 = t, y_2 = t^2$.

Example: Show that $y_1(t) \triangleq t$ and $y_2 \triangleq t^2$ on interval $(-1, 1)$ can *not* be solutions to differential equations $y'' + p(t)y' + q(t)y = 0$.

Proof: The Wronskian is $W(t) = y_1 y_2' - y_1' y_2 = t^2$, which is zero at $t = 0$ but non-zero otherwise. If they are solutions to the differential equation, we know the Wronskian should be either zero everywhere, or non-zero everywhere, which will lead to contradiction.

Exercise (Abel's Theorem): Suppose that y_1 and y_2 are two solutions of differential equation

$$y'' + p(t)y' + q(t)y = 0.$$

Show that function

$$e^{\int p(t)dt} W(t) \equiv \text{const}.$$

Proof: All we need to show is that

$$\left(e^{\int p(t)dt} W(t)\right)' \equiv 0$$

However, the left-hand side equals

$$e^{\int p(t)dt} \big(W' + p(t)W\big)$$

So it suffices to show $W' + p(t)W = 0$. Since $W = y_1 y_2' - y_1' y_2$, it follows that

$$\begin{aligned} W' &= y_1' y_2' + y_1 y_2'' - y_1' y_2' - y_1'' y_2 = y_1 y_2'' - y_1'' y_2 \\ &= y_1 \big[-p(t) y_2' - q(t) y_2\big] - \big[-p(t) y_1' - q(t) y_1\big] y_2 \\ &= -p(t) \cdot (y_1 y_2' - y_1' y_2) = -p(t) W \end{aligned}$$

This completes the proof.

Remark: This actually provide another way to show the equivalence of statement 2 and statement 3 in the second theorem.

One thing shall keep in mind though, fundamental set of solutions is *not* unique.

Example (Non-uniqueness of fundamental set of solutions): Suppose (y_1, y_2) is a fundamental set of solution to equation $y'' + p(t)y' + q(t)y = 0$. Show that $(y_1 + cy_2, y_2)$ is also a fundamental set of solutions, where c is an arbitrary constant.

2.2 Linear Independence

We can do an extension of the second theorem by introducing a new concept of linear independence.

Definition: Two function f and g are said to be *linearly dependent* on interval I if there exist two constants k_1 and k_2, not both zero, such that

$$k_1 f(t) + k_2 g(t) \equiv 0$$

for all $t \in I$. Otherwise, they are said to be *linearly independent*.

Example: Functions e^t and e^{-t} are linearly independent on any interval, while $\sin t$ and $\sin(t + \pi)$ are linearly dependent on any interval.

Example: Functions $|t|$ and t are linear dependent on interval $[0, 1)$, and linear dependent on interval $(-1, 0]$. However, they are linear independent on interval $(-1, 1)$.

Lemma: Suppose f and g are linear dependent on interval I, then $W(f, g)(t) \equiv 0$ for every $t \in I$.

Proof: Let k_1 and k_2 be two constants, not both zero, such that

$$k_1 f + k_2 g \equiv 0 \;\Rightarrow\; k_1 f' + k_2 g' \equiv 0$$

Without loss of generality, we assume $k_1 \neq 0$. We have

$$0 \equiv (k_1 f + k_2 g)g' - (k_1 f' + k_1 g')g = k_1 W \;\Rightarrow\; W \equiv 0$$

This completes the proof. □

Remark: The reverse of the lemma might not always hold true generally. For example, functions $f(t) \overset{\triangle}{=} t|t|$ and $g(t) \overset{\triangle}{=} t^2$ are linear independent on interval $(-1, 1)$. It is not difficult to show that $W(f, g)(t) \equiv 0$. Actually $f'(t) = 2|t|$, $g'(t) = 2t$ and

$$W(f, g) = t|t| \cdot 2t - 2|t| \cdot t^2 = 0$$

Remark: The reverse of the lemma holds true in case both f and g are solutions to same differential equation, as we shall see in next lemma.

Lemma: Suppose y_1 and y_2 are two solutions to differential equations

$$y'' + p(t)y' + q(t)y = 0$$

on interval I, here p and q are continuous. Then y_1 and y_2 are linear dependent if and only if $W(y_1, y_2) \equiv 0$.

Proof: We only need to prove the sufficiency (Necessity is proved in the preceding lemma). Fix any $t_0 \in I$. From assumption, $W(t_0) = 0$. Therefore, there exist two constants c_1 and c_2, not both zero, such that

$$\begin{aligned} c_1 y_1(t_0) + c_2 y_2(t_0) &= 0 \\ c_1 y_1'(t_0) + c_2 y_2(t_0) &= 0 \end{aligned}$$

Let $\phi \stackrel{\triangle}{=} c_1 y_1 + c_2 y_2$. It follows that ϕ is a solution to the initial value problem

$$y'' + p(t)y' + q(t)y = 0, \qquad y(t_0) = 0, \qquad y'(t_0) = 0$$

But constant 0 is also a solution to the initial value problem. By uniqueness, we have

$$\phi(t) = c_1 y_1(t) + c_2 y_2(t) \equiv 0, \qquad \forall\, t \in I$$

or y_1 and y_2 are linearly dependent. □

Remark: We have actually obtained another equilavent condition for fundamental solutions.

2.3 Conclusion

Let $p(t), q(t)$ be contiouous functions on interval I.

Fundamental theorem: There exists a unique solution to every initial value problem

$$y'' + p(t)y' + q(t)y = 0, \quad t \in I; \quad y(t_0) = y_0,\ y'(t_0) = y_0'$$

fundamental set of solutions: There exist a pair of solutions (y_1, y_2) to differential equation

$$y'' + p(t)y' + q(t)y = 0, \quad t \in I;$$

such that any other solutions to this differential equation can be represented as a linear combination of (y_1, y_2). In another word, the set

$$\{c_1 y_1 + c_2 y_2; \quad c_1, c_2 \text{ are arbitrary constants}\}.$$

is the *complete* set of solutions to the differential equation. We call $y = c_1 y_1 + c_2 y_2$ **general solutions** where c_1, c_2 are arbitrary constants.

Equivalence conditions for fundamental set of solutions: Consider a linear homogeneous equation

$$y'' + p(t)y' + q(t)y = 0, \quad t \in I.$$

Suppose (y_1, y_2) is a pair of solutions to this differential equation. Then the following statements are equilavent

1. (y_1, y_2) is a fundamental set of solutions.
2. The Wronskian of of (y_1, y_2), denoted by $W(t)$, is non-zero for all $t \in I$.
3. The Wronskian of of (y_1, y_2), denoted by $W(t)$, is non-zero for some $t_0 \in I$.
4. (y_1, y_2) are linear independent.

Corollary: Let (y_1, y_2) be a pair of solutions to differential equation

$$y'' + p(t)y' + q(t)y = 0, \quad t \in I.$$

Then the Wrnoskian of (y_1, y_2) is either zero for all $t \in I$, or non-zero for all $t \in I$. Indeed, we have the Abel Theorem:

$$W(y_1, y_2)(t) = ce^{-\int p(t)\,dt}$$

for some constant c.

Corollary: Let (y_1, y_2) be a pair of solutions to differential equation

$$y'' + p(t)y' + q(t)y = 0, \quad t \in I.$$

Then (y_1, y_2) are linear dependent on interval I if and only if the Wrnoskian of (y_1, y_2) are zero for all $t \in I$. Alternatively, (y_1, y_2) are linear independent on interval I if and only if the Wrnoskian of (y_1, y_2) are non-zero for all $t \in I$. (*Note:* the last claim is not true for a general pair of functions).

The following examples illustrate the idea of solving a homogeneous equation of second order.

Example: Consider a differential equation

$$y'' + 3y' + 2y = 0, \quad t \geq 0$$

Guess a result of form $y = e^{rt}$, we have

$$(r^2 + 3r + 2)e^{rt} \equiv \quad \Rightarrow \quad r^2 + 3r + 2 = 0 \quad \Rightarrow \quad r_1 = -1, \; r_2 = -2.$$

Therefore (e^{-t}, e^{-2t}) is a pair of solutions. However, their Wronskian is $-e^{-3t}$ in non-zero, so this a fundamental set of solutions. The general solutions to this equations can be written as

$$y = c_1 e^{-t} + c_2 e^{-2t}.$$

for arbitrary constants (c_1, c_2).

Now if we have initial conditions

$$y(0) = 0, \quad y'(0) = 1,$$

we can solve a particular pair of (c_1, c_2) that

$$c_1 + c_2 = 0, \quad -c_1 - 2c - 2 = 1 \quad \Rightarrow \quad c_1 = 1, \; c_2 = -1.$$

Or the solution to the initial value problem

$$y'' + 3y' + 2y = 0, \quad t \geq 0; \quad y(0) = 0, \quad y'(0) = 1$$

is

$$\phi(t) = e^{-t} - e^{-2t}.$$

3 Homogeneous Equations with Constant Coefficients

Second order linear differential equation with constant coefficients can be solved very easily, as we shall see below. Consider equation

$$ay'' + by' + cy = 0$$

where $a \neq 0$, b and c are constants. From the discussion in previous section, it is sufficient to obtain a set of fundamental solutions. We seek a solution of form $y = e^{rt}$, where r is a parameter to be determined. It follows that

$$ay'' + by' + cy = e^{rt}(ar^2 + br + c) = 0.$$

This implies that r is the root of algebraic equation

$$ar^2 + br + c = 0,$$

which is called *characteristic equation*. It follows that

$$r = r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

There are three possible scenarios, namely, the characteristic equation has two distinctive real roots ($b^2 - 4ac > 0$), two repeated real roots ($b^2 - 4ac = 0$), two complex conjuate roots ($b^2 - 4ac < 0$).

3.1 Two distinctive real roots ($b^2 - 4ac > 0$)

If two real roots $r_1 \neq r_2$, then $y_1 \triangleq e^{r_1 t}$ and $y_2 \triangleq e^{r_2 t}$ are both solutions to the differential equation. Furthermore, their Wronskian is

$$W(t) = y_1 y_2' - y_1' y_2 = e^{r_1 t} \cdot r_2 e^{r_2 t} - r_1 e^{r_1 t} \cdot e^{r_2 t} = (r_2 - r_1) e^{(r_1 + r_2)t} \neq 0$$

It follows that y_1 and y_2 form a fundamental set of solutions. The general solution is therefore

$$y = c_1 y_1 + c_2 y_2 = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

Example: Solve the following initial value problem

$$y'' - 3y' + 2y = 0, \qquad y(0) = 0, \ y'(0) = 1.$$

Solution: The characteristic equation is

$$r^2 - 3r + 2 = 0 \quad \Rightarrow \quad r_1 = 1, \ r_2 = 2.$$

Therefore the general solution is

$$y = c_1 e^t + c_2 e^{2t}.$$

Intial conditions implies

$$c_1 + c_2 = 0, \quad c_1 + 2c_2 = 1 \quad \Rightarrow \quad c_1 = -1, \ c_2 = 1$$

Therefore the solution to the initial value problem is

$$y = -e^t + e^{2t}.$$

Example: A particle of mass m moving in a straight line is repelled from origin 0 by force F. Assume the force is proportional to the distance of the particle from 0. Find the position of the particle as a function of time if at $t = 0$, the particle has velocity 0 and it is l ft from the origin.

Solution: Suppose at time t, the distance of the particle from origin is $x(t)$. We have

$$m\frac{d^2x}{dt^2} = F = kx.$$

The characteristic equation is

$$mr^2 - k = 0 \quad \Rightarrow \quad r_1 = \sqrt{\frac{k}{m}},\ r_2 = -\sqrt{\frac{k}{m}}.$$

Therefore the general solution for the above differential equation is

$$x(t) = c_1 e^{\sqrt{\frac{k}{m}}t} + c_2 e^{-\sqrt{\frac{k}{m}}t}.$$

However, initial conditions $x(0) = l$, $x'(0) = 0$ imply that

$$c_1 + c_2 = l, \quad \sqrt{\frac{k}{m}}(c_1 - c_2) = 0 \quad \Rightarrow \quad c_1 = c_2 = \frac{l}{2}.$$

Or

$$x(t) = \frac{l}{2}\left(e^{\sqrt{\frac{k}{m}}t} + e^{-\sqrt{\frac{k}{m}}t}\right).$$

Example (cont.): Suppose in the preceding example, the initial velocity of the particle is v (a positive v means the velocity is away from origin, and a negative v means the velocity is toward the origin). Find the critical value of v such that the particle will approach the origin but never reach it.

Solution: Let $x(t)$ be the distance of the particle from the origin at time t. It follows that

$$x(t) = c_1 e^{\sqrt{\frac{k}{m}}t} + c_2 e^{-\sqrt{\frac{k}{m}}t}$$

for some constants c_1, c_2. However, initial condition $x(0) = l, x'(0) = v$ yields that

$$c_1 + c_2 = l, \quad \sqrt{\frac{k}{m}}(c_1 - c_2) = v \quad \Rightarrow \quad c_1 = \frac{l + v\sqrt{\frac{m}{k}}}{2}, c_2 = \frac{l - v\sqrt{\frac{m}{k}}}{2}.$$

It follows that

$$x(t) = \frac{l + v\sqrt{\frac{m}{k}}}{2} e^{\sqrt{\frac{k}{m}}t} + \frac{l - v\sqrt{\frac{m}{k}}}{2} e^{-\sqrt{\frac{k}{m}}t}.$$

However,

$$\lim_{t\to\infty} x(t) = \left\{ \begin{array}{ccc} +\infty & ; & v > -l\sqrt{\frac{k}{m}} \\ -\infty & ; & v < -l\sqrt{\frac{k}{m}} \\ 0 & ; & v = -l\sqrt{\frac{k}{m}} \end{array} \right\}$$

Therefore, the critical velocity is $-l\sqrt{\frac{k}{m}}$. □

3.2 Two repeated real roots ($b^2 - 4ac = 0$)

In case of $b^2 - 4ac = 0$, we have $r_1 = r_2 = -\frac{2b}{a} := r$, and $y_1 = e^{rt}$ is a solution of the differential equation. There are several ways to obtain a second solution – you can just guess-and-verify.

Lemma: If $b^2 - 4ac = 0$, then $y_2(t) \triangleq te^{rt}$ is a solution of the differential equation. Here $r = -\frac{2b}{a}$.

Proof: We have $y_2' = rte^{rt} + e^{rt}$, $y_2'' = r^2te^{rt} + 2re^{rt}$. Therefore

$$ay'' + by' + cy = (ar^2 + br + c)te^{rt} + (2ra + b)e^{rt} = 0$$

since $2ra + b = 0$ and $ar^2 + br + c = 0$. $\square$

The Wroskian of y_1 and y_2 is

$$W(y_1, y_2)(t) = y_1y_2' - y_1'y_2 = e^{rt} \cdot (rte^{rt} + e^{rt}) - re^{rt} \cdot te^{rt} = e^{2rt} \neq 0.$$

It follows that y_1 and y_2 are a fundamental set of solutions, and the general solution is

$$y = c_1e^{rt} + c_2te^{rt}.$$

Example: The general solution to differential equation

$$y'' + 4y' + 4y = 0$$

is

$$y = c_1e^{-2t} + c_2te^{-2t}.$$

Reduction of Order: This is another method of obtaining y_2, *Reduction of Order*, which is of some general interest. Actually, suppose we know a non-zero solution y_1 to a general homogeneous linear differential equation

$$y'' + p(t)y + q(t)y = 0.$$

Assuming that $y_2(t) = v(t)y_1(t)$ is a second solution, we have

$$y_2' = v'y_1 + vy_1', \qquad y_2'' = v''y_1 + 2v'y_1' + vy_1''$$

and

$$v''y_1 + 2v'y_1' + vy_1'' + p(t)v'y_1 + p(t)vy_1' + q(t)vy = 0 \;\Rightarrow\; y_1v'' + \big[2y_1' + p(t)y_1\big]v' = 0.$$

Letting $w \triangleq v'$, we obtain a first order differential equation

$$y_1w' + \big[2y_1' + p(t)y_1\big]w = 0$$

This implies

$$\frac{w'}{w} = -\frac{2y_1'}{y_1} - p(t) \;\Rightarrow\; (\ln|w|)' = -\left[2\ln|y_1| + e^{\int p(t)\,dt}\right]'$$

It follows that

$$w = \frac{c}{e^{\int p(t)\,dt}y_1^2}$$

for some constant $c \neq 0$. It follows that $v = \int w\,dt$

Exercise: Prove that the Wronskian $W(y_1, y_2)(t) \neq 0$.

Proof: We have $y_2' = v'y_1 + vy_1' = wy_1 + vy_1'$. Hence

$$W(y_1, y_2) = y_1y_2' - y_1'y_2 = wy_1^2 = ce^{-\int p(t)\,dt} \neq 0.$$

This completes the proof. $\square$

Example: Find the general solution of

$$t^2y'' - ty' + y = 0, \qquad t > 0$$

given that $y_1 = t$ is a solution.

Solution: By Reduction of Order method, assume $y_2(t) = v(t)y_1(t) = tv(t)$. It follows that

$$t^2(tv'' + 2v') - t(tv' + v) + tv = 0 \quad \Rightarrow \quad t^3v'' + t^2v' = 0 \quad \Rightarrow \quad tv'' + v' = 0$$

Let $w = v'$, we have

$$tw' + w = 0 \quad \Rightarrow \quad w = \frac{1}{t}, \quad v = \ln t.$$

Therefore $y_2 = t\ln t$ is also solution. Therefore, the general solution is

$$y = c_1t + c_2t\log t.$$

Exercise: Use Reduction of Order to obtain y_2 again in case of two repeated real roots.

3.3 Two conjuate complex roots ($b^2 - 4ac < 0$)

If $b^2 - 4ac < 0$, the characteristic equation admits two complex roots

$$r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = -\frac{b}{2a} \pm \frac{\sqrt{4ac - b^2}}{2a}i := \lambda \pm \mu i$$

Fromally, we have $y_{1,2} = e^{(\lambda \pm \mu i)t}$.

3.3.1 Review of Complex Analysis

For a complex number $z = a + bi$, a is called the *real* part, and b is called the *imaginary* part. Here i is the *imaginary unit* with property $i^2 = -1$. The *conjuate* of z, denoted by $\overline{z}$, is defined as $\overline{z} \triangleq a - bi$. The arithmetic rules of complex numbers are

$$(a_1 + b_1i) + (a_2 + b_2i) = (a_1 + a_2) + (b_1 + b_2)i$$

$$(a_1 + b_1i) \cdot (a_2 + b_2i) = (a_1a_2 - b_1b_2) + (a_1b_2 + a_2b_1)i$$

Exercise: Show that

$$\overline{z_1 + z_2} = \overline{z_1} + \overline{z_2}, \qquad \overline{z_1 \cdot z_2} = \overline{z_1} \cdot \overline{z_2}$$

It will be convienent to represent a complex number on a plain geometrically; see the graph below.

Euler's Formula: We define that

$$e^{it} \stackrel{\triangle}{=} \cos t + i \sin t, \qquad \forall\, t \in \mathbb{R} \quad \text{(Euler's Formula)}$$

and

$$e^{z} = e^{a+bi} \stackrel{\triangle}{=} e^{a} \cdot e^{bi} = e^{a}(\cos b + i \sin b)$$

Example: $e^{i\frac{\pi}{4}} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$, $\quad e^{i\frac{\pi}{2}} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i$, $\quad e^{2+i} = e^{2}(\cos 1 + i \sin 1) = 7.39 \cdot (0.54 + 0.84i) = 3.99 + 6.22i$

Remark: (Motivation from Calculus) Taylor expansion yileds that

$$\begin{aligned} \cos t &= 1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots \\ \sin t &= t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots \end{aligned}$$

We also have expansion

$$e^{t} = 1 + \frac{t}{1!} + \frac{t^2}{2!} + \cdots + \frac{t^n}{n!} + \cdots$$

If we are willing to replace t by it in the above expansion, we have

$$\begin{aligned} e^{it} &= 1 + \frac{it}{1!} + \frac{(it)^2}{2!} + \cdots + \frac{(it)^n}{n!} + \cdots \\ &= \left(1 - \frac{t^2}{2!} + \frac{t^4}{4!} - \frac{t^6}{6!} + \cdots\right) + \left(t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots\right) i \\ &= \cos t + i \sin t \end{aligned}$$

Exercise:

1. Show that $e^{it_1} \cdot e^{it_2} = e^{i(t_1+t_2)}$, for all real numbers t_1, t_2. In general, $e^{z_1} \cdot e^{z_2} = e^{z_1+z_2}$, for allcomplex numbers z_1, z_2.
2. (de Moivre's Formula) Prove $(\cos\varphi + i\sin\varphi)^n = \cos n\varphi + i \sin n\varphi$ (This provide a very simple way to express $\cos n\varphi$ and $\sin n\varphi$ in terms of $\cos\varphi$ and $\sin\varphi$).

Derivative: The *derivative* of a complex function $z(t) = f(t) + ig(t)$ is naturally defined as

$$\frac{dz}{dt} \triangleq \frac{df}{dt} + i\frac{dg}{dt}$$

Exercise: Show that $\frac{d}{dt}e^{zt} = ze^{zt}$ for any fixed complex number z.

Finally, we wish to point out that usual calculus rules also hold in complex analysis.

3.3.2 Real-Valued solutions

It follows that

$$y_{1,2} = e^{(\lambda \pm \mu i)t} = e^{\lambda t}(\cos \mu_t \pm i \sin \mu_t)$$

are indeed two (complex) solutions of the differential equation (Exercise!). However, this implies the linear combinations of y_1 and y_2,

$$u(t) \triangleq \frac{y_1 + y_2}{2} = e^{\lambda t}\cos \mu t, \qquad v(t) \triangleq \frac{y_1 - y_2}{2i} = e^{\lambda t}\sin \mu t$$

are also solutions to the differential equations.

Exercise: Use direct calculation to verify u and v are really solutions to the differential equation.

Exercise: Show that the Wronskian $W(u, v) = \mu e^{2\lambda t}$.

Since $\mu = \sqrt{4ac - b^2} \neq 0$, the Wronskian $W \neq 0$, therefore u and v are a fundamental set of solutions. The general solution of the differential solution is

$$y = c_1 e^{\lambda t}\cos \mu t + c_2 e^{\lambda t}\sin \mu t$$

Here are several examples.

Example: Find the general solution to the differential equation

$$y'' - 4y' + 20y = 0, \qquad t \geq 0.$$

Show that any solution goes to 0 as $t \to \infty$.

Solution: The characteristic equation is

$$r^2 - 4r + 20 = 0 \quad \Rightarrow \quad b^2 - 4ac = 16 - 80 = -64 \quad \Rightarrow \quad r_{1,2} = \frac{-4 \pm \sqrt{-64}}{2} = -2 \pm 4i.$$

Therefore, the general solution is

$$y = c_1 e^{-2t}\cos 4t + c_2 e^{-2t}\sin 4t = e^{-2t}(c_1 \cos 4t + c_2 \sin 4t).$$

Since

$$|c_1 \cos 2t + c_2 \sin 4t| \leq |c_1| + |c_2|$$

We have $y(t) \to 0$ as $t \to \infty$. □

Example: A displaced simple pendulum of length l, is attached with weight $w = mg$. Let $\theta(t)$ be the angle of swing at time t.

1. Find the differential equation θ satisfies. Is this equation linear or nonlinear?
2. When the swing is very small, show that θ satisfies a linear ODE approximately.
3. Find the period of this oscillation approximately.

Solution: The direction of the net external force must be in the direction tangent to the arc of swing (there is should be no net force in the direction of the string). Therefore, the net force is $F = -mg\sin\theta$ (the negative sign is because the direction of F is always in the opposition direction of swing). We have, by Newton's law of force,

$$ma = -mg\sin\theta \qquad \text{where } a \text{ is the acceleration.}$$

However, if we let the s denote the distance the weight moves along the arc, we have $s = l\theta$, hence

$$a = \frac{d^2s}{dt^2} = l\frac{d^2\theta}{dt^2}$$

Therefore, the equation for θ is

$$(*) \qquad l\frac{d^2\theta}{dt^2} + g\sin\theta = 0, \quad \text{or} \quad l\theta'' + g\sin\theta = 0$$

It is nonlinear. However, when the swing is small, we have $\sin\theta \approx \theta$ (why?), therefore, the equation of θ is approximately

$$l\frac{d^2\theta}{dt^2} + g\theta = 0,$$

which is linear. For this equation, the characteristic equation is

$$lr^2 + g = 0 \quad \Rightarrow \quad r = \pm\sqrt{\frac{g}{l}}$$

Therefore, the general solution is

$$\theta = c_1\cos\sqrt{\frac{g}{l}}t + c_2\sin\sqrt{\frac{g}{l}}t.$$

The period is $2\pi\sqrt{\frac{l}{g}}$. □

Remark: Equation (*) can be reduced to an equation of first order. Actually, let $\omega = \frac{d\theta}{dt}$,

$$l\frac{d\omega}{dt} + g\sin\theta = 0.$$

Observe

$$\frac{d\omega}{dt} = \frac{d\omega}{d\theta}\frac{d\theta}{dt} = \omega\frac{d\omega}{d\theta}$$

we have

$$l\omega\frac{d\omega}{d\theta} + g\sin\theta = 0.$$

This is a first order nonlinear equation (seperable).

3.3.3 Premilinary Geometric Properties

The geometric properties of y are very different, depending on different parameters. To see it more clearly, we will reparametrize y as following.

$$\begin{aligned} y &= c_1e^{\lambda t}\cos\mu t + c_2e^{\lambda t}\sin\mu t \\ &= \sqrt{c_1^2+c_2^2}e^{\lambda t}\left(\frac{c_1}{\sqrt{c_1^2+c_2^2}}\cos\mu t + \frac{c_2}{\sqrt{c_1^2+c_2^2}}\sin\mu t\right) \\ &= Re^{\lambda t}\cdot(\cos\delta\cos\mu t + \sin\delta\sin\mu t) = Re^{\lambda t}\cos(\mu t - \delta). \end{aligned}$$

Here reparametrization $R \overset{\triangle}{=} \sqrt{c_1^2+c_2^2}$ (*amplitude*) and δ (*phase*) is determined by

$$\cos\delta = \frac{c_1}{\sqrt{c_1^2+c_2^2}}, \qquad \sin\delta = \frac{c_2}{\sqrt{c_1^2+c_2^2}}.$$

or $c_1 = R\cos\delta$, $c_2 = R\sin\delta$. We shall call $\frac{2\pi}{\mu}$ the *natural frequency.*

We have three possible cases.

$\lambda = 0$, or $b = 0, ac > 0$: (*Periodic Oscillation*) For example, equation

$$y'' + \mu^2 y = 0.$$

The general solution is

$$y = c_1\cos\mu t + c_2\sin\mu t = R\cos(\mu t - \delta)$$

$\boldsymbol{\lambda > 0}$, or $\boldsymbol{\frac{b}{2a} < 0, b^2 < 4ac}$: (*Amplifies oscillation*) The general solution is

$$y = Re^{\lambda t}\cos(\mu t - \delta)$$

$\boldsymbol{\lambda < 0}$, or $\boldsymbol{\frac{b}{2a} > 0, b^2 < 4ac}$: (*Damped oscillation*) The general solution is

$$y = Re^{\lambda t}\cos(\mu t - \delta)$$

Example: Consider differential equation

$$ay'' + by' + c = 0, \qquad t \geq 0$$

where $a, b, c > 0$ are all positive. Show that any solution of this differential equation goes to 0 as $t \to \infty$.

Proof: The characteristic equation is

$$ar^2 + br + c = 0 \qquad \Rightarrow \qquad r_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

If $b^2 - 4ac > 0$, both r_1, r_2 are real. Moreover, they are negative (why?). The general solution is

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t} \quad \Rightarrow \quad y(t) \to 0 \text{ as } t \to \infty.$$

If $b^2 - 4ac = 0$, $r = r_1 = r_2 = \frac{-b}{2a} < 0$. The general solution is

$$y = e^{rt}(c_1 + c_2 t) \quad \Rightarrow \quad y(t) \to 0 \text{ as } t \to \infty.$$

If $b^2 - 4ac < 0$, the two roots are complex, we have

$$r_{1,2} = e^{\lambda \pm \mu i} \quad \text{where } \lambda = \frac{-b}{2a} < 0,\ \mu = \frac{\sqrt{4ac - b^2}}{2a}$$

The general solution is

$$y = e^{\lambda t}(c_1 \cos \mu t + c_2 \sin \mu t) \quad \Rightarrow \quad y(t) \to 0 \text{ as } t \to \infty$$

since $|c_1 \cos \mu t + c_2 \sin \mu t| \leq |c_1| + |c_2|$ for all t. $\square$

4 Non-homogeneous Equations

Let us consider differential equation

$$y'' + p(t)y' + q(t)y = g(t),$$

where p, q, and g are continuous functions on interval I, with g not necessarily everywhere zero. To simplify notation, let us introduce *differential operator*

$$L[y] \stackrel{\triangle}{=} y'' + p(t)y' + q(t)y$$

Remark: L maps a twice-differentiable function to a function whose value at t is determined $L[y](t) = y''(t) + p(t)y(t) + q(t)y(t)$ for every $t \in I$. Furthermore, it is clear that L is a linear operator in the sense that

$$L[ay + bz] = aL[y] + bL[z]$$

for any twice-differentiable functions y, z and arbitrary constants a, b (Exercise!).

In another word, we want to find the solution to equation

$$L[y] = g(t).$$

We have the following theorem.

Theorem: If y_p is any particular solution satisfying $L[y_p] = g(t)$, then the general solution of the non-homogeneous equation $L[y] = g(t)$ is

$$y = y_p + c_1 y_1 + c_2 y_2$$

where y_1, y_2 are a fundamental set of solutions to the homogeneous equation $L[y] = 0$, and c_1, c_2 are arbitrary constants.

Proof: First we show that y is actually a solution to differential equation $L[y] = g(t)$. Actually,

$$L[y] = L\big[y_p + c_1 y_1 + c_2 y_2\big] = L\big[y_p\big] + c_1 L\big[y_1\big] + c_2 L\big[y_2\big] = 0.$$

It remains to show that any solution to the differential equation $L[y] = g(t)$ can be represented this way. To this end, suppose y is any solution to the non-homogeneous equation. We have

$$L\big[y - y_p\big] = L\big[y\big] - L\big[y_p\big] = g(t) - g(t) = 0$$

or, $y - y_p$ is a solution to the homogeneous equation $L[y] = 0$. Therefore

$$y - y_p = c_1 y_1 + c_2 y_2$$

for some constants c_1, c_2. This completes the proof. $\square$

This theorem provides a general method to solve a non-homogeneous linear equation.

4.1 Method of Undetermined Coefficients

This is essentially a *guess-and-verify* method. We make an assumption of the form of a particular solution of $L[y] = g(t)$, with some parameters to be determinded. Let illustrate by examples.

Example: Find the general solution of $y'' + 4y = x^2 + x$.

Solution: Try $y_p = Ax^2 + Bx + C$. We have

$$L\big[y_p\big] = 2A + 4(Ax^2 + Bx + C) = x^2 + x \;\Rightarrow\; 4A = 1, 4B = 1, 2A + 4C = 0$$

or $A = B = \frac{1}{4}$, $C = -\frac{1}{8}$. Therefore a particular solution

$$y_p = \frac{1}{4}x^2 + \frac{1}{4}x - \frac{1}{8}.$$

The corresponding homogeneous equation

$$y'' + 4y = 0$$

has general solution $c_1 \cos 2x + c_2 \sin 2x$. Therefore, the general solution to the non-homogeneous equation is

$$y = \frac{1}{4}x^2 + \frac{1}{4}x - \frac{1}{8} + c_1 \cos 2x + c_2 \sin 2x.$$

Exercise: Solve initial value problem

$$y'' + 4y = x^2 + x, \quad y(0) = 1,\; y'(0) = 0$$

Solution: We have

$$y(0) = -\frac{1}{8} + c_1 = 1, \quad y'(0) = \frac{1}{4} + 2c_2 = 0$$

Hence $c_1 = \frac{9}{8}$, $c_2 = -\frac{1}{8}$, and

$$y = \frac{1}{4}x^2 + \frac{1}{4}x - \frac{1}{8} + \frac{9}{8}\cos 2x - \frac{1}{8}\sin 2x.$$

Example: Find general solutions of $y'' + 4y = 10e^x$.

Solution: Try $y_p = Ae^x$, with A to be determined.

$$L\big[y_p\big] = Ae^x + 4Ae^x = 5Ae^x \;\Rightarrow 5A = 10 \;\Rightarrow\; A = 2.$$

The general solution is

$$y = 2e^x + c_1 \cos 2x + c_2 \sin 2x.$$

Exercise: Find general solutions of $y'' + 4y = x^2 + x + e^x$.

Solution: It follows from previous two examples that

$$L\Big[\frac{1}{4}x^2 + \frac{1}{4}x - \frac{1}{8}\Big] = x^2 + x \;\text{ and }\; L\Big[\frac{1}{5}e^x\Big] = e^x.$$

This implies a particular solution is

$$y_p = \frac{1}{4}x^2 + \frac{1}{4}x - \frac{1}{8} + \frac{1}{5}e^x$$

and the general solution is $y = y_p + c_1 \cos 2x + c_2 \sin 2x$.

Remark: Sometime $g(t)$ can be written as $g(t) = g_1(t) + \cdots + g_n(t)$ so that it is relatively easier to find a particular $y_p^{(i)}$ such that

$$L\big[y_p^{(i)}\big] = g_i(t), \qquad \forall\ i = 1, 2 \cdots, n$$

A particular solution to $L[y] = g(t)$ is given by

$$y_p = y_p^{(1)} + y_p^{(2)} + \cdots + y_p^{(n)}.$$

Exercise: Find general solutions to the following equations.

1. $y'' + 2y' + y = \sin x + 3\cos x \qquad$ (*Hint:* Try $y_p = A\cos x + B\sin x$)
2. $y'' - 3y' + 2y = xe^x \qquad$ (*Hint:* Try $y_p = Axe^x + Be^x$)
3. $y'' - 3y' + 2y = e^x \cos x \qquad$ (*Hint:* Try $y_p = Ae^x \cos x + Be^x \sin x$)

Sometime, a bit more guess-and-verify need to be done.

Example: Find general solution of equation $y'' - y = e^x$.

Solution: We guess a particular solution of form $y_p = Ae^x$, it follows that

$$L\big[y_p\big] = Ae^x - Ae^x \equiv 0.$$

It does not work. Make a second guess as $y_p = Axe^x$, we have

$$L\big[y_p\big] = 2Ae^x + Axe^x - Axe^x = 2Ae^x \ \Rightarrow\ A = \frac{1}{2} \ \Rightarrow\ y_p = \frac{1}{2}xe^x$$

Therefore, general solution is $y = \frac{1}{2}xe^x + c_1 e^x + c_2 e^{-x}$.

Example: Find general solution of $y'' + y = \cos x$.

Solution: Try a particular solution $y_p = A\cos x + B\sin x$, we have

$$L\big[y_p\big] = -A\cos x - B\sin x + A\cos x + B\sin x \equiv 0.$$

Try a second guess of $y_p = Ax\cos x + Bx\sin x$, and we have

$$L\big[y_p\big] = -2A\sin x - Ax\cos x + 2B\cos x - Bx\sin x + Ax\cos x + Bx\sin x = -2A\sin x + 2B\cos x$$

Therefore a particular solution is $y_p = \frac{1}{2}x\cos x$ and the general solution is

$$y = \frac{1}{2}x\cos x + c_1\cos x + c_2\sin x$$

Exercise: Find a particular solution $y'' + 2y' + 1 = e^{-x}$. (*Hint:* Try Ae^{-t}, then Ate^{-t}, and then At^2e^{-t})

4.2 Variation of Parameters

Although it is easy to carried out, Method of Undetermined Coefficients, which requires function $g(t)$ to have some specific form (to allow us to guess the correct form of the solution), has very limited application. In this section, we discuss method of Variation of Parameters. In principle, it can be applied to any form of $g(t)$ – actually we have an explicit integral formula for the solution with respect to any $g(t)$. However, the limitation if this method lies right here, since it asks us to evaluate some integral that might turn out to be rather difficult (or impossible).

Example: Solve differential equation

$$y'' + 3y' + 2y = \sin(e^x)$$

Solution: The characteristic equation is $r^2+3r+2=0$, which has two roots $r_1 = -1, r_2 = -2$. Hence the general solution to the corresponding homogeneous equation is

$$y = c_1 e^{-t} + c_2 e^{-2t}$$

Replacing constants c_1, c_2 by functions $u_1(t), u_2(t)$ respectively, we have

$$y = u_1(t)e^{-t} + u_2(t)e^{-2t}.$$

We wish to determine $u_1(t), u_2(t)$ so that y is a particular solution to the non-homogeneous equation. It follows that

$$y' = u_1' e^{-t} + u_2' e^{-2t} - u_1 e^{-t} - 2u_2 e^{-2t}.$$

We impose the first condition to require the term involving u_1' and u_2' to be zero, that is

$$u_1' e^{-t} + u_2' e^{-2t} = 0 \qquad \text{(Condition 1)}$$

Therefore $y' = -u_1 e^{-t} - 2u_2 e^{-2t}$, which implies

$$y'' = u_1 e^{-t} + 4u_2 e^{-2t} - u_1' e^{-t} - 2u_2' e^{-2t}$$

and

$$y'' + 3y' + 2y = -u_1' e^{-t} - 2u_2' e^{-2t}$$

Hence we obtain the second condition

$$-u_1' e^{-t} - 2u_2' e^{-2t} = \sin(e^t) \qquad \text{(Condition 2)}$$

Solving condition 1 and 2, we obtain

$$u_1' = e^t \sin(e^t), \qquad u_2' = -e^{2t}\sin(e^t).$$

It follows that

$$\begin{aligned}
u_1 &= \int e^t \sin(e^t)\,dt = \int \sin(e^t)\,d(e^t) = -\cos(e^t) \\
u_2 &= -\int e^{2t}\sin(e^t)\,dt = -\int x\sin x\,dx \quad (x = e^t) \\
&= x\cos x - \sin x \quad \text{(integration by parts)} \\
&= e^t\cos(e^t) - \sin(e^t)
\end{aligned}$$

Therefore, a particular solution is

$$y_p = -e^{-2t}\sin\left(e^t\right)$$

The general solution is

$$y = c_1 e^{-t} + c_2 e^{-2t} - e^{-2t}\sin\left(e^t\right).$$

Let us now give a general description of this method. Consider differential equation

$$y'' + p(t)y' + q(t)y = g(t)$$

and suppose that y_1, y_2 is a fundamental set of solutions to the corresponding homogeneous equation

$$y'' + p(t)y' + q(t)y = 0,$$

whose general solution is therefore $y = c_1 y_1 + c_2 y_2$.

Replace constants c_1, c_2 by functions $u_1(t), u_2(t)$. We wish to determine u_1, u_2 so that

$$y \stackrel{\triangle}{=} u_1 y_1 + u_2 y_2$$

is a particular solution to the non-homogeneous equation. Observe

$$y' = u_1' y_1 + u_2' y_2 + u_1 y_1' + u_2 y_2'.$$

We impose the first condition to erase the terms involving u_1', u_2':

$$u_1' y_1 + u_2' y_2 = 0 \qquad \text{(condition 1)}$$

Hence $y' = u_1 y_1' + u_2 y_2'$, $y'' = u_2' y_1' + u_2' y_2' + u_1 y_1'' + u_2 y_2''$, and

$$y'' + p(t)y' + q(t)y = u_1\left[y_1'' + p(t)y_1' + q(t)y_1\right] + u_2\left[y_2'' + p(t)y_2' + q(t)y_2\right] + u_2' y_1' + u_2' y_2'.$$

We obtain our second condition

$$u_2' y_1' + u_2' y_2' = g(t) \qquad \text{(condition 2)}$$

Solving condition 1 and 2, we have

$$u_1' = -\frac{y_2 g}{W}, \qquad u_2' = \frac{y_1 g}{W}$$

where W is the Wrondskian of y_1 and y_2. Therefore, a particular solution is

$$y_p = -y_1 \int \frac{y_2 g}{W} + y_2 \int \frac{y_1 g}{W}$$

and general solution is

$$y = c_1 y_1 + c_2 y_2 + y_p.$$

Exercise: Find the general solution of equation

$$y'' + 4y' + 4y = 3xe^{2x}$$

using both methods of Undetermined Coefficients and Variation of Parameters.

4.3 Reduction of Order (revisited)

Suppose we have been able to find a nontrivial solution y_1 to the homogeneous equation

$$y'' + p(t)y' + q(t)y = 0.$$

It is already shown that by the reduction of order method we shall obtain a second linear independent solution to this homogeneous equation, and thus the general solutions. Here we show that by this method we shall also be able to obtain from y_1 the general solutions of a non-homegenous equation

$$y'' + p(t)y' + q(t)y = g(t).$$

The basic procedure remains the same. Assuming that $y_2(t) = v(t)y_1(t)$ is a second solution, we have

$$y_2' = v'y_1 + vy_1', \qquad y_2'' = v''y_1 + 2v'y_1' + vy_1''$$

and

$$v''y_1 + 2v'y_1' + vy_1'' + p(t)v'y_1 + p(t)vy_1' + q(t)vy = g(t) \;\Rightarrow\; y_1v'' + \big[2y_1' + p(t)y_1\big]v' = g.$$

Letting $w \triangleq v'$, we obtain a first order differential equation

$$y_1w' + \big[2y_1' + p(t)y_1\big]w = 0$$

This is now a linear equation and can be solved by finding the integrating factor.

Example: Given $y_1 = t$ is a solution to equation

$$t^2y'' + ty' - y = 0, \qquad t > 0,$$

find the general solution of the non-homogeneous equation

$$t^2y'' + ty' - y = t, \qquad t > 0.$$

Solution: Let $y_2(t) = v(t)y_1(t) = tv(t)$. We have

$$y_2' = tv' + v, \qquad y_2'' = tv'' + 2v'$$

Substituting this in the non-homogeneous equation gives

$$t^2(tv'' + 2v') + t(tv' + v) - tv = t \quad\Rightarrow\quad t^3v'' + 3t^2v' = t.$$

Let $w = v'$, we have

$$t^3w' + 3t^2w = t \quad\Rightarrow\quad (t^3w)' = t \quad\Rightarrow\quad t^3w = \frac{1}{2}t^2 + c$$

or

$$w = \frac{1}{2t} + \frac{c}{t^3} \quad\Rightarrow\quad v = \int w\,dt = \frac{1}{2}\ln t - \frac{c}{2t^2} + c'$$

which is equivalent to

$$y_2(t) = \frac{1}{2}t\ln t - \frac{c_1}{t} + c_2t.$$

This is the general solution of the non-homogeneous equation.

Exercise: Given $y_1(x) = x^2$ is a solution to the homogeneous equation

$$x^2 y'' - 2y = 0.$$

Find the general solution of non-homogeneous equation

$$x^2 y'' - 2y = 2x.$$

(*Answer:* $y = c_1 x^2 + \frac{c_2}{x} + \frac{2}{3}x^2 \ln x$)

5 Vibrations and Equibilirium

Newton's Law of Motion says $F = ma$, where m is the mass of the particle, a is its acceleration, and F is net external force.

5.1 Free Motion

Here we shall consider two applications of second order linear equation to physics.

Example 1 (Elastic Spring. Hooke's Law): Suppose a mass m is attached to a verticle spring whose unstreched length is l. This mass is subject to two forces: its own weight mg, and force due to spring F_s. It follows from *Hooke's Law*, with the positive direction taken as downward,

$$F_s = -ky.$$

Here the positive constant k is called *spring constant.*

1. *Undamped Free Vibration:* Assuming for a moment that there is no damping force (e.g. air resistance), the net external force is therefore

$$F = mg + F_s = mg - ky \quad (g \text{ is the accerlation due to gravity})$$

which implies, by Newton's Law,

$$F = mg - ky = ma = m\ddot{y} \;\Rightarrow\; \ddot{y} + \frac{k}{m}y = g.$$

Equilibrium Position: By letting $\ddot{y} = 0$, we obtain the equilibrium position

$$y^* = \frac{mg}{k}.$$

At equilibrium position y^*, the mass has a net external force $F = mg - ky^* = 0$.

Let $u \stackrel{\triangle}{=} y - y^*$ (deviation from equilibrium). It is easy to see that u satisfies equation

$$\ddot{u} + \frac{k}{m}u = 0,$$

which has general solution, with $\omega_0 \stackrel{\triangle}{=} \sqrt{\frac{k}{m}}$

$$u = c_1 \cos \omega_0 t + c_2 \sin \omega_0 t = R\cos(\omega_0 t - \delta).$$

Here reparametrization R, δ are determined as before by

$$R = \sqrt{c_1^2 + c_2^2}, \qquad \cos\delta = \frac{c_1}{R}, \;\; \sin\delta = \frac{c_2}{R}.$$

It is easy to see that R is the maximal deviation from the equilibrium.

Terminology: The parameters ω_0, R and δ are called *natural frequency, amplitude*, and *phase* respectively. $T \stackrel{\triangle}{=} \frac{2\pi}{\omega_0}$ is called *period.*

Example: A 2-pound mass attached to a spring stretches it 4 inches. If the mass is displaced an additional 4 inches and is released. Determined the position of the mass at time t.

Solution: We first determine the mass m by

$$m = \frac{mg}{g} = \frac{2 \text{ lb}}{32 \text{ ft/sec}^2} = \frac{1}{16}\frac{\text{lb-sec}^2}{\text{ft}}$$

Since equilibrium position $y^* = \frac{mg}{k} = 2$, we obtain

$$k = \frac{mg}{4\text{-in}} = \frac{2 \text{ lb}}{\frac{1}{3} \text{ ft}} = 6\frac{\text{lb}}{\text{ft}}$$

Let u be the deviation from equilibrium. We have equation

$$0 = \ddot{u} + \frac{k}{m}u = \ddot{u} + 96u,$$

which hasgeneral solution $u = c_1 \cos\sqrt{96}t + c_2 \sin\sqrt{96}t$. The initial conditions are

$$u(0) = 4 \text{ in} = \frac{1}{3} \text{ ft}, \quad u'(0) = 0,$$

which yield $c_1 = \frac{1}{3}$, $c_2 = 0$, or

$$u = \frac{1}{3}\cos\sqrt{96}t.$$

The amplitude is $R = \frac{1}{3}$-ft, phase $\delta = 0$, natural frequency $\omega_0 = \sqrt{96} = 9.8\,\frac{\text{rad}}{\text{sec}}$ (hence the period is $T = \frac{2\pi}{\omega_0} = 0.64$-sec). $\square$.

2. *Damped Free Vibration:* Now let us take the damping force F_d into consideration. Suppose the mass is also subject to a damping force proportional to its velocity,

$$F_d = -\gamma u'(t).$$

Here $\gamma > 0$ is the constant of proportionality. The equilibrium position will NOT change. The net external force is therefore

$$mu'' = ma = F = mg - k(y^* + u) - \gamma u' \;\Rightarrow\; mu'' + \gamma u' + ku = 0$$

here u is the deviation from the equilibrium like before. The corresponding characteristic equation is

$$mr^2 + \gamma r + k = 0 \;\Rightarrow\; r_{1,2} = -\frac{\gamma}{2m} \pm \frac{\sqrt{\gamma^2 - 4km}}{2m}.$$

There are three cases.

1. Overdamped ($\gamma > 2\sqrt{km}$): In this case, $r_{1,2}$ are both real and *negative* (check!). The general solution is

$$u = Ae^{r_1 t} + Be^{r_2 t}$$

It is easy to see that the motion is *non-oscillatory* (see graph below). Moreover, as $t \to \infty$, the deviation will diminish.

2. Critically damped ($\gamma = 2\sqrt{km}$): In this case, $r_{1,2} = r = -\frac{\gamma}{2m}$ is real and *negative*. The general solution is

$$u = Ae^{rt} + Bte^{rt}.$$

Like in the overdamped case, the motion is *non-oscillatory* and diminishes as $t \to \infty$. The graphes are also similar to the overdamped case.

3. Underdamped ($\gamma < 2\sqrt{km}$): In this case, $r_{1,2}$ are conjuate complex numbers with *negative* real parts. The general solution is

$$u = e^{-\frac{\gamma}{2m}t}\big(A\cos\mu t + B\sin\mu t\big) = Re^{-\frac{\gamma}{2m}t}\cos(\mu t - \delta).$$

Here $\mu \triangleq \frac{\sqrt{4km-\gamma^2}}{2m} > 0$, and R, δ are determined by $A = R\cos\delta, B = R\sin\delta$. Clearly the motion is *oscillatory* (but *not* periodic), and diminishes as $t \to \infty$. It has a *damped amplitude* $Re^{-\frac{\gamma}{2m}t}$, which goes to zero as $t \to \infty$. See the following graph.

Qunatities $\mu = \frac{\sqrt{4km-\gamma^2}}{2m}$ and $T \triangleq \frac{2\pi}{\mu}$ are called *quasi-frequency* and *quasi-period* respectively.

Example 2 (Archimedes' Principle): Archimedes' Principle states that a body submerged in a fluid is subjected to an upward (*buoyant*) force with magnitude equal to the weight of displaced fluid.

Floating in a liquid is a l ft high cylindrical body with cross-sectional area S (sq ft). The mass densities of the body and the liquid are ρ and $\hat{\rho}$ per unit volume, respectively. Assume $r \triangleq \frac{\rho}{\hat{\rho}} < 1$ (otherwise, the body will sink), and neglect viscous damping forces. Let y stand for the distance from bottom of the body to the surface, with the positive direction taken as downward. The body has mass $m = \rho l S$, while the displaced liquid has mass $\hat{m} = \hat{\rho} y S$. Therefore, the net force on the body is

$$F = mg - \hat{m}g = Sg(\rho l - \hat{\rho} y) \quad \Rightarrow \quad y'' = \frac{F}{m} = \frac{F}{\rho l S} = g\left(1 - \frac{y}{rl}\right)$$

We obtain immediately the *equilibrium* position $y^* = rl$.

Let $u \triangleq y - y^*$ be the deviation from the equilibrium. It follows that u satisfies equation

$$u'' + \frac{g}{rl}u = 0.$$

Even though this is a very different physics problem, we end up with same type of differential equation. All the properties we have obtained in the spring example hold in this case. We leave the details to the interested students.

Exercise: Show that the period for the body motion is

$$T = 2\pi\sqrt{\frac{rl}{g}}.$$

5.2 Forced Motion

In this section, we will only consider the undamped motion. Interested readers can find more details in the textbook. Suppose now a periodic external force, say $F = F_0 \cos \omega t$, is imposed to the system. Here ω is the frequency of the external force.

Elastic Spring (continued): Let $u = y - y^*$ be the deviation from equilibrium. We have

$$mu'' + ku = F_0 \cos \omega t.$$

Exercise: Show that, with $u_0 \triangleq \sqrt{\frac{k}{m}}$ as the natural frequency,

$$u_p \triangleq \begin{cases} & ; \text{ if } \omega_0 \neq \omega \\ \frac{F_0}{2m\omega_0} t \sin \omega_0 t & ; \text{ if } \omega_0 = \omega \end{cases}$$

is a particular solution to the differential equation. Hence the general solution takes form

$$u = c_1 \cos \omega_0 t + \sin \omega_0 t + u_p.$$

¿From the form of u_p, we discuss the motion according to $\omega_0 \neq \omega$ or $\omega_0 = \omega$.

$\omega_0 \neq \omega$: In this case, the resulting oscillation of the body is the sum of two periodic displacements with different frequencies, namely ω_0 and ω. It is said to be a *stable motion* in the sense that the deviation remains bounded. This motion, however, might be non-periodic. Actually it is periodic if and only if $\frac{\omega_0}{\omega}$ is a rational number.

$\omega_0 = \omega$: (Resonance) The presence of variable t in the particular solution implies that the deviation goes unbounded as $t \to \infty$ (hence an *unstable* motion). In such cases, a mechinal breakdown of the system is bound to occur. This phenomenon, where the frequency of external force equals the natural frequency of system, is known as (undamped) *resonance*.

6 Weak Maximum Principle

Weak Maximum Principle: Suppose y satisfies the following differential inequality

$$y'' + p(t)y' \geq 0, \qquad a \leq t \leq b$$

where $p(t)$ is continuous on interval $[a, b]$. Show that

$$\max_{t\in[a,b]} y(t) = \max\{y(a), y(b)\}$$

That is, the maximum of y is achieved at the boundary of the interval.

Proof: To ease exposition, let

$$L[y] \stackrel{\triangle}{=} y'' + p(t)y'.$$

For any constant α, we have

$$L[e^{\alpha t}] = e^{\alpha t}(\alpha^2 + p(t)\alpha).$$

Since $p(t)$ is continuous on interval on interval $[a, b]$, it must be bounded. Fix any $\alpha > \max_{t\in[a,b]} |p(t)|$, we have

$$L[e^{\alpha t}] = e^{\alpha t}(\alpha^2 + p(t)\alpha) > 0$$

for all $t \in [a, b]$. Hence, for any $\varepsilon > 0$, let $u_\varepsilon = y + \varepsilon e^{\alpha t}$. We have

$$L[u_\varepsilon] = L[y + \varepsilon e^{\alpha t}] = L[y] + \varepsilon L[e^{\alpha t}] > 0.$$

for all $t \in [a, b]$. We claim that

$$\max_{t\in[a,b]} u_\varepsilon(t) = \max\{u_\varepsilon(a), u_\varepsilon(b)\}.$$

Otherwise, there exist $c \in (a, b)$ such that u_ε achieves maximum at $t = c$. However, this implies $u_\varepsilon'(c) = 0$ and $u_\varepsilon''(c) \leq 0$, or

$$L[u_\varepsilon](c) = u_\varepsilon''(c) + p(c)u_\varepsilon'(c) = u_\varepsilon''(c) \leq 0,$$

a contradiction.

Now letting $\varepsilon \to 0$, we have $u_\varepsilon \to y$ uniformly on interval$[a, b]$. Therefore

$$\max_{t\in[a,b]} y(t) = \max\{y(a), y(b)\}$$

This completes the proof. □

From now on we will let $L[y] \triangleq y'' + p(t)y'$. We have the following comparison principle. Note the difference here is that the initial conditions are the initial values of the solution at the two end-points.

Comparison principle: Suppose $L[y] \geq L[z]$ for all $t \in [a, b]$ and $y(t) \leq z(t)$ for $t = a, b$ (boundary of the interval). Then $y(t) \leq z(t)$ for all $t \in [a, b]$.

Proof: Let $w \triangleq y - z$. We have

$$L[w] = L[y] - L[z] \geq 0, \qquad w(a) \leq 0,\ w(b) \leq 0.$$

Therefore, by the weak maximum principle, we have

$$\max_{t \in [a,b]} w(t) = \max\{w(a), w(b)\} \leq 0.$$

This completes the proof. □

Corollary (Uniqueness): The initial value problem

$$L[y] = 0, \qquad t \in [a, b]$$

with initial condition $y(a) = y_0,\ y(b) = y_1$ has at most one solution.

Proof: Suppose y and z are both the solution to the above initial value problem, we have

$$L[y] = L[z], \qquad y(a) = z(a),\ y(b) = z(b).$$

It follows that $y \geq z$ and $y \leq z$ for all $t \in [a, b]$ (Comparison principle). Therefore $y(t) \equiv z(t)$ for all $t \in [a, b]$. □

Remark: the comparison principle and uniqueness hold for more general linear operator

$$L[y] = y'' + p(t)y' + q(t)y$$

where $q(t) \leq 0$. One can derive this general comparison principle from weak maximum principle, which is not terribly difficult.

7 Non-linear Second order ODE

Consider the following general nonlinear initial value problem of second order

$$y'' = f(t; y, y'), \qquad t \in I.$$

with initial condition $y(t_0) = y_0, y'(t_0) = z_0$. We have the following fundamental theorem regarding the existence and uniqueness of the solution.

Theorem: If functions $f(t; y, z)$, $f_y(t; y, z)$ and $f_z(t; y, z)$ are continuous in region

$$R = \left\{(t, y, z) /\ |t - t_0| \leq a,\ |y - y_0| \leq b,\ |z - z_0| \leq c\right\},$$

then there exists a unique solution $y = \phi(t)$ of the initial value problem

$$y'' = f(t; y, y'), \qquad y(t_0) = y_0,\ y'(t_0) = z_0$$

on an interval $|t - t_0| \leq h$ for some $h \leq a$.

Proof: Let $z \triangleq y'$, we have the *equivalent* system of first order equations

$$\begin{cases} y' &= z \\ z' &= f(t; y, z) \end{cases}$$

Write

$$W = \begin{bmatrix} y \\ z \end{bmatrix}, \qquad F(t; W) = \begin{bmatrix} z \\ f(t, y, z) \end{bmatrix}.$$

We have

$$\frac{dW}{dt} = F(t, W), \qquad W(t_0) = \begin{bmatrix} y_0 \\ z_0 \end{bmatrix}$$

Using the same iteration method as for the first order ODE and the Gronwall inequality, the existence and uniquess of the solution $W = \Phi(t) = [\Phi_1(t), \Phi_2(t)]^t$ follows. Therefore $\phi(t) = \Phi_1(t)$ is the solution to the original second order initial value problem. $\square$

In the following, we will collect miscellaneous problems and examples.

Boundedness: Suppose y solve the differential equation

$$y'' + V'(y) = 0, \qquad t \in \mathbb{R}$$

where V is a smooth function on $\mathbb{R}$ with

$$\lim_{z \to \pm\infty} V(z) = \infty.$$

Show that y is bounded.

Proof: Multiplying both side by y', we have

$$y'y'' + V'(y)y' = 0 \quad \Rightarrow \quad \frac{d}{dt}\left(\frac{1}{2}(y')^2 + V(y)\right) = 0$$

or

$$\frac{1}{2}(y')^2 + V(y) = c$$

for some constant c. However, this implies that $V(y)$ is bounded from above by the constant c. Since $\lim_{z \to \pm\infty} V(z) = \infty$, we must have that $y(t)$ is bounded. $\square$

Remark: The above method also provides a way to solve the type of differential equation

$$y'' + V'(y) = 0.$$

We should have

$$\frac{1}{2}(y')^2 + V(y) = c \quad \Rightarrow \quad y' = \pm\sqrt{2\big((c - V(y)\big)} \quad \Rightarrow \quad \frac{dy}{\sqrt{2\big((c - V(y)\big)}} = \pm dt$$

The following are two example of the applications of the nonlinear differential equation of second order: *populations of two interacting species* and *Newton's law of universal gravitation.*

A biological problem: There is a constant struggle for survival among different species. For example, certain birds live on fish. If there is an abundant supply of fish, the population of this bird species grows. When these birds become too numerous and consume too much fish, thus reducing the fish population, then their own population begins to diminish. When the number of birds decreases, the fish population increases, then the bird population starts increasing. This induces an endless cycle of periodic increases and decreases in the respective populations of the two species.

We will consider the problem of determining the populations of two interacting species: a parasitic species P hatches its eggs in a host species H. Unfortunately, the deposit of an egg in a member of H causes this member's death. Let

1. H_b denote the birth rate for the H species.
2. H_d denote the (natural) death rate for the H species, if no P species were present.
3. P_d denote the (natural) death rate for the P species.

Suppose (x, y) denote the population size for H and P species, respectively. Also assume that the number of eggs per year deposited by the P species is propotional to the probability that the members of the two species meet. Since this probability depends on the product xy of the two population size, we assume that, in time $\triangle t$, the approximate number of eggs deposited on H species is $kxy\triangle t$. We have

$$\begin{aligned} \triangle x &= H_b x\triangle t - H_d x\triangle t - kxy\triangle t \\ \triangle y &= kxy\triangle t - P_d y\triangle t. \end{aligned}$$

Letting $h = H_b - H_d$, $p = P_d$ and $\triangle t \to 0$, we have

$$\begin{aligned} x' &= hx - kxy = x(h - ky) \\ y' &= kxy - py = y(kx - p). \end{aligned}$$

This is a system of first order differential equations. It is possible to solve y as a function of x. Actually,

$$\frac{dy}{dx} = \frac{y'}{x'} = \frac{y(kx-p)}{x(h-ky)} \quad \Rightarrow \quad \left(\frac{h}{y} - k\right) dy = \left(k - \frac{p}{x}\right) dx,$$

which leads to, for some constant c,

$$\ln\left(cy^h x^p\right) = k(x+y) \quad \Rightarrow \quad cy^h e^{-ky} = x^{-p} e^{kx}$$

If $x(0) = x_0,\ y(0) = y_0$, then one can determine

$$c = x_0^{-p} y_0^{-h} e^{k(x+y)}.$$

However, such an implicit solution might not be intuitive to deal with.

We will give an approximation of the system (*linearization of first order systems*). The equilibrium solution to this system is

$$h - ky^* = 0, \quad kx^* - p = 0 \qquad \Rightarrow \qquad y^* = \frac{h}{k}, \quad x^* = \frac{p}{k}$$

Use the change of the variable,

$$x = x^* + X, \qquad y = y^* + Y.$$

We have (check!)

$$\frac{dX}{dt} = -pY - kXY, \qquad \frac{dY}{dt} = hX + kXY.$$

However, if the deviations from the equilibrium (X, Y) is small, we may, without serious error, discard the XY term. These equations then become

$$\frac{dX}{dt} = -pY, \quad \frac{dY}{dt} = hX \qquad \Rightarrow \qquad \frac{dY}{dX} = -\frac{hX}{pY}.$$

It follows that

$$\frac{X^2}{p} + \frac{Y^2}{h} = c$$

for some constant c. The graph is an ellipse (when $h = p$, it is a circle). □

New's inverse square law: Kepler's three laws of planetary motion are:

1. The planet moves in an elliptical orbit with the sun at one focus.
2. The radius vector connecting sun and the planet sweeps out equal areas in equal times.
3. The square of the period of the planet's movement is proportional to the cube of the semimajor axis of its orbit (a in the graph).

Newton's law of universal gravitation states that: if M and m are the mass of two planets, then the two bodies attracts each other with a central force

$$F = -\frac{GMm}{r^2}.$$

Here r is the distance between the two planets.

In the following, we are going to derive (partially) the Newton's law from Kepler's first two laws. Indeed we are going to show that F is a central force inversely proportional to the square of the distance r.

Proof: We will divide the proof into several steps.

Step 1: We are going to use the polar coordinates (r, θ) to denote the position of the planet (see the graph). It follows from analytical geometry that

$$r = \frac{c}{1 + k\cos\theta}$$

for some constant c and some constant $k < 1$. The constant k is called *eccentricity* (indeed, if $k = 1$, this equation corresponds to a parabola; if $k > 1$, it corresponds to a hyperbola). Here we give a short proof. We have

$$r + \bar{r} = \text{constant} := l \qquad \Rightarrow \qquad r + \sqrt{(r\sin\theta)^2 + (r\cos\theta + d)^2} = l,$$

which implies

$$(l - r)^2 = r^2 + 2rd\cos\theta + d^2 \quad \Rightarrow \quad l^2 - d^2 = r(2l + 2d\cos\theta) \quad \Rightarrow \quad r = \frac{\frac{l^2 - d^2}{2l}}{1 + \frac{d}{l}\cos\theta}$$

But $k = \frac{d}{l} < 1$. This completes the proof.

Step 2: Suppose at time t, the planet is at position $\big(r(t), \theta(t)\big)$. The speed and acceleration of the body can both be decomposed into two components: one along the radius r, the other in a direction perpendicular to it (see the graph below). We claim

$$v_r = r', \quad v_\theta = r\theta'; \qquad a_r = r'' - r(\theta')^2, \quad a_\theta = 2r'\theta' + r\theta''.$$

Actually, since $x = r\cos\theta, y = r\sin\theta$, we have

$$\begin{aligned} \text{speed along } x\text{-axis} &= x' = \cos\theta \cdot r' - \sin\theta \cdot r\theta' = \cos\theta \cdot v_r - \sin\theta \cdot v_\theta \\ \text{speed along } y\text{-axis} &= y' = \sin\theta \cdot r' + \cos\theta \cdot r\theta' = \sin\theta \cdot v_r + \cos\theta \cdot v_\theta \end{aligned}$$

which clearly implies that $v_r = r',\ v_\theta = r\theta'$. Similarly,

$$\begin{aligned} \text{accleration along } x\text{-axis} &= x'' = \cos\theta \cdot \left(r'' - r(\theta')^2\right) - \sin\theta \cdot (2r'\theta' + r\theta'') = \cos\theta \cdot a_r - \sin\theta \cdot a_\theta \\ \text{accleration along } y\text{-axis} &= y'' = \sin\theta \cdot \left(r'' - r(\theta')^2\right) + \cos\theta \cdot (2r'\theta' + r\theta'') = \sin\theta \cdot a_r + \cos\theta \cdot a_\theta \end{aligned}$$

which gives $a_r = r'' - r(\theta')^2,\ a_\theta = 2r'\theta' + r\theta''$.

Step 3: By Kepler's second law, if we denote $A(t)$ the area that the radius vector connecting sun and the planet sweeps up to time t, we have $\frac{dA}{dt}$ is a constant, say $\frac{p}{2}$. However, $dA = \frac{1}{2}r^2\,d\theta$ (why?), which implies that

$$r^2\frac{d\theta}{dt} = p \quad \Rightarrow \quad 2rr'\theta' + r^2\theta'' = 0 \quad \Rightarrow \quad 2r'\theta' + r\theta'' = 0.$$

In other words, the acceleration perpendicular to the radius a_θ equals 0. Since this component is zero, the force acting on the planet must be a central one.

Step 4: As we have already seen, the equation for an elliptical orbit in polar coordinates is

$$r = \frac{c}{1 + k\cos\theta}$$

for some constant c and eccentricity e. We have

$$r' = \frac{ce\sin\theta}{(1 + k\cos\theta)^2}\theta' = r^2\frac{k\sin\theta}{c}\frac{p}{r^2} = \frac{pk}{c}\sin\theta.$$

Therefore

$$r'' = \frac{pk}{c}\cos\theta \cdot \theta' = \frac{p^2}{cr^2}(k\cos\theta) = \frac{p^2}{cr^2}\left(\frac{c}{r} - 1\right).$$

The last equality follows from the equation for the ellipse. The component of the force in the radius direction is

$$F_r = ma_r = m\left(r'' - r(\theta')^2\right) = m\left(\frac{p^2}{r^3} - \frac{p^2}{cr^2} - r\frac{p^2}{r^4}\right) = -\frac{mp^2}{cr^2}.$$

This completes the proof. $\square$