Introduction

In this chapter, we will consider ordinary differential equations of first order, that is,

$$F(t; y, y') = 0.$$

Unfortunately, only very few special types of first order differential equations admit solutions that can be expressed in terms of elementary functions. In fact, one could say that *almost all* first order differential equations can not be expressed. In the following, we will discuss several special classes of first order ODE that are solvable, illustrated by "textbook" examples, as well as some general existence and uniqueness results.

1 Linear First Order ODE

In this section we will focus on linear first order ODE of following type:

$$\frac{dy}{dt} + p(t)y = g(t); \qquad \text{sometime} \;\; \dot{y} + p(t)y = g(t).$$

This class of differential equations are easy to solve. Indeed, here we have our first existence and uniqueness result:

Theorem: Provided $p(t)$ and $g(t)$ are continuous functions on an open interval $I = (\alpha, \beta)$, there exists a *unique* solution to the initial value problem

$$\frac{dy}{dt} + p(t)y = g(t); \qquad y(t_0) = y_0$$

on *whole* interval I, for every $t_0 \in I$ and $y_0 \in \mathbb{R}$. Actually, the solution is given by

$$y(t) = \frac{\int_{t_0}^{t} \mu(s)g(s)\, ds + y_0}{\mu(t)}, \qquad \text{where} \;\; \mu(t) \triangleq e^{\int_{t_0}^{t} p(s)\, ds}; \qquad \forall\, t \in I.$$

Idea of the proof: The trick of solving this differential equation is the following: Suppose we can find a function $\mu(t)$ such that

$$\dot{\mu}(t) = \mu(t)p(t).$$

It follows that, by multiplying $\mu(t)$ on both sides of the equation,

$$\mu(t)\dot{y} + \mu(t)p(t)y = \mu(t)g(t) \;\Rightarrow\; \mu(t)\dot{y} + \dot{\mu}(t)y = \frac{d}{dt}[\mu(t)y] = \mu(t)g(t)$$

which yields

$$\mu(t)y = \int \mu(t)g(t)\,dt + c \;\Rightarrow\; y = \frac{\int \mu(t)g(t)\,dt + c}{\mu(t)}.$$

However, such $\mu(t)$ alwasy exists as

$$\frac{\dot{\mu}(t)}{\mu(t)} = \frac{d}{dt}\ln \mu(t) = p(t) \;\Rightarrow\; \mu(t) = e^{\int p(t)\,dt + c}.$$

All such functions differs up to a multiplicative factor. A possible choice is

$$\mu(t) = e^{\int_{t_0}^{t} p(s)\,ds} \;\Rightarrow\; y = \frac{\int_{t_0}^{t} \mu(s)g(s)\,ds + y_0}{\mu(t)},$$

which is a solution to the differential equation.

Proof: Existence: It is straight-forward to verify that function y defined above is a solution to the differential equation.

Uniqueness: Suppose both $y_1(t)$ and $y_2(t)$ are solutions to the differential equations. That is

$$y_i' + p(t)y_i = g(t), \qquad y_i(t_0) = y_0; \quad \forall\, i = 1, 2$$

Define $z(t) \overset{\triangle}{=} y_1(t) - y_2(t)$. It follows that

$$z' + p(t)z = 0, \qquad z(t_0) = 0;$$

Multiplying both side by $\mu(t)$ we obtain

$$\mu(t)z' + \mu(t)p(t)z = \frac{d}{dt}[\mu(t)z] = 0,$$

which implies that $\mu(t)z(t) \equiv const$. In particular, $\mu(t)z(t) \equiv \mu(t_0)z(t_0) = 0$. It follows that $z(t) \equiv 0$ since $\mu(t) > 0$. Therefore $y_1(t) \equiv y_2(t)$. $\square$.

Corollary: Suppose $p(t) \equiv r \neq 0$, $g(t) \equiv b$ are both constant. The differential equation

$$y' = -ry + b, \qquad y(t_0) = y_0;$$

has a unique solution

$$y(t) = \left(y_0 - \frac{b}{r}\right) e^{-r(t-t_0)} + \frac{b}{r}.$$

Remark: Here the existence and unique result is a *global* result, since the solution we find is defined on the whole interval. Later we will see that a *local* existence and uniqueness result for non-linear first order differential equation.

Remark: The multiplicator $\mu(t)$ is called *integrating factor.*

Example (Comparison Principle): Suppose x is a solution to the initial value problem

$$\frac{dx}{dt} + p(t)x = g_1(t), \qquad x(t_0) = x_0$$

while y is a solution to the initial value problem

$$\frac{dy}{dt} + p(t)y = g_2(t), \qquad y(t_0) = y_0.$$

If $g_1(t) \geq g_2(t)$, $\forall t \geq t_0$ and $x_0 \geq y_0$, then $x(t) \geq y(t)$ for all $t \geq t_0$.

Proof: Without loss of generality, assume $t_0 = 0$. Define $z(t) = x(t) - y(t)$. It is easy to see that z satisfies differential equation

$$z' + p(t)z = g_1(t) - g_2(t).$$

Multiplying both sides by integrating factor $\mu = e^{\int p(t)\,dt}$, we have

$$\mu(t)z + p(t)\mu(t)z = \frac{d}{dt}\big[\mu(t)z\big] = \mu(t) \cdot (g_1 - g_2) \geq 0$$

This implies that $\mu(t)z(t)$ is a non-decreasing function. In particular,

$$\mu(t)z(t) \geq \mu(0)z(0) \geq 0 \quad \Rightarrow \quad z(t) \geq 0$$

for all t. This completes the proof. □

Exercise: (Comparsion Principle) Following the preceding example, but assume $x(t)$ is instead a solution to a general the initial value problem

$$\frac{dx}{dt} = f(t,x), \qquad x(t_0) = x_0$$

where $f(t,x)$ is a continuous function such that $f(t,x) \geq -p(t)x + g_2(t)$ for all x and all $t \geq t_0$. Show the comparison inequality still holds, i.e. $x(t) \geq y(t)$ for all $t \geq t_0$.

Example:

1. Solve the following differential equation

$$y' + \frac{1}{x}y = 1; \qquad \forall\, x > 0.$$

2. If we have initial condition $y(1) = 1$, determine y.

Solution:

1. The integrating factor $\mu(x)$ is determined by

$$\mu'(x) = \frac{\mu(x)}{x} \;\Rightarrow\; \frac{d}{dx}\ln\mu = \frac{1}{x} \;\Rightarrow\; \ln\mu(x) = \ln x + c \;\Rightarrow\; \mu(x) = cx.$$

Any choice of c will yield a integrating factor, and any one will do. We just pick $c = 1$ and we have

$$xy' + y = x \;\Rightarrow\; (xy)' = x \;\Rightarrow\; xy = \frac{1}{2}x^2 + c \;\Rightarrow\; y = \frac{x}{2} + \frac{c}{x}.$$

2. $y(1) = 1$ yields $c = \frac{1}{2}$. Hence $y(x) = \frac{1}{2}\left(x + \frac{1}{x}\right)$. $\square$

Exercise: Solve the following differential equation

$$xy' + 2y = x^4 + 1, \qquad x > 0$$

Solution: Multiply both sides by $\mu(x) = x$, we obtain that

$$x^2y' + 2xy = x^5 + x \ \Rightarrow\ (x^2y)' = x^5 + x \ \Rightarrow\ x^2y = \frac{x^6}{6} + \frac{x^2}{2} + c$$

Therefore the general solution of the equation is

$$y = \frac{x^4}{6} + \frac{1}{2} + \frac{c}{x^2}.$$

Example (A geometric problem) Find the family of curves with the property that the segment of a tangent line drawn between a point of tangency and Y-axis is bisected by the X-axis.

Solution: Suppose $P = (x, y)$ is a point on the curve, and the tangent line drawn from point P will intersect Y-axis at point Q whose coordinates are, say $(0, z)$. Since the mid-point of P and Q has coordinates

$$\left(\frac{x+0}{2}, \frac{y+z}{2}\right)$$

we must have $z = -y$, which implies that

$$\frac{dy}{dx} = \text{slope of tangent line} = \frac{-y-y}{0-x} = \frac{2y}{x}.$$

Multiplying integrating factor $\mu(x) = \frac{1}{x^2}$, we have

$$\frac{d}{dx}\left(\frac{y}{x^2}\right) = 0 \quad \Rightarrow \quad y = cx^2 \ \text{ for some constant } c.$$

The family of curves we are looking for is $y = cx^2$. $\square$

Exercise: Find the family of curves such that the tangent line drawn between X-axis and Y-axis is bisected by the point of contact. That is, point P is the mid-point of Q and R.

Solution: Suppose $P = (x, y)$, $Q = (0, q)$ and $R = (r, 0)$. It is very easy to see that

$$\text{slope of the tangent line} = \frac{dy}{dx} = \frac{y-q}{x-0} = \frac{y-0}{x-r}$$

this implies that

$$q = y - xy', \qquad r = x - \frac{y}{y'}.$$

Since P is a mid-point of Q and R, we have

$$\frac{1}{2}(0+r) = x \quad \text{and} \quad \frac{1}{2}(q+0) = y$$

(actually, these two equations are equivalent), which imply that

$$xy' + y = 0 \quad \Rightarrow \quad (xy)' = 0 \quad \Rightarrow \quad xy = c$$

That is, the family of curves are hyperbolae. □

Example: (A mixing problem) A tank contains 100 gallons of water. In error 300 pounds of salt are poured into the tank instead of 200 pounds. To correct this error, a stopper is removed from the bottom of the tank allowing 3 gallons of brine to flow out per miniute. At same time, 3 gallon of fresh water per miniute are poured into the tank. If the mixture is kept uniform by constant stirring, how long will it take for the brine to contain the desire amount of salt?

Solution: Let $x(t)$ denote the amount of salt in the tank at time t. Since the mixture is well-stirred, the concentration of salt at time t would be $\frac{x}{100}$. In a small time interval $[t, t+dt)$, $3\,dt$ gallon brine will flow out of the tank, which means that $\frac{x}{100} \cdot 3\,dt$ will be the loss of salt from the solution. Therefore, we have

$$dx = -\frac{x}{100} \cdot 3\,dt \quad \Rightarrow \quad \frac{dx}{dt} + 0.03x = 0.$$

Solving this equation, we obtain

$$x(t) = ce^{-0.03t} \qquad \text{for some constant } c$$

But $x(0) = 300$, hence $c = 300$ and $x(t) = 300e^{-0.03t}$. The time t^* that the amount of salt will reach the desired is

$$x(t^*) = 300e^{-0.03t^*} = 200 \quad \Rightarrow \quad t^* = 13.5 \text{ (min)}$$

That is, it will take 13.5 miniutes for the solution to reach the desired concentration. □

Exercise: A tank contains 100 gallon of brine whose salt concentration is 3 pounds per gallon. Three gallons of brine whose salt concentration is 2 pounds per gallon flow into the tank per miniute, and at the same time 3 gallons of the mixture flow out each miniute. If the mixture is kept uniform via constant stirring, determine the salt concentration of the solution as a function time t. (*Answer:* $x(t) = 100 \cdot \left(2 + e^{-0.03t}\right)$)

Example (Bernoulli Equation): A special type of non-linear first-order differential equations, named after the Swiss mathematician Jacob Bernoulli, can be converted into a linear equation. This type of equations take form

$$y' + p(t)y = q(t)y^n,$$

where n is an integer (could be negative). The equation is solvable immediately if $n = 0, 1$ (Exercise). Otherwise, we can multiply both sides by $(1-n)y^{-n}$ to obtain that

$$(1-n)y^{-n}y' + p(t)y^{1-n} = q(t) \quad \Rightarrow \quad \frac{d}{dx}\left(y^{1-n}\right) + p(t)y^{1-n} = q(t).$$

Letting $u \triangleq y^{1-n}$, we obtain a linear equation for v, namely,

$$v' + p(t)v = q(t)$$

which is then solvable using integrating factor.

For example, solve the initial value problem

$$y' + 2xy = \frac{x}{y}, \qquad y(0) = -1.$$

We have

$$yy' + 2xy^2 = x \quad \Rightarrow \quad v' + 4xv = 2x, \quad \text{where } v \triangleq y^2.$$

It follows that

$$\frac{d}{dx}\left(e^{2x^2}v\right) = 2xe^{2x^2}$$

which implies that

$$v = e^{-2x^2}\int 2xe^{2x^2}\,dx \quad \Rightarrow \quad y^2 = \frac{e^{-2x^2}}{2}\left(e^{2x^2} + c\right) = \frac{1}{2}\left(1 + ce^{-2x^2}\right)$$

for some constant c. However, $v(0) = y^2(0) = 1$, we have $c = 1$, or

$$y = \pm\sqrt{\frac{1}{2}\left(1 + e^{-2x^2}\right)}$$

But $y(0) = -1$, so the solution to the initial value problem is

$$y = -\sqrt{\frac{1}{2}\left(1 + e^{-2x^2}\right)}$$

2 Seperable Equations

There is no general metholodogy to solve non-linear first order differential equation. In the following we will consider the class of *seperable* equations; namely, those that can be written as

$$M(x) + N(y)\frac{dy}{dx} = 0.$$

Here y is usually the dependent variable and x the independent variable.

This class of equations are easy to solve. Actually, define

$$H_1(x) \stackrel{\triangle}{=} \int M(x)\,dx \qquad \text{and} \qquad H_2(y) = \int N(y)\,dy.$$

It follows that

$$H_1'(x) = M(x) \qquad \text{and} \qquad H_2'(y) = N(y),$$

and, by chain rule,

$$H_1'(x) + H_2'(y)\frac{dy}{dx} = \frac{d}{dx}\left[H_1(x) + H_2(y)\right] = 0.$$

This yields the *general solution* of seperable is

$$H_1(x) + H_2(y) = c, \qquad \text{for some constant } \; c.$$

If, in addition, an initial condition $y(x_0) = y_0$ is prescribed, we have

$$c = H_1(x_0) + H_2(y_0) \;\Rightarrow\; \int_{x_0}^{x} M(s)\,ds + \int_{y_0}^{y} N(s)\,ds = H_1(x) - H_1(x_0) + H_2(y) - H_2(y_0) = 0.$$

Hence the solution to the initial value problem is given by

$$\int_{x_0}^{x} M(s)\,ds + \int_{y_0}^{y} N(s)\,ds = 0$$

Remark: The solution we obtain is generally an *implicit* one, since usually we can not solve the above equation to get an explicit solution $y = f(x)$.

Exercise: Solve the initial value problem

$$2x - 3y^2y' = 0, \qquad y(0) = 1$$

Solution: We have

$$0 = 2x\,dx - 3y^2\,dy = d\left(x^2 - y^3\right)$$

or

$$x^2 - y^3 = c \quad\Rightarrow\quad c = -1 \quad\Rightarrow\quad y = \sqrt[3]{x^2+1}$$

Example: Solve initial value problem

$$y' = e^{x+y}, \qquad y(0) = 0$$

Determine the valid interval of definition.

Solution: We have

$$0 = e^x\,dx - e^{-y}dy = d\left(e^x + e^{-y}\right)$$

Hence teh solution is

$$e^x + e^{-y} = c \quad\Rightarrow\quad c = 2 \quad\Rightarrow\quad e^x + e^{-y} = 2 \quad\Rightarrow\quad y = -\ln\left(2 - e^x\right)$$

The solution is well-defined on $x < \ln 2$. □

Example: Find solution to the initial value problem

$$xy' + (1 + y^2) = 0, \qquad y(1) = 0$$

Note where the solution is well-defined (i.e. valid interval of definition).

Solution: Divide both sides by $x(1 + y^2)$, we have

$$0 = \frac{dy}{1 + y^2} + \frac{dx}{x} = d\left(\arctan y + \ln |x|\right)$$

So the solution is

$$\ln |x| + \arctan y = c \quad \Rightarrow \quad c = 0 \quad \Rightarrow \quad \arctan y = -\ln |x| \quad \Rightarrow \quad y = -\tan(\ln |x|)$$

However, since x can never be zero (otherwise the right-hand side would be ill-defined), or go beyond $e^{\frac{\pi}{2}}$ (otherwise, we have $y = \infty$ at $x = e^{\frac{\pi}{2}}$). Therefore, the solution is

$$y = \tan(\ln x), \qquad \text{for } \; 0 < x < e^{\frac{\pi}{2}}$$

2.1 Homogeneous Equations

A special class of first order differential equation can be written as a seperable equation after appropriate transformations. Here we consider a class of *homogeneous* equations.

Definition: A function $z = f(x, y)$ is said to be a *homogeneous* function of *order* n if

$$f(tx, ty) = t^n f(x, y),$$

for all t, x, y. Alternatively, it can be written as

$$f(x, y) = x^n f\left(1, \frac{y}{x}\right) := x^n F\left(\frac{y}{x}\right)$$

Example: Consider the following functions.

1. $f(x, y) = x^2 + 4xy + 2y^2$ is homogeneous with order 2.
2. $f(x, y) = \frac{1}{x} + \frac{2}{y}$ is homogeneous with order -1.
3. $f(x, y) = \frac{3x+4y}{x-y} + \frac{x^2-3y^2}{x^2+y^2}$ is homonegeous with order 0.
4. $f(x, y) = x^2 + y^2 + 2xy^3$ is *not* homogeneous.

Definition: A first order differential equation $\dot{y} = f(x, y)$ is said to be *homonegeous* if $f(x, y)$ is a homogeneous function with order 0. The equation can therefore be written as

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right)$$

Consider the following simple transformation, namely $v = \frac{y}{x}$. It follows that

$$dy = F(v)\,dx \;\Rightarrow\; d(xv) = F(v)\,dx \;\Rightarrow\; x\,dv + v\,dx = F(v)\,dx \;\Rightarrow\; \frac{1}{F(v) - v}\,dv + \frac{1}{x}\,dx = 0,$$

which is a *seperable* equation.

Example: Find the solution of the following initial value problem

$$\frac{dy}{dx} = e^{\frac{y}{x}} + \frac{y}{x}, \qquad y(1) = 0$$

Solution: Let $v = \frac{y}{x}$, or $y = xv$, we have

$$\frac{dy}{dx} = v + x\frac{dv}{dx} = v + e^v \quad \Rightarrow \quad -e^{-v}dv + \frac{1}{x}dx = d\left(e^{-v} + \ln|x|\right) = 0$$

Therefore, the solution takes form

$$\ln|x| + e^{-\frac{y}{x}} = c \quad \Rightarrow \quad c = 1 \quad \Rightarrow \quad \ln|x| + e^{-\frac{y}{x}} = 1$$

However, it is easy to see that the solution will never cross $x = 0$, or the valid definition of interval is $x > 0$. Thus the solution is

$$\ln x + e^{-\frac{y}{x}} = 1, \qquad x > 0$$

Example (linear coefficient): Consider the following initial value problem with linear coefficients.

$$(x + y + 2)\,dx - (x - y - 4)\,dy = 0, \qquad y(1) = 0$$

Solution: Using change of variable

$$u = x + y + 2, \quad v = x - y - 4 \qquad \text{or} \qquad x = \frac{u+v}{2} + 1, \quad y = \frac{u-v}{2} + 3$$

we have

$$dx = \frac{du + dv}{2}, \quad dy = \frac{du - dv}{2}$$

But $dy = \frac{u}{v}\,dx$ implies that

$$\frac{du}{dv} = \frac{u+v}{v-u}$$

which is a homogeneous equation, and can be solved with solution takes form

$$\arctan\frac{u}{v} - \frac{1}{2}\ln\left(u^2 + v^2\right) = c \quad \Rightarrow \quad \arctan\frac{x+y+2}{x-y-4} - \frac{1}{2}\ln\left[(x-1)^2 + (y+3)^2\right] - \frac{1}{2}\ln 2 = c$$

Using initial condition $y(1) = 0$ we obtain $c = -\frac{\pi}{4} - \ln 3 - \frac{1}{2}\ln 2$. It follows that the solution is

$$2\arctan\frac{x+y+2}{x-y-4} - \ln\left[(x-1)^2 + (y+3)^2\right] = -2\ln 3 - \frac{\pi}{2}.$$

Actually, it can be further shown that

$$\arctan\frac{x+y+2}{x-y-4} + \frac{\pi}{4} = -\arctan\frac{x-1}{y+3}$$

(Exercise!), so the solution can also be written as

$$\arctan\frac{x-1}{y+3} + \ln\left[(x-1)^2 + (y+3)^2\right] = 2\ln 3$$

Remark: Same change of variable can be used to solve general differentail equation with linear coefficients.

$$(a_1x + b_1y + c_1)\,dx + (a_2x + b_2y + c_2)\,dy = 0$$

Remark: Another method to solve differential equation with linear coefficient can be found in Problem 2.9.15 in textbook.

Exercise: Find the solution of the differential equation

$$(2x - y + 1)\,dx + (x + y)\,dy = 0$$

3 Exact Differential Equation and Integrating Factor

Seperable equations are just a special case of *exact* differential equations.

3.1 Exact Differential Equations

Definition: A first order differential equation

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0 \quad \text{or} \quad M(x,y)\,dx + N(x,y)\,dy = 0$$

is said to be an *exact* differential equation, if there exists a function $\phi(x,y)$ such that

$$\frac{\partial\phi(x,y)}{\partial x} = M(x,y), \qquad \frac{\partial\phi(x,y)}{\partial y} = N(x,y).$$

Notation: Sometime, we use notation ϕ_x (resp. ϕ_y) to denote partial derivative $\frac{\partial\phi}{\partial x}$ (resp. $\frac{\partial\phi}{\partial y}$).

An exact differential equation will generally admit an implicit solution. Actually

$$d\phi(x,y) = \phi_x(x,y)\,dx + \phi_y(x,y)\,dy = M(x,y)\,dx + N(x,y)\,dy = 0,$$

which gives an implicit solution to the differential equation, namely,

$$\phi(x,y) = c$$

where c is an arbitary constant.

Example: Find the solution of the initial value problem

$$\sin x \cos y + \cos x \sin y \frac{dy}{dx} = 0, \qquad y\left(\frac{\pi}{4}\right) = \frac{\pi}{4}$$

Solution: It is easy to see that the differential equation can be written as

$$-\cos y\, d(\cos x) + \cos x\, d(-\cos y) = 0 \quad \Rightarrow \quad d\left(-\cos x \cos y\right) = 0$$

(or $\phi(x,y) = -\cos x \cos y$), which implies that $\cos x \cos y = c$, for some constant c. But $y\left(\frac{\pi}{4}\right) = \frac{\pi}{4}$, so $c = \frac{1}{2}$. Therefore the solution is

$$2\cos x \cos y = 1.$$

Exercise: Find the solution of the following differential equation

$$2xy + (x^2 + y^2)\frac{dy}{dx} = 0$$

Solution: The differential equation is equivalent to

$$y\,d\left(x^2\right) + x^2\,dy + \frac{1}{3}d\left(y^3\right) = d\left(x^2 y + \frac{1}{3}y^3\right) = 0$$

Therefore the solution is

$$x^2 y + \frac{1}{3}y^3 = c$$

for some constant c. (In this problem, $\phi(x,y) = x^2 y + \frac{1}{3}y^3$) □

We can ask two questions. First, how could we identify an exact differential equation, and second, how would we find the function ϕ so as to obtain a solution? Before we write out the theorem that will address both question, let us make a small observation.

If the differential equation is exact, we have

$$\frac{\partial}{\partial x}\phi(x,y) = M(x,y) \;\Rightarrow\; \frac{\partial}{\partial y}\frac{\partial}{\partial x}\phi(x,y) = \frac{\partial}{\partial y}M(x,y)$$

and

$$\frac{\partial}{\partial y}\phi(x,y) = N(x,y) \;\Rightarrow\; \frac{\partial}{\partial x}\frac{\partial}{\partial y}\phi(x,y) = \frac{\partial}{\partial x}N(x,y).$$

Under very mild conditions, we know

$$\frac{\partial}{\partial y}\frac{\partial}{\partial x}\phi(x,y) = \frac{\partial}{\partial x}\frac{\partial}{\partial y}\phi(x,y) \;\Rightarrow\; \frac{\partial}{\partial y}M(x,y) = \frac{\partial}{\partial x}N(x,y).$$

With some effort, we can show that it is not only *necessary*, but also *sufficient* for the differential equation to be exact. We have the following result.

Theorem: Suppose M, N, M_y and N_x are continuous in a simple connected region R. Then the differential equation

$$M(x,y) + N(x,y)\frac{dy}{dx} = 0$$

is exact if and only if

$$M_y(x,y) = N_x(x,y), \qquad \forall\ (x,y) \in R.$$

Remark: Intuitively speaking, a *simply connected region* is a region with no "hole" in its interior. Formally, a region is simply connected if every simple closed curve lying entirely in the region encloses only points of the region. A rectangular region is simply connected. See the following two graphs.

Proof: Here we will give a short proof when region R is *rectangular*. It remains to show the sufficiency. Fix any point $(x_0, y_0) \in R$. It follows that, for a fixed y,

$$\phi(x,y) = \int_{x_0}^{x} \phi_x(x,y)\,dx + K(y) = \int_{x_0}^{x} M(x,y)\,ds + K(y)$$

Here $K(y)$ is an arbitrary constant of integration. However,

$$\begin{aligned} N(x,y) &= \phi_y(x,y) = \frac{\partial}{\partial y}\int_{x_0}^{x} M(x,y)\,dx + K'(y) \\ &= \int_{x_0}^{x} \frac{\partial}{\partial y} M(x,y)\,dx + K'(y) = \int_{x_0}^{x} \frac{\partial}{\partial x} N(x,y)\,dx + K'(y) \\ &= N(x,y)\Big|_{x_0}^{x} + K'(y) = N(x,y) - N(x_0,y) + K'(y) \end{aligned}$$

which implies that

$$K'(y) = N(x_0,y) \quad \Rightarrow \quad K(y) = \int_{y_0}^{y} N(x_0,y)\,dy + k$$

for some constant k. Therefore, we find a function

$$\phi = \int_{x_0}^{x} M(x,y)\,dx + \int_{y_0}^{y} N(x_0,y)\,dy + k$$

It is easy to verify that $\phi_x = M$ and $\phi_y = N$ indeed (Exercise!). □

Counter-Example: Consider the following differential equation

$$\frac{-y}{x^2+y^2} + \frac{x}{x^2+y^2}\frac{dy}{dx} = 0$$

with region $R = \{(x,y);\ 1 \le x^2+y^2 \le 4\}$, which is *not* simply connected. We want to show that there does *not* exist a function $\phi(x,y)$ defined on region R, such that

$$\phi_x = M = -\frac{y}{x^2+y^2}, \qquad \phi_y = N = \frac{x}{x^2+y^2}$$

even though

$$M_y = N_x = \frac{y^2-x^2}{(x^2+y^2)^2}.$$

Now suppose that there exists such a function $\phi(x,y)$ defined on region R. We wish to get a contradiction. Consider the following curve

$$L \overset{\triangle}{=} \{(x,y);\quad x^2+y^2 = r^2\} = \{(x,y);\quad x = r\cos\theta,\ y = r\sin\theta,\ 0 \le \theta < 2\pi\}$$

for any $r \in [1,2]$. If such ϕ exists, we must have

$$\int_L d\phi = 0 \quad \text{(the integral of } d\phi \text{ over cure } L)$$

(Why?). However,

$$d\phi = M\,dx + N\,dy = -\frac{r\sin\theta}{r^2}d(r\cos\theta) + \frac{r\cos\theta}{r^2}d(r\sin\theta) = d\theta$$

which implies that

$$\int_L d\phi = \int_0^{2\pi} d\theta = 2\pi = 0$$

a contradiction. □

Remark: The proof provide a formula to find $\phi(x,y)$. But it would be preferable to just go through the process for each problem instead of memorizing the formula.

Example: Show that the following differentiable equation is exact and solve the initial value problem.

$$(x - 2xy + e^y) + (y - x^2 + xe^y)\frac{dy}{dx} = 0, \qquad y(0) = 2$$

Solution: We see that

$$M(x,y) = x - 2xy + e^y, \quad N(x,y) = y - x^2 + xe^y.$$

It is easy to see that $M_y = -2x + e^y = N_x$. Therefore the equation is exact. We wish to find a function $\phi(x,y)$ such that

$$\phi_x = M(x,y) = x - 2xy + e^y, \qquad \phi_y = N(x,y) = y - x^2 + xe^y$$

The first equation yields that (regarding y as fixed)

$$\phi = \frac{1}{2}x^2 - x^2y + xe^y + h(y).$$

for some function $h(y)$. Substituing this formula into the second equation, we obtain

$$-x^2 + xe^y + h'(y) = y - x^2 + xe^y \quad \Rightarrow \quad h'(y) = y \quad \Rightarrow \quad h(y) = \frac{1}{2}y^2 + c$$

for some constant c. Choosing $c = 0$ (*any* choice of c will do), we have

$$\phi(x,y) = \frac{1}{2}x^2 - x^2y + xe^y + \frac{1}{2}y^2$$

Hence the solution is

$$\phi(x,y) = c \quad \Rightarrow \quad c = \phi(0,2) = 2$$

or the solution is

$$\frac{1}{2}x^2 - x^2y + xe^y + \frac{1}{2}y^2 = 2$$

Exercise: Determine whether each of the following differential equations is exact. If so, give its solution.

1. $\cos y - (x\sin y - y)\frac{dy}{dx} = 0$.

2. $(1+2xy)+\left(\frac{y-x}{y}\right)\frac{dy}{dx}=0.$

3. $\left(\frac{2xy+1}{y}\right)+\left(\frac{y-x}{y^2}\right)\frac{dy}{dx}=0.$

Hint: (1) and (3) are exact, while (2) is not. *Note*: Even though (2) is not exact, it is indeed equivalent to equation (3), which is exact.

Remark: A seperable differential equation is a special case of exact equation in that

$$M_y(x,y)=M_y(x)=0=N_x(y)=N_x(x,y).$$

Moreover, $\phi(x,y)=H_1(x)+H_2(y)$.

3.2 Integrating Factor

Consider the differential equation $y-x\frac{dy}{dx}=0$, which is not exact since

$$M(x,y)=y,\ N(x,y)=-x\ \Rightarrow\ M_y=1\neq -1=N_x.$$

However, by multipying $\frac{1}{xy}$, we obtain $\frac{1}{x}-\frac{1}{y}\frac{dy}{dx}=0$, which is a seperable equation, hence *exact*. We have the following definition

Definition: A multiplying factor is said to be an *integrating factor* if it can convert an inexact differential equation into an exact one.

We assume that

$$M(x,y)+N(x,y)\frac{dy}{dx}=0$$

is not an exact equation, and that $\mu(x,y)$ is an integrating factor we wish to determine. Clearly, it follows from the previous theorem that

$$\frac{\partial}{\partial y}\left[\mu(x,y)M(x,y)\right]=\frac{\partial}{\partial x}\left[\mu(x,y)N(x,y)\right]$$

This will leads to the following partial differential equation

$$M\mu_y-N\mu_x+(M_y-N_x)\mu=0.$$

However, this differential equation is often *more* difficult to solve than the original first order ODE. In the following, we consider two possibilities.

μ is a function only of x, or $\mu=\mu(x)$. In this case, $\mu=\mu(x)$ shall satisfy equation

$$\mu'(x)=\frac{M_y-N_x}{N}\mu(x)$$

Therefore, we need condition

$$\frac{M_y-N_x}{N}\quad\text{is a function of } x \text{ only}$$

to obtain a integrating factor $\mu=\mu(x)$ which is also a function of x only.

μ is a function only of y, or $\mu = \mu(y)$. Similarly, $\mu = \mu(y)$ shall satisfy equation

$$\mu'(y) = -\frac{M_y - N_x}{M}\mu(y)$$

Therefore, we need condition

$$\frac{M_y - N_x}{M} \quad \text{is a function of } y \text{ only}$$

to obtain a integrating factor $\mu = \mu(y)$ which is also a function of y only.

Let us work on several examples.

Example: Solve the following differential equation

$$xy + (1 + x^2)\frac{dy}{dx} = 0.$$

Solution: We have $M(x, y) = xy$, $N(x, y) = 1 + x^2$. Since $M_y = x \neq 2x = N_x$, the equation is not exact. However,

$$\frac{M_y - N_x}{M} = \frac{1}{y}$$

is a function of y only. So we can find a integrating factor $\mu = \mu(y)$ which shall satisfy equation

$$\mu'(y) = \frac{1}{y}\mu(y) \quad \Rightarrow \quad \mu(y) = cy$$

for some constant c. Choosing $c = 1$ (or *any* c), we have $\mu(y) = y$, and the equation become

$$xy^2 + (x^2y + y)\frac{dy}{dx} = \frac{dy}{dx}\left(\frac{(x^2 + 1)y^2)}{2}\right) = 0$$

or, the solution is

$$(1 + x^2)y^2 = c$$

for some constant c.

Exercise: Show that the preceding example also admit an integrating factor of form $\mu = \mu(x) = \frac{1}{\sqrt{1+x^2}}$. Using this integrating factor, redo the example.

Example: Solve the following differential equation using integrating factor of form $\mu = \mu(xy)$.

$$(x^4y^2 - y) + (x^2y^4 - x)\frac{dy}{dx} = 0$$

Solution: We have

$$M = x^4y^2 - y, \quad N = x^2y^4 - x \quad \Rightarrow \quad M_y = 2x^4y - 1, \; N_x = 2xy^4 - 1$$

It is not difficult to check that there is no integrating factor that is a function of x only or of y only (check!). However, let us for the moment assume that there is an integrating factor $\mu = \mu(xy)$. Then μ must satisfy equation

$$M\mu_y - N\mu_x + (M_y - N_x)\mu = (xM - yN)\mu' + (M_y - N_x)\mu = 0.$$

In another word, we need condition

$$\frac{M_y - N_x}{xM - yN} \quad \text{is a function of } xy \text{ only}$$

to guarntee the existence of such integrating factor.

In our problem, we have

$$\frac{M_y - N_x}{xM - yN} = \frac{2}{xy} = \frac{\mu'(xy)}{\mu(xy)}$$

This yields an integrating factor $\mu = \frac{1}{(xy)^2}$. The differential equation becomes

$$\left(x^2 - \frac{1}{x^2 y}\right) dx + \left(y^2 - \frac{1}{xy^2}\right) dy = 0$$

which is equivalen to

$$d\left(\frac{1}{3}(x^3 + y^3) + \frac{1}{xy}\right) = 0$$

or, the solution is

$$\frac{x^3 + y^3}{3} + \frac{1}{xy} = c$$

for some constants c. □

4 Autonomous Equations and Stability

In this section, we shall discuss the *autonomous* first order differential equations, which have the form

$$\frac{dy}{dt} = f(y)$$

for some continuous function $f(y)$. Even this is a seperable equation, the explicit solution might be elusive because it is difficult to find $\int \frac{1}{f(y)}\, dy$. However, we shall rely on gemetric methods to obtain certain qualitative information of the differential equation. We first introduce the following definition.

Definition: An *equilibrium point* (or *critical point*, or *singular point*) is a point y^* such that $f(y^*) = 0$. Other points are said to be *regular* points.

Remark: Suppose y^* is an equilibrium point, then $y(t) \equiv y^*$ is a solution of the differential equation. This is said to be an *equilibrium solution* of the differential equation.

In the following, we shall assume that

Assumption: *no two solutions can intersect.*

For all the examples we shall discuss below, this assumption is satisfied. Indeed, according to the fundamental existence and uniqueness theorem, which we will discuss in the next section, a sufficient condition for the above assumption is that $f'(y)$ is *continuous.*

Example: Consider the following initial value problem

$$\frac{dy}{dt} = ay - b, \qquad y(0) = y_0$$

where $a \neq 0$ and b are constants. The only equilibrium point of this equation is $y^* = \frac{b}{a}$. Recall that the solution of this linear differential equation is

$$y(t) = \left(y_0 - \frac{b}{a}\right) e^{at} + \frac{b}{a} = (y_0 - y^*)\, e^{at} + y^*.$$

As we have discussed, if $y_0 = y^*$, then $y(t) \equiv y^*$ for all $t \geq 0$.

The asymptotic behavior of the solution heavily depends on the sign of a. If $a < 0$, the limit of $y(t)$ as $t \to \infty$ equals y^* for any initial condition y_0. However, if $a > 0$, the deviation from equilibrium $y(t) - y^* \to +\infty$ (resp. $-\infty$) whenever $y_0 > y^*$ (resp. $y_0 < y^*$), as $t \to \infty$.

To conclude, if $a < 0$, then every solution will converge to the equilibrium point as $t \to \infty$ (in this case, the equilibrium point is said to be *asymptotically stable*). If $a > 0$, every solution except the equilibrium solution $y(t) \equiv y^*$, will depart away from the equilibrium point (in this case, the equilibrium point is said to be *unstable*).

Example: Consider the following Gompertz model for population growth

$$\frac{dy}{dt} = ry \ln \frac{K}{y}; \qquad y(0) = y_0 > 0$$

Here r, K are positive constants.

Solution: Here $f(y) = ry \ln \frac{K}{y}$. Letting $f(y) = 0$, we obtain the unique equilibrium point $y^* = K$. We now study the stability of this equilibrium solution. Actually we can solve this initial value problem as follows

$$\frac{1}{y}\frac{dy}{dt} = \frac{d(\ln y)}{dt} = r \ln K - r \ln y.$$

Making a change of variable $u = \ln y$, we have

$$\frac{du}{dt} = r \ln K - ru \quad \Rightarrow \quad \frac{d}{dt}\left(e^{rt} u\right) = r \ln K e^{rt} \quad \Rightarrow \quad e^{rt} u = \ln K e^{rt} + c$$

for some constant c. Since $u(0) = \ln y(0) = \ln y_0$, we have

$$c = \ln y_0 - \ln K \quad \Rightarrow \quad u = \ln y = \ln K + \ln \frac{y_0}{K} \cdot e^{-rt}$$

It follows that no matter what value y_0 takes,

$$\ln y(t) \to \ln K \quad \text{or} \quad y(t) \to K \qquad \text{as } t \to \infty$$

since $r > 0$. Therefore $y(t) \equiv K$ is a stable equilibrium.

Example: Consider the following differential equation

$$\frac{dy}{dt} = ry \ln \frac{K}{y}; \qquad y(0) = y_0 > 0$$

Here $r < 0$, $K > 0$ are two constants.

Solution: The equilibrium solution is still $y(t) \equiv K$, and we still have

$$\ln y = \ln K + \ln \frac{y_0}{K} \cdot e^{-rt}$$

If $y_0 > K$, we have

$$\lim_{t\to\infty} \ln y(t) = +\infty \;\text{ or }\; y(t) \to \infty, \qquad \text{as } \; t \to \infty$$

If $y_0 < K$, we have

$$\lim_{t\to\infty} \ln y(t) = -\infty \;\text{ or }\; y(t) \to 0, \qquad \text{as } \; t \to \infty$$

Therefore, the equilibrium solution is unstable. □

Let us give the formal definition of stability

Definition: Suppose $f(y^*) = 0$. It follows that $y(t) \equiv y^*$ is an equilibrium solution

1. The equilibrium solution $y(t) \equiv y^*$ is said to be *(asymptotically) stable* if there exist a neighborhood of y^*, say N, such that the solution to differential equation $y' = f(y)$ with starting value $y(0) = y_0 \in N$ will have property $y(t) \to y^*$ as $t \to \infty$. Intuitively speaking, stablility means a solution starting nearby the equilibrium point will always converge to equilibrium.
2. The equilibrium solution $y(t) \equiv y^*$ is said to be it unstable if there exist a neighborhood of y^*, say N, such that the solution to differential equation $y' = f(y)$ with starting value $y(0) = y_0 \in N$ will diverge away from y^* as $t \to \infty$.
3. The equilibrium solution $y(t) \equiv y^*$ is said to be it semi-stable if the solution to differential equation $y' = f(y)$ with starting value $y(0) = y_0 \in N$ will diverge away from y^* when $y_0 > y^*$ (resp. $y_0 < y^*$) and converge to y^* when $y_0 < y^*$ (resp. $y_0 > y^*$), as $t \to \infty$. Intuitively speaking, semi-stablility means one-side stability.

It is not difficult to see that the stability of equilibrium solution $y(t) \equiv y^*$ are determined the change of sign of function $f(y)$ at y^*. We have the following result.

Rule of Thumb: Suppose that y^* is an equilibrium point of differential equation $\dot{y} = f(y)$, where $f(y)$ is a continuous function. We have

1. If $f(y)$ change from "$+$" to "$-$" as y increases across y^*, then $y(t) \equiv y^*$ is an asymptotically stable equilibrium solution.
2. If $f(y)$ change from "$-$" to "$+$" as y increases across y^*, then $y(t) \equiv y^*$ is an unstable equilibrium solution.
3. If $f(y)$ does not change sign as y increases across y^*, then $y(t) \equiv y^*$ is an semi-stable equilibrium solution.

Proof: Since the proofs for three cases are similar, we will only show the third case here. Suppose $f(y)$ is non-negative across y^* with $f(y^*) = 0$. Consider the solution of the following initial value problem

$$y' = f(y), \qquad y(0) = y_0.$$

Here y_0 is a point near y^* such that $y'(0) = f(y_0) > 0$. It is not difficult to show that $y(t)$ is an *increasing* function. Actually, $y'(t) = f\big(y(t)\big) \geq 0$ for all $t \geq 0$ – otherwise, there must exist a point t_0 such that $y'(t_0) = f\big(y(t_0)\big) = 0$. Hence this solution *intersects* with the equilibrium solution $y(t) \equiv y(t_0)$, which is a contradiction. Therefore, if $y_0 > y^*$, $y(t)$ will never converge to y^*. However, if $y_0 < y^*$, assume $y(t) \to y_*$ as $t \to \infty$. It follows that $\lim_{t\to\infty} y'(t) = f(y_*) = 0$, which implies $y^* = y_*$. We complete the proof. $\square$.

Example: Consider the following differential equation

$$y' = y \sin y := f(y).$$

Find all the equilibrium points and determine their stability.

Solution: The equation $f(y) = 0$ has multiple roots $y^* = n\pi$ where $n = 0, \pm 1, \pm 2, \cdots$. Therefore the differential equation has multiple equilibrium solutions $y(t) \equiv 0, \pm\pi, \pm 2\pi, \cdots$.

1. Since $f(y)$ does not change sign across $y^* = 0$, the equilibrium solution $y(t) \equiv 0$ is semi-stable.
2. Since $f(y)$ changes from "+" to "−" as y increases across $y^* = \pi, 3\pi, \cdots$ or $y^* = -2\pi, -4\pi, \cdots$ respectively, these equilibrium solutions are asymptotically stable.
3. Since $f(y)$ changes from "−" to "+" as y increases across $y^* = 2\pi, 4\pi, \cdots$ or $y^* = -\pi, -3\pi, \cdots$ respectively, these equilibrium solutions are unstable.

Exercise: Consider differential equation $y' = f(y)$. Assume $f(y)$ is twice continuously differentiable and $f(y^*) = 0$. Show that

1. If $f'(y^*) > 0$, then the equilibrium solution $y(t) \equiv y^*$ is unstable.
2. If $f'(y^*) < 0$, then the equilibrium solution $y(t) \equiv y^*$ is stable.
3. If $f'(y^*) = 0$ and $f''(y^*) \neq 0$, then the equilibrium solution $y(t) \equiv y^*$ is semi-stable.

5 Stability for General Non-linear Equations

In this section, we will briefly discuss the stability of general differential equations of form

$$\frac{dy}{dt} = f(t, y);$$

As before, we have the following definition of equilibrium point.

Definition: An *equilibrium point* is a point y^* such that $f(t, y^*) \equiv 0$ for all $t \geq 0$. In particular, $y(t) \equiv y^*$ is an *equilibrium solution* to the differential equation $y' = f(t, y)$.

The stability can be defined in exactly the same way. Here we only give the definition of (asymptotic) stability, which is our main interest.

Definition: The equilibrium solution $y(t) \equiv y^*$ is said to be *(asymptotic) stable* if there exists a neighborhood of y^*, say N, such that the solution to differential equation $y' = f(t, y)$ with starting value $y(0) = y_0 \in N$ will converge to y^* as $t \to \infty$.

Example: Consider the linear differential equation

$$\frac{dy}{dt} + p(t)y \equiv 0; \qquad t \geq 0$$

Show that equilibrium solution $y(t) \equiv 0$ is stable if and only if

$$\int_0^{\infty} p(s)\, ds \triangleq \lim_{t \to \infty} \int_0^t p(s)\, ds = +\infty.$$

Proof: Clearly $y^* = 0$ is an equilibrium point. Consider the following initial value problem

$$\frac{dy}{dt} + p(t)y \equiv 0; \qquad y(0) = y_0$$

Multiplying both sides by integrating factor $\mu(t) = e^{\int p(t)\,dt}$, we have

$$\frac{d}{dt}[\mu(t)y(t)] = 0 \quad \Rightarrow \quad \mu(t)y(t) \equiv \text{const} \quad \Rightarrow \quad y(t) = \frac{y_0}{\mu(t)}$$

Therefore, the equilibrium solution is stable if and only if

$$\frac{y_0}{\mu(t)} \to 0 \quad \text{as} \quad t \to \infty$$

for y_0 nearby 0. Or, if and only if

$$\lim_{t\to\infty} \mu(t) = \infty \quad \Leftrightarrow \quad \lim_{t\to\infty} \int_0^t p(s)\,ds = +\infty.$$

This completes the proof. □

Example: Consider the following differential equation

$$\frac{dy}{dt} + y = e^{-t}y^2; \qquad t \geq 0$$

Show that the equilibrium solution $y(t) \equiv 0$ is stable.

Proof: This is an Bernoulli equation. Dividing both sides by $\frac{1}{y^2}$ we have

$$-\frac{d}{dt}\left(\frac{1}{y}\right) + \frac{1}{y} = e^{-t}$$

It follows that

$$\frac{e^{-t}}{y} = \frac{1}{2}e^{-2t} + c \quad \Rightarrow \quad y(t) = \frac{e^{-t}}{\frac{1}{2}e^{-2t} + c}$$

for some constant c. If $y(0) = y_0$, we have

$$y(t) = \frac{e^{-t}}{\frac{1}{2}(e^{-2t} - 1) + \frac{1}{y_0}}$$

As $t \to \infty$, we have $y(t) \to 0$, or $y(t) \equiv 0$ is a stable equilibrium point. □

Unfortunately, there are only very few non-linear equations we can solve. We need some general criteria to establish stability. Below we shall give two easy theorems, one is based on the so-called *Lyapunov function*, while the other is a natural extension of our result for autonomous equations.

Without loss of generality, we assume that $y^* = 0$ (otherwise, we can always make a change of variable $u = y - y^*$).

Theorem: Consider the differentiable equation

$$\frac{dy}{dt} = f(t, y); \qquad t \geq 0$$

with equilibrium solution $y(t) \equiv 0$, that is $f(t, 0) \equiv 0$ for all t. A function V is said to be a *Lyapunov* function if

1. V is continuously differentiable.
2. $V(z) > 0$ when $z \neq 0$ and $V(0) = 0$.
3. $V'(z)f(t,z) \leq -kV(z)$ for all $t \geq 0$ and z, for some constant $k > 0$.

Show that the equilibrium solution $y(t) \equiv 0$ is stable if such a Lyapunov function exists.

Proof: Suppose V is a Lyapunov function. By definition, we can always find a positive number, say ϵ, such that

$$z_+ \stackrel{\triangle}{=} \inf\{z \geq 0;\ V(z) \geq \epsilon\} \quad \text{and} \quad z_- \stackrel{\triangle}{=} \inf\{z \geq 0;\ V(-z) \geq \epsilon\}$$

are both finite. Suppose now $y(t)$ is the solution to the initial value problem

$$\frac{dy}{dt} = f(t,y); \quad y(0) = y_0 \in (-z_-, z+).$$

Define $\phi(t) \stackrel{\triangle}{=} V\big(y(t)\big)$. It follows that

$$\frac{d\phi}{dt} = V'\big(y(t)\big) \cdot \frac{dy}{dt} = V'\big(y(t)\big) \cdot f\big(t, y(t)\big) \leq -kV\big(y(t)\big) = -k\phi(t).$$

In another word,

$$\frac{d}{dt}\left(e^{kt}\phi(t)\right) \leq 0 \quad \Rightarrow \quad e^{kt}\phi(t) \leq e^{k\cdot 0}\phi(0) = V(y_0)$$

for all $t \geq 0$. It follows that

$$\phi(t) \leq e^{-kt}V(y_0) < \epsilon \quad \text{for all } \ t \geq 0.$$

In particular, we have

$$\phi(t) = V\big(y(t)\big) \ \rightarrow \ 0 \quad \text{as} \quad t \rightarrow \infty.$$

To prove $\lim_{t\rightarrow\infty} y(t) = 0$, we only need to observe that

$$y(t) \in (-z_-, z_+)$$

for all $t \geq 0$. Indeed, otherwise there must exists t^* such that $y(t^*) = -z_-$ or $y(t^*) = z_+$, which implies that $\phi(t^*) = V\big(y(t^*)\big) = \epsilon$, a contradiction. However, since $V(z) > 0$ for $z \neq 0$, we must have

$$\lim_{t\rightarrow\infty} y(t) = 0.$$

We completes the proof. □

Example: Reconsider the linear differential equation

$$\frac{dy}{dt} + p(t)y = 0$$

with $p(t) \geq p^* > 0$ for all t, and some positive constant p^*. Let $V(z) = z^2$. We have

$$V'(z)f(t,z) = 2z \cdot (-p(t)z) = -2p(t)z^2 \leq -2p^*z^2$$

for all t, z. It follows that V is a Lyapunov function, and $y(t) \equiv 0$ is a stable equilibrium solution.

Theorem: Consider the differential equation

$$\frac{dy}{dt} = -ry + g(t, y); \qquad t \geq 0$$

where r is a positive constant, and g is a continuous function with $g(t, 0) \equiv 0$ for all t. If

$$\lim_{z\to 0} \sup_{t\geq 0} \frac{|g(t,z)|}{z} = 0$$

Then $y(t) \equiv 0$ is a stable equilibrium solution.

Proof: Choose any $\epsilon \in (0, r)$. By assumption, there exists $\delta > 0$ such that

$$\sup_{t\geq 0} |g(t,z)| \leq \epsilon |z|, \qquad \text{for all} \quad |z| \leq \delta.$$

Let $y_0 \in (0, \delta)$. Consider the initial value problem

$$\frac{dy}{dt} = -ry + g(t, y); \qquad y(0) = y_0.$$

Since no two solutions can intesect, we have $y(t) \geq 0$ for all $t \geq 0$. Observing $-ry + g(t, y) \leq -ry + \epsilon y < 0$ for all $y \in (0, \delta)$, it is not difficult to see that the solution $y(t)$ is *decreasing.* Therefore, we have

$$\frac{dy}{dt} \leq -(r-\epsilon)y$$

for all $t \geq 0$. Equivalently, we have

$$\frac{d}{dt}\left(e^{(r-\epsilon)t}y(t)\right) \leq 0 \quad \Rightarrow \quad e^{(r-\epsilon)t}y(t) \leq y(0) = y_0$$

or

$$y(t) \leq y_0 e^{-(r-\epsilon)t} \;\to\; 0 \quad \text{as} \quad t \to \infty$$

Similarly, we can prove the $y(t) \to 0$ as $t \to \infty$ if $y_0 \in (-\delta, 0)$. Hence the equilibrium solution $y(t) \equiv 0$ is stable. □

Example: Re-consider the following differential equation

$$\frac{dy}{dt} + y = e^{-t}y^2; \qquad t \geq 0$$

Show that the equilibrium solution $y(t) \equiv 0$ is stable.

Proof: Here $r = 1$ and $g(t, y) = e^{-2t}y^2$. It is clear that

$$\lim_{z\to 0} \sup_{t\geq 0} \frac{|g(t,z)|}{|z|} = \lim_{z\to 0} \frac{z^2}{|z|} = \lim_{z\to 0} |z| = 0$$

Therefore, the equilibrum solution $y(t) \equiv 0$ is stable.

Exercise: Show that $y(t) \equiv 0$ is a stable equilibrium solution to the differential equation

$$\frac{dy}{dt} = ry + q(t)y^2; \qquad t \geq 0.$$

Here r is a positive constant, and q is a bounded function (i.e. $|q(t)| \leq K$ for some K and all $t \geq 0$).

Proof: Here $g(t,z) = q(t)y^2$, and

$$\lim_{z\to 0}\sup_{t\geq 0}\frac{|g(t,z)|}{|z|} \leq \lim_{z\to 0}\frac{Kz^2}{|z|} = \lim_{z\to 0} K|z| = 0.$$

The stability follows from the preceding theorem. □

6 Non-linear First Order ODE

We shall consider first order differential equation of the following general form

$$y' = f(t,y)$$

with initial conditions $y(t_0) = y_0$. Unlike the linear case, where *globle* existence and uniqueness results can be obtained under very mild condition, we have the following fundamental (*local*) existence and uniqueness theorem.

Picard-Lindelöf Theorem: If $f(t,y)$ and $f_y(t,y)$ are continuous in region $R \triangleq \{(t,y)/\ |t-t_0| \leq a,\ |y-y_0| \leq b\}$, then there exists a unique solution $y = \phi(t)$ of initial value problem

$$y' = f(t,y), \qquad y(t_0) = y_0$$

on an interval $|t - t_0| < h$ for some $h \leq a$.

Proof: First, we introduce an integral equation that is equivalent to the initial value problem. Actually, if $\phi(t)$ solve the initial value problem on an interval I, we have

$$\phi(t) = \phi(t_0) + \int_{t_0}^{t} \phi'(s)\,ds = y_0 + \int_{t_0}^{t} f\big(s,\phi(s)\big)\,ds; \quad \forall\ t \in I$$

Conversely, if $\phi(t)$ is a solution of the above integral equation, we have $\phi(t_0) = y_0$ and it follows from differentiating the equation that $\phi' = f(t,\phi)$ for all $t \in I$. Therefore, the integral equation is equivalent to the initial value problem. It remains to show that there exists a unique solution to the integral equation on an interval $|t - t_0| < h$ for some $h \leq a$. More precisely, we want to show the existence and uniqueness for

$$h \triangleq \min\left(a, \frac{b}{M}\right); \qquad \text{here } M \triangleq \max_{(t,y)\in R} |f(t,y)|$$

We make a useful observation that, for some constant $C > 0$,

$$|f(t,y_1) - f(t,y_2)| \leq C|y_1 - y_2|; \qquad \forall\ (t,y_i) \in R,\ i = 1,2.$$

Indeed, you can choose $C = \max_{(t,y)\in R} |f_y(t,y)|$. In the following, we prove the existence and uniqueness of a solution on the half interval $t \in [t_0, t_0 + h]$. The proof for the other half $t \in [t_0 - h, t_0]$ is very similar, and thus omitted.

Existence: Here we shall use the *iteration method* to prove the existence of a solution, which also gives a constructive method to obtain a solution. Let

$$\phi_0(t) = y_0; \qquad \forall\ t \in [t_0, t_0 + h]$$

$$\phi_{n+1}(t) = y_0 + \int_{t_0}^{t} f\big(s, \phi_n(s)\big)\, ds; \qquad \forall\ t \in [t_0, t_0 + h]$$

First we observe that function $\phi_{n+1}(t)$ will always stay in region R as long as $t \in [t_0, t_0 + h]$, with $\phi_n(t_0) = y_0$. Actually,

$$|\phi_{n+1}(t) - y_0| = \left| \int_{t_0}^{t} f\big(s, \phi_n(s)\big)\, ds \right| \leq \int_{t_0}^{t} \big|f\big(s, \phi_n(s)\big)\big|\, ds \leq M(t - t_0) \leq Mh \leq b$$

Now we show that functions $\{\phi_n(t);\ n \geq 0\}$ converge (uniformly) as $n \to \infty$ on inerval $t \in [t_0, t_0 + h]$. We have

$$|\phi_1(t) - \phi_0(t)| = \left| \int_{t_0}^{t} f(s, \phi_0(s))\, ds \right| \leq \int_{t_0}^{t} |f(s, \phi_0(s))|\, ds \leq M(t - t_0), \qquad \forall\ t \in [t_0, t_0 + h]$$

and similarly

$$\begin{aligned} |\phi_2(t) - \phi_1(t)| &\leq \int_{t_0}^{t} |f(s, \phi_1(s)) - f(s, \phi_0(s))|\, ds \leq \int_{t_0}^{t} C\, |\phi_1(s) - \phi_0(s)|\, ds \\ &\leq C \int_{t_0}^{t} M(s - t_0)\, ds = MC \frac{(t - t_0)^2}{2}, \qquad \forall\ t \in [t_0, t_0 + h] \end{aligned}$$

and an easy induction yileds that

$$|\phi_{n+1}(t) - \phi_n(t)| \leq MC^n \frac{(t - t_0)^n}{(n+1)!} \leq M \frac{(Ch)^n}{(n+1)!}; \qquad \forall\ n,\ t \in [t_0, t_0 + h].$$

However,

$$\phi_{n+1}(t) = \phi_0(t) + \sum_{i=0}^{n} [\phi_{i+1}(t) - \phi_i(t)]$$

and

$$\sum_{n=0}^{\infty} M \frac{(Ch)^n}{(n+1)!} = \frac{M}{Ch} \left(e^{Ch} - 1 \right) < \infty.$$

This implies that $\{\phi_n(t);\ n \geq 0\}$ (uniformly) converges to some continuous function, say $\phi(t)$, on interval $[t_0, t_0 + h]$. It follows that

$$\phi(t) = y_0 + \int_{t_0}^{t} f(s, \phi(s))\, ds,$$

or equivalently,

$$\phi'(t) = f(t, \phi), \qquad \phi(t_0) = y_0$$

for all $t \in [t_0, t_0 + h]$.

Uniqueness: Suppose that there is another solution $\varphi(t)$. Since

$$\varphi(t) = y_0 + \int_{t_0}^{t} f(s, \varphi(s))\, ds,$$

we can similarly show that, for $t \in [t_0, t_0 + h]$,

$$|\varphi(t) - \phi_0(t)| \le M(t - t_0)$$

$$|\varphi(t) - \phi_1(t)| \le \int_{t_0}^{t} MC(s - t_0)\, ds = MC\frac{(t - t_0)^2}{2},$$

and so on. An easy induction yields that

$$|\varphi(t) - \phi_n(t)| \le MC^n \frac{(t - t_0)^n}{(n+1)!} \le M\frac{(Ch)^n}{(n+1)!}, \qquad \forall\ n,\ t_0 \le t \le$$

Letting $n \to \infty$, we obtain

$$\varphi(t) = \lim_{n\to\infty} \phi_n(t) = \phi(t).$$

This proves the uniqueness. □

Another method to prove the uniquesness is using one of the simplest and most useful *Gronwall's Inequality.*

Gronwall's Inequality: Let $v(t), k(t)$ be non-negative, continuous functions on $[a, b]$, and

$$v(t) \le K + \int_a^t k(s)v(s)\, ds \qquad \text{for } \ a \le t \le b$$

where K is a non-negative constant. Show that

$$v(t) \le Ke^{\int_a^t k(s)\, ds} \qquad \text{for } \ a \le t \le b.$$

In particular, if $K = 0$, then $v(t) \equiv 0$.

Proof: Let $u(t) \overset{\triangle}{=} K + \int_a^t k(s)v(s)\, ds$. We have

$$u'(t) = k(t)v(t) \le k(t)u(t) \quad \Rightarrow \quad \frac{d}{dt}\left(e^{-\int_a^t k(s)\, ds}u(t)\right) \le 0$$

In particular, we have

$$e^{-\int_a^t k(s)\, ds}u(t) \le u(a) = K \qquad \text{for all } a \le t \le b.$$

Therefore,

$$v(t) \le u(t) \le Ke^{\int_a^t k(s)\, ds}$$

for all $a \le t \le b$. □

Exercise: Prove the uniqueness part of Picard-Lindelöf theorem, using the Gronwall's Inequality.

Proof: Suppose $\phi(t)$ and $\varphi(t)$ are two solutions on interval $|t - t_0| \leq h$. We wish to prove that $\phi(t) = \varphi(t)$. Assume $t \in [t_0, t_0 + h]$ (we can prove similarly for $t \in [t_0 - h, t_0]$). We have

$$|\phi(t) - \varphi(t)| \leq \int_{t_0}^{t} |f(s, \phi(s)) - f(s, \varphi(s))| \, ds \leq \int_{t_0}^{t} C \, |\phi(s) - \varphi(s)| \, ds.$$

Define $v(t) = |\phi(t) - \varphi(t)|$, we have $v(t) \geq 0$ and

$$v(t) \leq \int_{t_0}^{t} Cv(s) \, ds; \quad \forall \; t_0 \leq t \leq t + h$$

Therefore, it follows from Gronwall's Inequality that $v(t) \equiv 0$, or $\phi(t) = \varphi(t)$ for $t_0 \leq t \leq t+h$.
□

Example (Blow-up): The following example shows that a *global* solution might not exist. Consider the following differential equation

$$\frac{dy}{dt} = y^2 := f(y), \qquad y(0) = 1.$$

Function $f(y) = y^2$ is continuously differentiable on the whole real line $\mathbb{R}$. This equation is easy to solve.

$$\frac{1}{y^2} \frac{dy}{dt} = -\frac{d}{dt} \left(\frac{1}{y} \right) = 1 \;\Rightarrow\; \frac{1}{y} = -t + c,$$

for some constant c. Initial condition $y(0) = 1$ implies $c = 1$. Hence

$$y = \frac{1}{1 - t}, \qquad t < 1.$$

As $t \to 1$, $y(t) \to \infty$ (*Blow-up*). There exists no global solution for this initial value problem.
□.

Counter-Example: The following example shows that there might exist more than one solution if the conditions of the theorem are violated. Consider the following initial value problem.

$$\frac{dy}{dt} = \sqrt{y}, \qquad y(0) = 0$$

In this case $f(t, y) \equiv \sqrt{y}$ is continuous at $y \geq 0$, but not differentiable at $y = y_0 = 0$. Clearly, $y(t) \equiv 0$ is a solution to this initial value problem. We claim the following function

$$y(t) \triangleq \left\{ \begin{array}{ccc} \frac{t^2}{4} & ; & t \geq 0 \\ 0 & ; & t \leq 0 \end{array} \right.$$

is another solution to the initial value problem. Actually, it is easy to verify that

$$y'(t) = \left\{ \begin{array}{ccc} \frac{1}{2}t & ; & t > 0 \\ 0 & ; & t < 0 \end{array} \right\} = \sqrt{y}; \qquad \text{for} \;\; t \neq 0$$

It remains to show that y is differentiable at $t = 0$ and $y'(0) = 0$. This is left as an exercise.
□

Counter-Example: The following example shows that there might exist *no* solution if the conditions of the preceding are violated. Consider the initial value problem

$$\frac{dy}{dt} = f(t) \triangleq \left\{ \begin{array}{ccc} 1 & ; & t \geq 0 \\ 0 & ; & t < 0 \end{array} \right\}; \qquad y(0) = 0$$

Here $f(t, y) \equiv f(t)$ is not continuous at $t = 0$. It is not difficult to show that

$$y(t) = y(t) - y(0) = \int_0^t dt = t \quad \text{for} \ \ t \geq 0$$

and

$$y(t) = y(t) - y(0) = \int_0^t 0\, dt = 0 \quad \text{for} \ \ t \leq 0$$

But the resulting function $y(t)$ is not differentiable at $t = 0$. So there is no solution to this initial value problem. □

Conuter-Example: The conditions in the preceding theorem is by no means *necessary.* That is, there might exist a unique solution for some differential equation even if the conditions are violated.

Consider the following initial value problem

$$\frac{dy}{dt} = |y|, \qquad y(0) = 0$$

Function $|y|$ is not differentiable at $y = 0$. However, the above initial value equation has a *unique* solution $y(t) \equiv 0$. Actually, any solution of the initial value problem must be increasing since $\frac{dy}{dt} \geq 0$. In another word, $y(t) \geq 0$ as $t \geq 0$ and $y(t) \leq 0$ as $t \leq 0$. It follows that

$$\frac{dy}{dt} = y \ \ \text{for} \ \ t \geq 0, \qquad \text{and} \qquad \frac{dy}{dt} = -y \ \ \text{for} \ \ t \leq 0.$$

However, this implies that

$$\frac{d}{dt}\left(e^{-t}y\right) = 0 \ \ \text{for} \ \ t \geq 0, \qquad \text{and} \qquad \frac{d}{dt}\left(e^{t}y\right) = 0 \ \ \text{for} \ \ t \leq 0.$$

But $y(0) = 0$, hence

$$e^{-t}y(t) \equiv 0 \ \ \text{for} \ \ t \geq 0, \qquad \text{and} \qquad e^{t}y(t) \equiv 0 \ \ \text{for} \ \ t \leq 0,$$

or $y(t) \equiv 0$. □

7 Elementary Difference Equation

We will exclusively consider the following form of *difference equation*, namely

$$y_{n+1} = F(y_n), \qquad n = 0, 1, 2, \cdots$$

where $F(y)$ is a continuous function. Given *initial condition* y_0, every y_n $(n = 1, 2, \cdots)$ could be determined iteratively. Like that of differential equation, we define

Definition: An *equilibrium solution* of the difference equation is a point y^* such that $y^* = F(y^*)$. Trivially, if $y_0 = y^*$, then $y_n = y^*$ for all n.

Example: Suppose $f(y) = \rho y + b$ for some constants $\rho \neq 1$ and b. The difference equation is $y_{n+1} = \rho y_n + b$. The trick here is to find a to-be-determined parameter α such that

$$y_{n+1} - \alpha = \rho(y_n - \alpha) \;\Rightarrow\; \alpha = \frac{b}{\rho - 1},$$

which implies, if we define $z_n = y_n - \alpha$,

$$z_{n+1} = \rho z_n = \rho^{n+1} z_0 \;\Rightarrow\; y_{n+1} = \rho^{n+1}\left(y_0 - \frac{b}{1-\rho}\right) + \frac{b}{1-\rho}.$$

Remark: It is easy to see that $y_n \to y^* \triangleq \frac{b}{1-\rho}$ regardless of the value of y_0, provided $|\rho| < 1$. Clearly y^* satisfies $y^* = f(y^*)$ is an (stable) *equilibrium solution.* When $|\rho| > 1$, $|y_n| \to \infty$ unless $y_0 = y^*$ (y^* is an unstable equilibrium solution). See the following graph.

Almost all the difference equations do not admit explicit general solution like we see in the previous example. We are interested in some qualitative information, especially the *stability* of equilibrium solutions.

Definition: The definition of stability of equilibrium point is very similar to that of differential equation. Suppose $y_n \equiv y^*$ is an equilibrium solution.

1. The equilibrium solution $y_n \equiv y^*$ is said to be *stable*, if there exists a neighborhood of y^*, say N, such that $y_n \to y^*$ as $n \to \infty$ if $y_0 \in N$.

2. The equilibrium solution $y_n \equiv y^*$ is said to be *unstable* if there exist a neighborhood of y^*, say N, such that the solution to the sequence $\{y_n;\ n \geq 0\}$ with starting value $y_0 \in N$ will diverge away from y^* as $t \to \infty$.
3. The equilibrium solution $y_n \equiv y^*$ is said to be *semi-stable* if the sequence $\{y_n;\ n \geq 1\}$ with starting value $y_0 \in N$ will diverge away from y^* when $y_0 > y^*$ (resp. $y_0 < y^*$) and converge to y^* when $y_0 < y^*$ (resp. $y_0 > y^*$), as $t \to \infty$.

The following theorem is quite useful in determining the stability of an equilibrium solution.

Rule of Thumb: Assume that $F(y)$ is continuously differentiable and that y^* is an equilibrium solution of difference equation $y_{n+1} = F(y_n)$. It follows that

1. the equilibrium solution $y_n \equiv y^*$ is asympotically stable if $\left|F'(y^*)\right| < 1$.
2. the equilibrium solution $y_n \equiv y^*$ is unstable if $\left|F'(y^*)\right| > 1$.

Proof: Suppose now $\left|F'(y^*)\right| < 1$. By continuity of F', there must exist a neighborhood of y^*, say $N = (y^* - \delta, y^* + \delta)$, such that $\left|F'(y)\right| \leq \epsilon < 1$ whenever $y \in N$ for some $\epsilon \in (0,1)$. We claim that $y_n \in N$ for all n if $y_0 \in N$. Indeed, It can be shown by an easy induction. Suppose $y_n \in N$, we have

$$|y_{n+1} - y^*| = |F(y_n) - F(y^*)| = \left|F'(\xi) \cdot (y_n - y^*)\right| \leq \epsilon\delta < \delta$$

Here the second equality follows from the mean-value theorem in Calculus, and ξ is some number between y^* and y_n. In another word, $y_{n+1} \in N$. Therefore, we have

$$|y_{n+1} - y^*| \leq \epsilon\,|y_n - y^*| \leq \cdots \leq \epsilon^{n+1}\,|y_0 - y^*| \quad \to \quad 0$$

as $n \to \infty$.

Now assume $\left|F'(y^*)\right| > 1$. Similarly, we know there exists a neighborhood, still denoted by $N = (y^* - \delta, y^* + \delta)$, such that $\left|F'(y)\right| > 1$ whenever $y \in N$. The idea is that, whenever y_n falls into the neighborhood N, we have

$$|y_{n+1} - y^*| = |F(y_n) - F(y^*)| = |F'(\xi) \cdot (y_n - y^*)| > |y_n - y^*|,$$

where ξ is some number between y_n and y^*. In another word, y_{n+1} is pushed further away from the equilibrium point. Therefore, the equilibrium solution is not stable. □

Rule of Thumb: Assume that y^* is an equilibrium solution of difference equation $y_{n+1} = F(y_n)$ such that $F'(y^*) = 1$, $F''(y^*) \neq 0$. Show that the equilibrium solution $y_n \equiv y^*$ is semi-stable.

Proof: Without loss of generality, let us assume $y^* = 0$ and $F''(y^*) > 0$. It follows that there exists a neighnorhood of $y^* = 0$, say $(-\delta, \delta)$, such that F is strictly increasing in this neighborhood, and

$$F(y) \geq y + \epsilon y^2, \qquad \forall\ y \in (-\delta, \delta).$$

for some $0 < \epsilon < \frac{1}{2}F''(0)$. If $y_n \in (-\delta, 0)$, we have $y_{n+1} = F(y_n) \geq y_n$, but $F(y_n) \leq F(y^*) = y^*$ since F is increasing. This means that $\{y_n;\ n \geq 0\}$ is going to be an increasing sequence in interval $(-\delta, 0)$. Suppose $y_n \to Y$ for some constant Y, we must have $F(Y) = Y$, and $Y \in (-\delta, 0]$. It follows that $Y = 0 = y^*$. Therefore, the sequence is stable if it starts from $y_0 \in (-\delta, 0)$. However, if $y_n \in (0, \delta)$, we can see that $y_{n+1} = F(y_n) \geq y_n$ will further diverges away from 0, so it is unstable if $y_0 \in (0, \delta)$. Therefore, the equilibrium solution y^* is semi-stable.

Now let us consider the following example, in which we use *graphic* method to verify the stability.

Example: Consider the difference equation

$$u_{n+1} = \rho u_n (1 - u_n)$$

Here $\rho > 0$ is a constant.

Solution: Clearly, the equilibrium points satisfy

$$u^* = \rho u^* (1 - u^*) \quad \Rightarrow \quad u^* = 0, \; u^* = \frac{\rho - 1}{\rho}$$

Note when $\rho = 1$, we only have one equilibrium solution.

1. *Stability of $u^* = 0$:* since $F(u) = \rho u(1-u)$ have derivative $F'(u) = \rho(1 - 2u)$, we have

$$F'(u^*) = \rho$$

Therefore, the equilibrium solution $u^* = 0$ is stable if $0 < \rho < 1$, and unstable if $\rho > 1$. When $\rho = 1$, we have

$$F''(u^*) = -2\rho = -2 \neq 0$$

so the equilibrium solution is semistable.

2. *Stability of $u^* = \frac{\rho - 1}{\rho}$, $\rho \neq 1$:* We have

$$F'(u^*) = \rho \left(1 - 2\frac{\rho - 1}{\rho}\right) = 2 - \rho$$

Therefore, this equilibrium solution is stable if $1 < \rho < 3$, and unstable if $0 < \rho < 1$ or $\rho > 3$.