Introduction

Method of Laplace transform is very useful in solving differential equations. Its main feature is to convert a differential equation into an algebric equation, which allows us to obtain the Laplace transform of the solution. There immediately lies the main difficulty – to obtain solution from its Laplace transform (*Inversion* problem) is not easy at all except for some special forms. We shall start with a brief review of improper integral.

1 Improper Integral and Laplace Transform

Suppose $f(t)$ is a function on inteval $0 \le t < \infty$ (we can regard t as time).

Improper Integral: If the limit of integral over finite intervals

$$\lim_{h\to\infty} \int_0^h f(t)\,dt$$

exists, then we say *improper integral* $\int_0^\infty f(t)\,dt$ *converges* and define it by

$$\int_0^\infty f(t)\,dt \triangleq \lim_{h\to\infty} \int_0^h f(t)\,dt.$$

Otherwise, we say the improper integral *diverges*.

Example: Evaluate the improper integral

$$\int_0^\infty \frac{1}{(1+t)^p}\,dt; \qquad \text{here } \; p > 0$$

Solution: First assume $p \neq 1$. For $h > 0$, we have

$$\int_0^h \frac{1}{(1+t)^p}\,dt = \left.\frac{(1+t)^{1-p}}{1-p}\right|_0^h = \frac{(1+h)^{1-p}}{1-p} - \frac{1}{1-p},$$

which implies that

$$\lim_{h\to\infty} \int_0^h \frac{1}{(1+t)^p}\,dt = \begin{cases} \frac{1}{p-1} & ; \quad p > 1 \\ \infty & ; \quad p < 1 \end{cases}$$

Therefore the improper integral converges for $p > 1$, but diverges for $p < 1$. In case of $p = 1$, we have

$$\int_0^h \frac{1}{(1+t)}\,dt = \log(1+h) \;\to\; \infty \qquad \text{as } \; h \to \infty,$$

and the improper integral diverges. In one word, the improper integral diverges for $p \le 1$, and convergess for $p > 1$ with value

$$\int_0^\infty \frac{1}{(1+t)^p}\,dt = \frac{1}{p-1} \qquad \text{for} \;\; p > 1.$$

Example: Evaluate improper integral

$$\int_0^\infty \frac{1}{1+t^2}\,dt$$

Solution: It follows that

$$\int_0^h \frac{1}{1+t^2}\,dt = \arctan t\big|_0^h = \arctan h \;\; \to \;\; \frac{\pi}{2}, \quad \text{as } h \to \infty.$$

Therefore, the improper integral converges with value

$$\int_0^\infty \frac{1}{1+t^2}\,dt = \frac{\pi}{2}$$

We state without proof a theorem that is very useful in determining the convergence of improper integral.

Theorem: If $|f(t)| \le g(t)$ for all $t \ge 0$ and improper integral $\int_0^\infty g(t)\,dt$ converges, then improper integral $\int_0^\infty f(t)\,dt$ also converges. On the other hand, if $f(t) \ge g(t) \ge 0$ and $\int_0^\infty g(t)\,dt$ diverges, then $\int_0^\infty f(t)\,dt$ diverges.

Example: Prove that the improper integral

$$\int_0^\infty e^{-t^2}\,dt$$

converges. Actually, it can be shown that

$$\int_0^\infty e^{-t^2}\,dt = \frac{\sqrt{\pi}}{2}.$$

Solution: It is easy to see that $e^{-t^2} \le e^{-t}$ for $t \ge 1$, and $e^{-t^2} \le 1$ for $0 \le t < 1$. Hence $e^{-t^2} \le g(t)$ with

$$g(t) \triangleq \begin{cases} 1 & ; \quad 0 \le t < 1 \\ e^{-t} & ; \quad t \ge 1 \end{cases}$$

But $\int_0^\infty g(t)\,dt = 1 + \frac{1}{e}$ converges. Therefore $\int_0^\infty e^{-t^2}\,dt$ converges.

To calculate the improper integral, it suffices to show that

$$A \triangleq \int_{-\infty}^\infty e^{-t^2}\,dt = \sqrt{\pi}$$

However, we have

$$A^2 = \int_{-\infty}^\infty e^{-x^2}\,dx \cdot \int_{-\infty}^\infty e^{-y^2}\,dy = \int_{-\infty}^\infty \int_{-\infty}^\infty e^{-(x^2+y^2)}\,dxdy.$$

Using Polar coordinates, $x = r\cos\theta, y = r\sin\theta$, with $r \geq 0, 0 \leq \theta < 2\pi$, we have $dxdy = rdrd\theta$ and

$$A^2 = \int_0^\infty \int_0^{2\pi} e^{-r^2} r\, drd\theta = 2\pi \int_0^\infty re^{-r^2}\, dr == -\pi \cdot e^{-r^2}\Big|_0^\infty = \pi.$$

This completes the proof. □

Exercise: Determine whether the following improper integrals converge or diverge.

1. $\int_0^\infty \frac{t}{\sqrt{t^2+2}}\, dt$. (*Hint:* Diverge, since the integrand goes to 1 as $t \to \infty$)
2. $\int_0^\infty e^{-st} \sin t\, dt$. Here $s > 0$. (*Hint:* Converge, since $\left|e^{-st}\sin t\right| \leq e^{-st}$)
3. $\int_0^\infty \frac{1+t}{1+t^2}\, dt$. (*Hint:* Diverge, since $\frac{1+t}{1+t^2} \geq \frac{1}{1+t}$)

Definition of Laplace Transform: Suppose $f(t)$ is defined for $0 \leq t < \infty$. The *Laplace transform* of f is defined as the improper integral

$$\mathcal{L}\{f\} = F(s) \triangleq \int_0^\infty e^{-st} f(t)\, dt$$

for all s such that the improper integral converges.

Remark: The Laplace transform of $f(t)$ is a function of s.

The following theorem will give a satisfying description regarding the domain on which the Laplace transform is well-defined.

Theorem: If improper integral $\int_0^\infty e^{-st} f(t)\, dt$ converges for some $s = s_0$, then it converges for all $s > s_0$.

Example: Find Laplace transform of $f(t) = e^{at}$. Here a is a constant.

Solution: The Laplace transform

$$\mathcal{L}\left\{e^{at}\right\} = \int_0^\infty e^{-st} e^{at} = \frac{1}{s-a}, \quad \text{for } s > a.$$

Example: Find Laplace transform of $f(t) = \frac{1}{\sqrt{t}}$.

Solution: The Laplace transform is

$$\mathcal{L}\left\{\frac{1}{\sqrt{t}}\right\} = \int_0^\infty e^{-st} \frac{1}{\sqrt{t}}\, dt = \frac{2}{\sqrt{s}} \int_0^\infty e^{-x^2}\, dx = \frac{\sqrt{\pi}}{\sqrt{s}}.$$

Here the second equality follows from change of variable $t = \frac{x^2}{s}$ (we can assume $s > 0$, otherwise the Laplace transform is clearly ∞).

Example: Find Laplace transform of $f(t) = \sin at$, where a is a constant.

Solution: One way to find its Laplace transform is using formula

$$\sin at = \frac{e^{iat} - e^{-iat}}{2i}.$$

However, for $s > 0$,

$$\mathcal{L}\left\{e^{iat}\right\} = \int_0^\infty e^{-(s-ia)t}\,dt = \frac{1}{s-ia}.$$

We have, for $s > 0$,

$$\mathcal{L}\{\sin at\} = \frac{1}{2i}\left(\frac{1}{s-ia} - \frac{1}{s+ia}\right) = \frac{a}{s^2+a^2}$$

Exercise: Find the Laplace transform of $f(t) = \cos at$. (*Answer:* $\frac{s}{s^2+a^2}$)

There are many theorems that facilitate the computations of Laplace transforms of more complicated functions. We shall state below a relatively simple one without proof.

Theorem: Suppose Laplace transform $F(s) = \mathcal{L}\{f(t)\}$ is well-defined for $s > s_0$. We have

$$\mathcal{L}\left\{tf(t)\right\} = -F'(s), \quad \mathcal{L}\left\{t^2 f(t)\right\} = F''(s), \quad \cdots, \quad \mathcal{L}\left\{t^n f(t)\right\} = (-1)^n F^{(n)}(s), \quad \cdots$$

for $s > s_0$.

Intuition: We have $F(s) = \int_0^\infty e^{-st} f(t)\,dt$. If we can exchange the order of differentiation and integration, then

$$\frac{dF}{ds} = \frac{d}{ds}\int_0^\infty e^{-st} f(t)\,dt = \int_0^\infty \frac{d}{ds}\left(e^{-st} f(t)\right)\,dt = -\int_0^\infty e^{-st}\cdot tf(t)\,dt = -\mathcal{L}\left\{tf(t)\right\}.$$

The derivation of $F''(s), \cdots$ follows similarly. $\square$.

Let us use this theorem to calculate several examples.

Example: Find Laplace transform of $f(t) = t^n e^{at}$. Here $n \geq 0$ is an integer.

Solution: We know $F(s) \triangleq \mathcal{L}\left\{e^{at}\right\} = \frac{1}{s-a}$. Hence

$$\mathcal{L}\left\{t^n e^{at}\right\} = (-1)^n F^{(n)}(s) = (-1)^n \cdot (-1)^n \frac{n!}{(s-a)^{n+1}} = \frac{n!}{(s-a)^{n+1}}.$$

Example: Find Laplace transform of $f(t) = t^{n-\frac{1}{2}}$ for all integer $n \geq 0$.

Solution: We know $F(s) \triangleq \mathcal{L}\left\{\frac{1}{\sqrt{t}}\right\}$. Hence

$$\mathcal{L}\left\{t^{n-\frac{1}{2}}\right\} = (-1)^n F^{(n)}(s) = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^n}\cdot\frac{\sqrt{\pi}}{s^{n+\frac{1}{2}}}.$$

2 Properties of Laplace Transform

Laplace transform is a *linear operator*, in the sense that

$$\mathcal{L}\{c_1 f_1 + c_2 f_2\} = c_1\mathcal{L}\{f_1\} + c_2\mathcal{L}\{f_2\}$$

for all constants c_1, c_2. More precisely, we have

Theorem: If Laplace transforms of $f_1(t)$ and $f_2(t)$ converges for $s > s_1$ and $s > s_2$ respectively. Then for $s > \max\{s_1, s_2\}$,

$$\mathcal{L}\{c_1 f_1 + c_2 f_2\} = c_1 \mathcal{L}\{f_1\} + c_2 \mathcal{L}\{f_2\},$$

where c_1, c_2 are constants.

Proof: Exercise.

Another extremely important property of Laplace transform is the one-to-one respondence between functions and their Laplace transforms. More precisely, we have

Theorem: If $f_1 = f_2$, then $\mathcal{L}\{f_1\} = \mathcal{L}\{f_2\}$. Conversely, if $\mathcal{L}\{f_1\} = \mathcal{L}\{f_2\}$, then $f_1 = f_2$ provided that f_1, f_2 are both continuous.

The proof of this theorem is beyond the scope of this text. More details can be found in D.V. Widder "*The Laplace Transform*".

This theorem enable us to define the *inverse Laplace transform.*

Definition: Suppose F is the Laplace transform of a continuous funtion f, that is

$$F = \mathcal{L}\{f\}.$$

then the *inverse Laplace transform* of F, written as $\mathcal{L}^{-1}\{F\}$, is f. In another word

$$f = \mathcal{L}^{-1}\{F\}.$$

Remark: The inverse is well-defined and unambiguous by the previous theorem.

Example: Find the inverse Laplace transform of $\frac{1}{s^2}$.

Solution: Since $\mathcal{L}\left\{t\right\} = \frac{1}{s^2}$, we have

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t.$$

Exercise: Find the inverse Laplace transform of $\frac{1}{\sqrt{s}}$. (*Answer:* $\frac{1}{\sqrt{\pi t}}$).

Theorem: The inverse Laplace transform is a *linear opeartor.* That is

$$\mathcal{L}^{-1}\left\{c_1 F_1 + c_2 F_2\right\} = c_1 \mathcal{L}^{-1}\{F_1\} + c_1 \mathcal{L}^{-1}\{F_2\}$$

for all constants c_1, c_2.

Proof: Let $F_1 = \mathcal{L}\{f_1\}$ and $F_2 = \mathcal{L}\{f_2\}$, where f_1, f_2 are continuous functions. We have

$$\mathcal{L}\{c_1 f_1 + c_2 f_2\} = c_1 \mathcal{L}\{f_1\} + c_2 \mathcal{L}\{f_2\} = c_1 F_1 + c_2 F_2$$

By definition, we have

$$\mathcal{L}^{-1}\left\{c_1 F_1 + c_2 F_2\right\} = c_1 f_1 + c_2 f_2 = c_1 \mathcal{L}^{-1}\{F_1\} + c_1 \mathcal{L}^{-1}\{F_2\}.$$

This completes the proof. $\square$

Remark: It follows easily that

$$\mathcal{L}^{-1}\left\{c_1F_1 + c_2F_2 + \cdots + c_nF_n\right\} = c_1\mathcal{L}^{-1}\{F_1\} + c_1\mathcal{L}^{-1}\{F_2\} + \cdots + c_n\mathcal{L}^{-1}\{F_n\}.$$

for all n and constants $c_1, c_2, \cdots, c_n$.

The linear property of inverse Laplace transform helps to deal with more complicated Laplace transforms. Before doing some examples, we shall briefly review **Partial Fraction Expansion**, which is also very useful in solving initial value problems.

2.1 Partial Fraction Expansion

A quotient of form $\frac{P(x)}{Q(x)}$, where $P(x), Q(x)$ are both polynomials and $P(x)$ is of degree *less* than $Q(x)$, can be expanded into partial fractions. For example, we can expand quotient

$$\frac{3x-1}{x^2-1} = \frac{3x-2}{(x+1)(x-1)}$$

into the sum of two partial fractions

$$\frac{A}{x+1} + \frac{B}{x-1},$$

where A, B are two constants yet to be determined. It is easy to see that

$$\frac{A}{x+1} + \frac{B}{x-1} = \frac{(A+B)x + (B-A)}{x^2-1}$$

which implies that

$$A + B = 3, \quad B - A = 1 \quad \Rightarrow \quad A = 1,\ B = 2 \quad \Rightarrow \quad \frac{3x-1}{x^2-1} = \frac{1}{x+1} + \frac{2}{x-1}$$

and we complete the partial fraction expansion.

In general, the form of each partial fraction in the expansion depends *only* on $Q(x)$. It shall be clear from the following example. Suppose

$$Q(x) = (x+a)(x^2+b)(x^2+c)^2(x+d)^3$$

and $P(x)$ is some polynomial with degree less than $Q(x)$. The partial fraction expansion of $\frac{P(x)}{Q(x)}$ will take form

$$\frac{P(x)}{Q(x)} = \frac{A}{x+a} + \frac{Bx+C}{x^2+b} + \frac{Dx+E}{x^2+c} + \frac{Fx+G}{(x^2+c)^2} + \frac{H}{x+d} + \frac{I}{(x+d)^2} + \frac{H}{(x+d)^3}$$

The general rule can be concluded as

1. Term such as $(x+a) = (x+a)^1$ has exponent 1 *outside* the parenthesis. Hence term $(x+d)$ will once in the denominators of the expansion – as $(x+a)$.

2. Term such as $(x^2+b) = (x^2+b)^1$ has exponent 1 *outside* the parenthesis. Hence term $(x+d)$ will once in the denominators of the expansion – as (x^2+b).

3. Term such as $(x+d)^3$ has exponent 3 *outside* the parenthesis. Hence term $(x+d)$ will appear three times in the denominators of the expansion – as $(x+d)$, $(x+d)^2$, and $(x+d)^3$ respectively.

4. Term such as $(x^2+c)^2$ has exponent 2 *outside* the parenthesis. Hence term (x^2+c) will appear twice in the denominators of the expansion – as (x^2+c) and $(x^2+c)^2$ respectively.

5. Each numerator is a polynomial of degree *one* less than that of the term *inside* the parenthesis of its denominator.

Let us study the following examples.

Example: With the aid of partial fraction expansion, find the Inverse Laplace Transforms of the following functions.

1. $\frac{3s+2}{s(s+1)(s+2)}$ 2. $\frac{s}{s^2+2s+1}$ 3. $\frac{3s^2+s+1}{s(s^2+1)}$

Solution: From the general rule of determining the form of each term in partial fraction expansion, it follows that

$$\begin{aligned}
\frac{2s-3}{s(s+1)(s+2)} &= \frac{A}{s}+\frac{B}{s+1}+\frac{C}{s+2} \Rightarrow A=B=1, C=-2 \\
\frac{s+2}{s^2+2s+1} &= \frac{s+2}{(s+1)^2}=\frac{A}{s+1}+\frac{B}{(s+1)^2} \Rightarrow A=1, B=-1 \\
\frac{3s^2+s+1}{s(s^2+1)} &= \frac{A}{s}+\frac{Bs+C}{s^2+1} \Rightarrow A=1, B=2, C=1
\end{aligned}$$

Therefore, the inverse Laplace transform are

$$(1)\ \ 1+e^{-t}-2e^{-2t} \qquad (2)\ \ e^{-t}-te^{-t} \qquad (3)\ \ 1+2\cos t+\sin t$$

respcetively. □

3 Solutions to Initial Value Problems with Constant Coefficients

Consider the following linear differential equation

$$L[y] \stackrel{\triangle}{=} a_0y^{(n)}+a_1y^{(n-1)}+\cdots+a_{n-1}y'+a_ny=f(t),$$

where $a_0, a_1, \cdots, a_n$ are constants with $a_n \neq 0$. An advantange of Laplace transform method is that non-homogeneous ($g(t)\equiv 0$) and non-homogeneous ($g(t)\neq 0$) equations are handled exactly the same way.

Taking Laplace transform on both sides, we have

$$a_0\mathcal{L}\left\{y^{(n)}\right\}+a_1\mathcal{L}\left\{y^{(n-1)}\right\}+\cdots+a_{n-1}\mathcal{L}\left\{y'\right\}+a_n\mathcal{L}\{y\}=\mathcal{L}\{f(t)\}:=F(s)$$

Our next task is to evaluate $\mathcal{L}\left\{y^{(n)}\right\}$ in terms of $Y(s)\stackrel{\triangle}{=}\mathcal{L}\left\{y\right\}$. We have the following important theorem.

Theorem: Let $Y(s) \stackrel{\triangle}{=} \mathcal{L}\{y(t)\}$. We have, for $n \geq 1$,

$$\mathcal{L}\left\{y^{(n)}\right\} = s^n Y(s) - s^{n-1}y(0) - \cdots - sy^{(n-2)}(0) - y^{(n-1)}(0)$$

under some reguarlity conditions.

Proof: We first assume $n = 1$. In this case,

$$\begin{aligned} \mathcal{L}\{y'\} &= \int_0^\infty e^{-st} y'(t)\, dt = \int_0^\infty e^{-st}\, dy(t) \\ &= e^{-st}y(t)\big|_0^\infty + \int_0^\infty y(t)se^{-st}\, dt = sY(s) - y(0). \end{aligned}$$

(In the last equality, we implicitly assume that $\lim_{t\to\infty} e^{-st}y(t) = 0$). Similarly

$$\mathcal{L}\{y''\} = s\mathcal{L}\{y'\} - y'(0) = s\big[sY(s) - y(0)\big] - y'(0) = s^2Y(s) - sy(0) - y'(0).$$

Here we implicitly assume $\lim_{t\to\infty} e^{-st}y'(t) = 0$ for the first equality and $\lim_{t\to\infty} e^{-st}y(t) = 0$ for the second equality. In general, it can be shown that

$$\mathcal{L}\left\{y^{(n)}\right\} = s^n Y(s) - s^{n-1}y(0) - \cdots - sy^{(n-2)}(0) - y^{(n-1)}(0)$$

under condition $\lim_{t\to\infty} e^{-st}y^{(k)}(t) = 0$ for $k = 0, 1, \cdots, n-1$. □

Remark: The condition that ensure the theorem to hold is

$$\boldsymbol{\lim_{t\to\infty} e^{-st}y^{(k)}(t) = 0, \qquad k = 0, 1, \cdots, n-1.}$$

This condition usually holds, especially, it always holds for a solution obtained by Laplace transform method; see D. Widder, *Advanced Calculus* (1961), for details. In the following examples, we will omit the details of verifying this condition for the solutions we obtained.

Example: Use the Laplace transform method to solve initial value problem

$$y' + 2y = e^{-x}, \qquad y(0) = 0$$

Solution: Observe that $F(s) \stackrel{\triangle}{=} \mathcal{L}\{e^{-x}\} = \frac{1}{s+1}$. Taking Laplace transform on both sides, and letting $Y(s) \stackrel{\triangle}{=} \mathcal{L}\{y\}$, we have

$$\mathcal{L}\{y'\} + 2Y(s) = \frac{1}{s+1} \;\Rightarrow\; sY(s) + 2Y(s) = \frac{1}{s+1} \;\Rightarrow\; Y(s) = \frac{1}{(s+1)(s+2)}$$

To determine the inverse Laplace transform of $\frac{1}{(s+1)(s+2)}$, we expand it in partial fraction. It follows that

$$Y(s) = \frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2} = \frac{A(s+2) + B(s+1)}{(s+1)(s+2)}$$

which yileds that

$$A + B = 0, \quad 2A + B = 1 \quad \Rightarrow \quad A = 1, \; B = -1$$

or

$$Y(s) = \frac{1}{s+1} - \frac{1}{s+2} \;\Rightarrow\; y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}$$

But $\mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\} = e^{at}$, we have

$$y(t) = e^{-t} - e^{-2t}.$$

Exercise: Solve initial value problem

$$y' + 2y = e^{-x}, \qquad y(0) = 4$$

Solution: It is very similar to the preceding example – the only difference is that $\mathcal{L}\{y'\} = sY(s) - y(0) = sY(s) - 4$. We obtain that

$$(s+2)Y(s) - 4 = \frac{1}{s+1} \;\Rightarrow\; Y(s) = \frac{4s+5}{(s+1)(s+2)} = \frac{1}{s+1} + \frac{3}{s+2}$$

We have $y(t) = e^{-t} + 3e^{-2t}$.

Example: Use Laplace transform method to solve initial value problem

$$y'' - 2y' + y = e^{t}, \quad y(0) = 1, \; y'(0) = 0$$

Solution: Let $Y(s) = \mathcal{L}\{y(t)\}$. It follows that

$$\begin{aligned} \mathcal{L}\{y''\} &= s^2Y(s) - sy(0) - y'(0) = s^2Y(s) - s \\ \mathcal{L}\{y'\} &= sY(s) - y(0) = sY(s) - 1 \end{aligned}$$

Taking Laplace transform on both sides, we have

$$s^2Y(s) - s - 2sY(s) + 2 + Y(s) = \frac{1}{s-1}$$

which implies that

$$Y(s) = \frac{1}{(s-1)^3} + \frac{s-2}{(s-1)^2} = \frac{1}{(s-1)^3} + \frac{(s-1)-1}{(s-1)^2} = \frac{1}{(s-1)^3} - \frac{1}{(s-1)^2} + \frac{1}{s-1}$$

Therefore $y(t) = (t^2 - t + 1)e^t$ (exercise!).

Exercise: Solve initial value problem

$$y^{(4)} - y = 0, \quad y(0) = 1, \; y'(0) = 0, \; y''(0) = 2, \; y'''(0) = 0.$$

Solution: Let $Y(s) \triangleq \mathcal{L}\{y(t)\}$, we have

$$s^4Y(s) - s^3 - 2s - Y(s) = 0 \;\Rightarrow\; Y(s) = \frac{s^3+2s}{s^4-1} = \frac{s^3+2s}{(s+1)(s-1)(s^2+1)}.$$

Usinf partial fraction, we have

$$Y(s) = \frac{A}{s+1} + \frac{B}{s-1} + \frac{Cs+D}{s^2+1} \;\Rightarrow\; A = B = -\frac{1}{4}, C = \frac{3}{2}, D = 0$$

which implies that

$$y(t) = -\frac{1}{4}\left(e^{-t} + e^{t}\right) + \frac{3}{2}\cos t.$$

Example (Resonance): Consider initial value problem

$$y'' + \omega_0^2 y = \frac{F_0}{m}\cos\omega t, \qquad y(0) = 0,\ y'(0) = 0$$

Solution: Taking Laplace transform on both sides, we have

$$Y(s) = \frac{F_0}{m}\frac{s}{(s^2+\omega_0^2)(s^2+\omega^2)}$$

In case $\omega \neq \omega_0$, we have

$$Y(s) = \frac{As}{s^2+\omega^2} - \frac{Bs}{s^2+\omega_0^2},$$

where $A = B = \frac{F_0}{m(\omega_0^2-\omega^2)}$. This implies that

$$y(t) = \frac{F_0}{m(\omega_0^2-\omega^2)}\left(\cos\omega t - \cos\omega_0 t\right)$$

In case that $\omega = \omega_0$ (*Resonance*), we have

$$Y(s) = \frac{F_0}{m}\frac{s}{(s^2+\omega_0^2)^2} = -F'(s)$$

where

$$F(s) \triangleq \frac{F_0}{2m}\cdot\frac{1}{s^2+\omega_0^2} = \mathcal{L}\left\{\frac{F_0}{2m\omega_0}\sin\omega_0 t\right\}$$

Therefore,

$$y(t) = \frac{F_0}{2m\omega_0}t\sin\omega_0 t$$

We recover the same results. □

Example: (*linear system*) Solve the following system of first order equations:

$$\begin{aligned} z_1' &= 2z_1 + 4z_2 \\ z_2' &= -z_1 - 3z_2 \end{aligned}$$

with initial condition $z_1(0) = 0$, $z_2(0) = 1$.

Solution: Let $Z_i(s) = \mathcal{L}\{z_i(t)\}$ for $i = 1, 2$. We have

$$\mathcal{L}\{z_1'\} = sZ_1(s) - z_1(0) = sZ_1, \quad \mathcal{L}\{z_2'\} = sZ_2(s) - z_2(0) = sZ_2 - 1.$$

Therefore

$$\begin{aligned} sZ_1 &= 2Z_1 + 4Z_2 \\ sZ_2 - 1 &= -Z_1 - 3Z_2 \end{aligned}$$

or

$$Z_1 = \frac{4}{(s+2)(s-1)} = \frac{-\frac{4}{3}}{s+2} + \frac{\frac{4}{3}}{s-1}, \qquad Z_2 = \frac{s-2}{(s+2)(s-1)} = \frac{\frac{4}{3}}{s+2} + \frac{-\frac{1}{3}}{s-1},$$

which implies

$$z_1 = \frac{4}{3}\left(e^t - e^{-2t}\right), \qquad z_2 = \frac{1}{3}\left(-e^t + 4e^{-2t}\right)$$

The main difficulty lies in the fact that the inverse Laplace transform is not that easy to calculate, contrary to what we have shown in examples. We have to resort to numerical schemes to obtain the inverse in most cases.

Remark: In above discussions, the initial conditions are evaluated at $t = 0$. There is no loss of generality, indeed. For example, if the initial value problem takes form

$$L[y] = f(t), \qquad y(t_0) = y_0,\ y'(t_0) = y_0',\ \cdots,\ y^{(n-1)}(t_0) = y_0^{(n-1)}$$

We can define $\tilde{y}(t) \stackrel{\triangle}{=} y(t + t_0)$, we have an equivalent initial value problem

$$L[\tilde{y}](t) = f(t + t_0), \qquad \tilde{y}(0) = y_0,\ \tilde{y}'(0) = y_0',\ \cdots,\ \tilde{y}^{(n-1)}(0) = y_0^{(n-1)}.$$

For example, consider initial value problem

$$y'' + 4y = -3\sin t, \quad y(\pi) = 0,\ y'(\pi) = 1.$$

Let $\tilde{y}(t) \stackrel{\triangle}{=} y(t + \pi)$. It follows that

$$\tilde{y}'' + 4\tilde{y} = -3\sin(t + \pi) = 3\sin t, \qquad \tilde{y}(0) = 0,\ \tilde{y}'(0) = 1.$$

It is not difficult to find the solution as

$$\tilde{y}(t) = \sin t \qquad \text{(exercise)}$$

Hence, $y(t) = \tilde{y}(t - \pi) = -\sin t$.

Exercise: To find $\tilde{Y}(s) = \mathcal{L}\{\tilde{y}\}$, show that

$$\tilde{Y}(s) = \frac{1}{s^2 + 1},$$

which implies that $\tilde{y} = \sin t$.

4 Miscellaneous Results of Laplace Transform

In this section we collect some useful results about Laplace transform. Let us start with the following lemma.

Lemma: Suppose $F(s) = \mathcal{L}\{f(t)\}$, or $f(t) = \mathcal{L}^{-1}\{F(s)\}$. We have

$$\mathcal{L}\left\{e^{ct}f(t)\right\} = F(s-c) \quad \text{or} \quad \mathcal{L}^{-1}\left\{F(s-c)\right\} = e^{ct}f(t)$$

Proof: Use definition. □

Example: Solve the following initial value problem

$$y'' - 2y' + 5y = 0, \qquad y(0) = 1,\ y'(0) = 0.$$

Solution: Let $Y(s) \stackrel{\triangle}{=} \mathcal{L}\{y\}$. We have

$$\mathcal{L}\{y''\} = s^2 Y(s) - s, \qquad \mathcal{L}\{y'\} = sY(s) - 1.$$

It follows that

$$Y(s) = \frac{s-2}{s^2-2s+5} = \frac{s-2}{(s-1)^2+4} = F(s-1)$$

where

$$F(s) \stackrel{\triangle}{=} \frac{s-1}{s^2+4} \ \Rightarrow\ \mathcal{L}^{-1}\{F\} = \cos 2t - \frac{1}{2}\sin 2t$$

Hence $y(t) = e^t\left(\cos 2t - \frac{1}{2}\sin 2t\right)$.

Remark: We can also use partial fraction expansion to solve the inverse problem in this example with the aid of complex variable. Indeed

$$Y(s) = \frac{s-2}{(s-1)^2+4} = \frac{s-2}{\big(s-(1+2i)\big)\big(s-(1-2i)\big)} = \frac{A}{s-(1+2i)} + \frac{B}{s-(1-2i)}$$

It follows that

$$A + B = 1, \qquad A(1-2i) + B(1+2i) = 2 \quad \Rightarrow \quad A = \frac{1}{2} + \frac{1}{4}i,\ B = \frac{1}{2} - \frac{1}{4}i.$$

Therefore,

$$\begin{aligned} y(t) &= Ae^{(1+2i)t} + Be^{(1-2i)t} = \left(\frac{1}{2}+\frac{1}{4}i\right)\cdot e^t(\cos 2t + i\sin 2t) + \left(\frac{1}{2}-\frac{1}{4}i\right)\cdot e^t(\cos 2t - i\sin 2t) \\ &= e^t\left(\cos 2t - \frac{1}{2}\sin 2t\right) \end{aligned}$$

Exercise: Solve the linear system

$$\begin{aligned} z_1' &= -3z_1 - 5z_2 \\ z_2' &= z_1 + z_2 \end{aligned}$$

with initial condition $z_1(0) = 1$, $z_2(0) = 1$. (*Answer:* $z_1 = e^{-t}(\cos t - 7\sin t)$, $z_2 = e^{-t}(\cos t + 3\sin t)$.)

4.1 Convolution and Integral Equation

The **Faltung Theorem**, which is very useful helping evaluating integrals, involves the *convolution* of two functions. Let us first give the definition.

Definition of Convolution: The *convolution* of two functions f and g, denoted by $f * g$, is a function whose value at t is determined by

$$(f * g)(t) \triangleq \int_0^t f(t-s)g(s)\,ds$$

This integral is called *convolution integral*.

Example: Let $f(t) \triangleq t$, $g(t) \triangleq \cos t$. Determine $f * g$.

Solution: By definition, we have

$$(f * g)(t) = \int_0^t (t-s)\cos s\,ds = (t-s)\sin s\big|_0^t + \int_0^t \sin s\,ds = 1 - \cos t$$

Here we state several properties of convolution product, which resemble those of ordinary product.

Lemma: For any functions f, g, h, we have

1. (*Commutative Law*) $\quad f * g \equiv g * f$.
2. (*Distributive Law*) $\quad f * (g + h) \equiv f * g + f * h$.
3. (*Associative Law*) $\quad (f * g) * h \equiv f * (g * h)$.

Proof: We will only give the proof of Associative Law. The Commutative and Distributive Laws are left as exercises. By definition, we have

$$\big((f * g) * h\big)(t) = \int_0^t (f * g)(t-u)h(u)\,du = \int_0^t \left[\int_0^{t-u} f(t-u-w)g(w)h(u)\,dw\right] du$$

For the integral in the bracket, make change of variable $w = s - u$. We have

$$\big((f * g) * h\big)(t) = \int_0^t \left[\int_u^t f(t-s)g(s-u)h(u)\,ds\right] du$$

This multiple integral is carried out over the region

$$\{(s, u);\quad 0 \le u \le s \le t\}$$

as depicted by shaded region in the following graph.

Change the order of integration, we have

$$\big((f*g)*h\big)(t) = \int_0^t \left[\int_0^s f(t-s)g(s-u)h(u)\,du\right] ds = \int_0^t f(t-s)(g*h)(s)\,ds = \big(f*(g*h)\big)(t)$$

This completes the proof. $\square$

The importance of convolution shows in the following theorem.

Faltung theorem: Let $F(s) = \mathcal{L}\{f(t)\}$, $G(s) = \mathcal{L}\{g(t)\}$. We have

$$F \cdot G = \mathcal{L}\left\{f * g\right\} \qquad \text{or} \qquad \mathcal{L}^{-1}\{F \cdot G\} = f * g = \mathcal{L}^{-1}\{F\} * \mathcal{L}^{-1}\{G\}$$

Proof: By definition,

$$\mathcal{L}\{f * g\}(s) = \int_0^\infty e^{-st}\left[\int_0^t f(t-u)g(u)\,du\right]\,dt = \int_0^\infty \left[\int_0^t e^{-st} f(t-u)g(u)\,du\right]\,dt$$

The multiple integral is carried out over region

$$\{(t,u); \quad 0 \le u \le t\}$$

as depicted by the shaded region in the following graph.

Changing the order of integration, we have

$$\begin{aligned}
\mathcal{L}\{f * g\}(s) &= \int_0^\infty \left[\int_u^\infty e^{-st} f(t-u)g(u)\,dt\right]\,du = \int_0^\infty s^{-su} g(u)\left[\int_u^\infty e^{-s(t-u)} f(t-u)\,dt\right]\,du \\
&= \int_0^\infty e^{-su} g(u)\left[\int_0^\infty e^{-sw} f(w)\,dw\right]\,du = \int_0^\infty e^{-su} g(u)\cdot F(s)\,du \\
&= F(s)\int_0^\infty e^{-su} g(u)\,du = F(s)\cdot G(s)
\end{aligned}$$

This completes the proof. □

An immediate application of Faltung theorem is finding a particular solution of the non-homogeneous linear equation. We will illustrate with a second-order linear equation, but same methodology works for higher order linear equations too. Consider non-homogeneous equation

$$ay'' + by' + cy = g(t).$$

We wish to find a particular solution. Actually, it suffices to solve the following initial value problem

$$ay'' + by' + cy = g(t), \qquad y(0) = 0,\ y'(0) = 0.$$

Suppose the solution to this initial value problem is $y_p(t)$. Let $Y_p(s) = \mathcal{L}\{y_p(t)\}$. We have

$$Y_p(s) = \frac{G(s)}{as^2 + bs + c}, \qquad \text{here } G(s) = \mathcal{L}\{g(t)\}$$

Letting $h(t) \stackrel{\triangle}{=} \mathcal{L}^{-1}\left\{\frac{1}{as^2+bs+c}\right\}$, we have that

$$y_p(t) = (h * g)(t) = \int_0^t h(t-s)g(s)\,ds.$$

Remark: Function $h(t)$ is easy to obtain. Let $r_{1,2}$ stand for the two roots of the characteristic equation $as^2 + bs + c = 0$. It follows that $as^2 + bs + c = a(s - r_1)(s - r_2)$.

1. $r_1 \neq r_2$ *are both real*: Using partial fraction expansion, we have
$$\frac{1}{as^2 + bs + c} = \frac{1}{a(r_1 - r_2)} \left(\frac{1}{s - r_1} - \frac{1}{s - r_2} \right)$$
or
$$h(t) = \frac{1}{a(r_1 - r_2)} \left(e^{r_1 t} - e^{r_2 t} \right).$$

2. $r_1 = r_2 := r$ *are both real*: In this case
$$\frac{1}{as^2 + bs + c} = \frac{1}{a} \frac{1}{(s - r)^2} = -F'(s)$$
where $F(s) \stackrel{\triangle}{=} \frac{1}{a} \frac{1}{s-r}$. Therefore,
$$h(t) = \frac{1}{a} t e^{rt}.$$

3. $r_1 \neq r_2$ *are complex*: Write $r_{1,2} = \lambda \pm i\mu$. Using partial expansion, we still have that
$$h(t) = \frac{1}{a(r_1 - r_2)} \left(e^{r_1 t} - e^{r_2 t} \right) = \frac{1}{a\mu} e^{\lambda t} \sin \mu t.$$

Example: Write the solution of the initial value problem
$$y'' + 4y' + 5y = f(t), \qquad t \geq 0; \qquad y(0) = 0, \quad y'(0) = 1,$$
in term of a definite integral.

Solution: Let $F = \mathcal{L}\{f\}$ and $Y = \mathcal{L}\{y\}$. We have
$$\mathcal{L}\{y'\} = sY(s), \qquad \mathcal{L}\{y''\} = s^2 Y(s) - 1$$
and
$$\mathcal{L}\{y''\} + 4\mathcal{L}\{y'\} + 5\mathcal{L}\{y\} = \mathcal{L}\{f\} \quad \Rightarrow \quad s^2 Y(s) - 1 + 4sY(s) + 5Y(s) = F(s).$$
Hence
$$Y(s) = \frac{1 + F(s)}{s^2 + 4s + 5} = \frac{1}{(s+2)^2 + 1} + \frac{1}{(s+2)^2 + 1} \cdot F(s)$$
Note that
$$\mathcal{L}^{-1} \left\{ \frac{1}{(s+2)^2 + 1} \right\} = e^{-2t} \sin t.$$
It follows that
$$y(t) = e^{-2t} \sin t + \int_0^t e^{-2(t-s)} \sin(t - s) f(s)\, ds \qquad \square$$

Example (Beta Functions): Show that

$$\int_0^1 (1-t)^m t^n \, dt = \frac{m!n!}{(m+n+1)!}$$

for all non-negative integers m, n.

Proof: Let $f(x) \triangleq x^m$ and $g(x) \triangleq x^n$. It follows that

$$h(x) \triangleq (f * g)(x) = \int_0^x f(x-t)g(t)\, dt = \int_0^x (x-t)^m t^n \, dt.$$

However, Faltung theorem yields that

$$H(s) \triangleq \mathcal{L}\{h(x)\} = \mathcal{L}\{f(x)\} \cdot \mathcal{L}\{g(x)\} = \frac{m!}{s^{m+1}} \cdot \frac{n!}{s^{n+1}} = \frac{m!n!}{s^{m+n+2}} = \frac{m!n!}{(m+n+1)!} \frac{(m+n+1)!}{s^{m+n+2}}$$

Therefore, for all x,

$$h(x) = \mathcal{L}^{-1}\{H(s)\} = \frac{m!n!}{(m+n+1)!} t^{m+n+1}.$$

In particular,

$$h(1) = \int_0^1 (1-t)^m t^n \, dt = \frac{m!n!}{(m+n+1)!}.$$

This completes the proof. □

4.2 Elementary Volterra Integral Equation

Convolution is very useful in the study of Volterra integral equations, which take form

$$y(t) = \int_0^t y(t-s) g(s)\, ds + y_0(t), \qquad t \geq 0.$$

Here $g(t), y_0(t)$ defined on $t \geq 0$ are given functions. We shall assume throughout this section that $g(t)$ and $y_0(t)$ are both *non-negative* functions, which implies the solution $y(t)$ is also a *non-negative* function.

Population Model (with age structure): Suppose in a population, the fraction of individuals that will survive to age s is $\rho(s)$, while each individual of age s will produce offspring at rate $\beta(s)$. We are interested in the dynamics of $y(t)$, the overall birth rate at time t for the population. That is, in a small time interval $[t, t+dt)$, the total popaultion will produce $y(t)\, dt$ new offsprings.

Now at time t, the number of individuals aged from s to $s + ds$ (where $0 \leq s \leq t$) is $y(t-s)\rho(s)\, ds$. These individuals will produce offspring $y(t-s)\rho(s)\, ds \cdot \beta(s)\, dt$ in time interval $[t, t+dt)$. Let $y_0(t)$ denote the birth rate of individuals with age greater than t (or born before time 0). We have

$$y(t)\, dt = \int_0^t y(t-s)\rho(s)\, ds \cdot \beta(s)\, dt + y_0(t)\, dt$$

which implies that

$$y(t) = \int_0^t y(t-s)g(s)\,ds + y_0(t)$$

where $g(s) \stackrel{\triangle}{=} \rho(s) \cdot \beta(s)$. $\square$

The integral can be rewritten as

$$y(t) = (y * g)(t) + y_0(t).$$

Taking Laplace transform on both sides, we obtain

$$Y(s) = Y(s)G(s) + Y_0(s) \qquad \Rightarrow \qquad Y(s) = \frac{Y_0(s)}{1 - G(s)}$$

Here $Y(s) = \mathcal{L}\{y(t)\}$, $G(s) = \mathcal{L}\{g(t)\}$ and $Y_0(s) \stackrel{\triangle}{=} \mathcal{L}\{y_0(t)\}$. We will first take a look of a few examples that are explicitly solvable.

Example: If $y_0(t) \equiv 0$, then $Y_0(s) \equiv 0$, which in turn imples that $Y(s) \equiv 0$, or $y(t) \equiv 0$.

Example: Solve the integral equation

$$y(t) = \int_0^t \sin(t-s)y(s)\,ds + 1.$$

Solution: Taking Laplace transform on both sides, we have

$$Y(s) = \frac{1}{s^2+1}Y(s) + \frac{1}{s} \qquad \Rightarrow \qquad Y(s) = \frac{s^2+1}{s^3} = \frac{1}{s} + \frac{1}{s^3}$$

Therefore

$$y(t) = 1 + \frac{1}{2}t^2.$$

Example: Solve the following integrodifferential equation

$$y''(t) - 2\int_0^t \sin(t-s)y(s)\,ds = 0; \qquad t \geq 0$$

with initial condition $y(0) = 0, y'(0) = 1$.

Solution: Let $Y(s) = \mathcal{L}\{y(t)\}$. We have

$$\mathcal{L}\{y''\} = s^2Y(s) - 1.$$

Hence,

$$s^2Y(s) - 1 - 2\frac{1}{s^2+1}Y(s) = 0 \qquad \Rightarrow \qquad Y(s) = \frac{s^2+1}{s^4+s^2-2} = \frac{s^2+1}{(s^2+2)(s+1)(s-1)}.$$

Partial fraction expansion yields

$$Y(s) = \frac{1}{3}\frac{1}{s^2+2} + \frac{1}{3}\frac{1}{s-1} - \frac{1}{3}\frac{1}{s+1}$$

Therefore,

$$y(t) = \frac{1}{3\sqrt{2}}\sin\sqrt{2}t + \frac{1}{3}(e^t - e^{-t}).$$

In general it is impossible to find explicit inverse Laplace transform. However, we can obtain an asymptotic result as $t \to \infty$ using Laplace transform. From now on, we assume that function g is non-negative and $Y_0 = \mathcal{L}\{y_0\}$ is bounded.

It can be shown that $y(t) \sim Ce^{rt}$ for some constant C and r, as $t \to \infty$. In another word,

$$\lim_{t\to\infty} \frac{y(t)}{Ce^{rt}} = 1$$

for some constants $C > 0$ and r. This asymptotic result shows that $y(t)$ increases (or decay) exponentially with exponent r. The question now is how to determine C and r using Laplace transform. Actually, we have

$$r \textit{ is the unique solution to equation } G(r) = 1, \textit{ while } C = -\tfrac{Y_0(r)}{G'(r)}.$$

We shall illustrate the main idea without being rigorous.

The following lemma plays a key role in the development, which is also of its own interest.

Lemma: Let $F(s) = \mathcal{L}\{f(t)\}$. If $f(t) \to c$ as $t \to \infty$, then $sF(s) \to c$ as $s \to 0$.

Proof: For any $\epsilon > 0$, there exists $T > 0$ such that $|f(t) - c| < \epsilon$ for all $t \geq T$. It follows that, for all $s < \min\left(\frac{\epsilon}{\int_0^T |f(t)|\,dt}, -\frac{\log(1-\epsilon/|c|)}{T}\right)$,

$$\begin{aligned}
\big|sF(s) - c\big| &= \left|s\int_0^T e^{-st}f(t)\,dt + s\int_T^\infty e^{-st}f(t)\,dt - c\right| \\
&\leq s\int_0^T |f(t)|\,dt + s\int_T^\infty e^{-st}|f(t)-c|\,dt + \left|s\int_T^\infty e^{-st}c\,dt - c\right| \\
&\leq \epsilon + \epsilon\cdot s\int_T^\infty e^{-st}\,dt + \left(1 - e^{-sT}\right)|c| \\
&\leq \epsilon + \epsilon + \epsilon = 3\epsilon.
\end{aligned}$$

This completes the proof. □

Now suppose $y(t) \sim Ce^{rt}$ or

$$\lim_{t\to\infty} e^{-rt}y(t) = C.$$

It follows from the preceding lemma that

$$s\mathcal{L}\left\{e^{-rt}y(t)\right\} = sY(s+r) = \frac{sY_0(s+r)}{1 - G(s+r)} \quad \to \quad C \qquad \text{as} \quad s \to 0.$$

However, since $sY_0(s+r) \to 0$, we have $1 - G(s+r) \to 0$ as $s \to 0$, or $G(r) = 1$. But G is a decreasing function. It follows that r is uniquely determined. By L'Hospital rule, we have

$$C = \lim_{s\to0}\frac{sY_0(s+r)}{1 - G(s+r)} = \lim_{s\to0}\frac{Y_0(s+r) + sY_0'(s+r)}{-G'(s+r)} = -\frac{Y_0(r)}{G'(r)}$$

This completes our discussion.

Example: Suppose that $k(t)$ is positive and continuous, with $\int_0^\infty k(t)\,dt < 1$. Let $y(t)$ be a smooth function that solve the Volterra Equation

$$y(t) = 1 + \int_0^t k(t-s)y(s)\,ds, \qquad t \geq 0$$

Show that $y(t)$ is non-decreasing and $\lim_{t\to\infty} y(t) = \big(1 - \int_0^\infty k(s)\big)^{-1}$.

Proof: We have

$$y(t) = 1 + \int_0^t y(t-s)k(s)\,ds.$$

Taking derivatives on both sides, we obtain

$$y'(t) = y(0)k(t) + \int_0^t y'(t-s)k(s)\,ds$$

However, $y(0) = 1$, therefore, with $\phi(t) \stackrel{\triangle}{=} y'(t)$, we have

$$\phi(t) = k(t) + \int_0^t \phi(t-s)k(s)\,ds \qquad \Rightarrow \qquad \phi(0) = k(0) > 0.$$

It follows that $\phi(t) > 0$ for all $t \geq 0$ (why?) and $y(t)$ is increasing. Hence, $\lim_{t\to\infty} y(t)$ must exist, say c (could be $+\infty$). We have

$$c = \lim_{s\to 0} s\mathcal{L}\{y\}(s) := \lim_{s\to 0} sY(s)$$

However, taking Laplace transform on both sides of the integral equation,

$$Y(s) = \frac{1}{s} + K(s)Y(s) \qquad \Rightarrow \qquad sY(s) = \frac{1}{1-K(s)} \qquad \Rightarrow \qquad c = \frac{1}{1 - \lim_{s\to 0} K(s)}$$

By definition

$$K(s) = \int_0^\infty e^{-st}k(t)\,dt \qquad \Rightarrow \qquad \lim_{s\to 0} K(s) = \int_0^\infty \lim_{s\to 0} e^{-st}k(t)\,dt = \int_0^\infty k(t)\,dt.$$

This completes the proof. $\square$

Example (Abel Equation): Formally solve the Abel equation

$$\varphi(x) = \int_0^x \frac{y(s)}{\sqrt{x-s}}\,ds, \qquad t \geq 0.$$

Here φ is a given non-negative, smooth function with $\varphi(0) = 0$. See the following graph:

Solution: Note the equation is indeed $\varphi = y * \sqrt{\frac{1}{x}}$. Taking Laplace transform on both sides, we have, with $\Phi = \mathcal{L}\{\varphi\}$, that

$$\Phi(s) = Y(s) \cdot \frac{\sqrt{\pi}}{\sqrt{s}} \qquad \Rightarrow \qquad Y(s) = \frac{1}{\sqrt{\pi}}\Phi(s) \cdot \sqrt{s}$$

But there is no function with Laplace transform as $\sqrt{s}$. However, note

$$\mathcal{L}\{\varphi'\} = s\Phi(s) - s\varphi(0) = s\Phi(s).$$

It follows that

$$Y(s) = \frac{1}{\pi} s\Phi(s) \cdot \frac{\sqrt{\pi}}{\sqrt{s}} = \frac{1}{\pi}\mathcal{L}\{\varphi'\} \cdot \mathcal{L}\{\frac{1}{\sqrt{x}}\},$$

which implies that

$$y(t) = \frac{1}{\pi}\varphi' * \frac{1}{\sqrt{x}} = \frac{1}{\pi}\int_0^x \frac{\varphi'(s)}{\sqrt{x-s}}\, ds.$$

5 Discontinuous and Impulse forcing funcitons

The same idea of Laplace transform can be carried out to solve differential equations with discontinuous non-homogeneous terms.

Example: Consider the following initial value problem

$$y'' + 2y' + 5y = g(t) = \begin{cases} 5 & ; \quad 0 \le t < 1 \\ 0 & ; \quad t \ge 1. \end{cases}$$

with initial condition $y(0) = 0$, $y'(0) = 0$.

Solution: Let $Y(s) = \mathcal{L}\{y\}$. It follows that

$$\mathcal{L}y' = sY - y(0) = sY; \qquad \mathcal{L}y'' = s^2Y - sy(0) - y'(0) = s^2Y.$$

Therefore, we have

$$(s^2 + 2s + 5)Y = \mathcal{L}\{g\} = \int_0^\infty e^{-st} g(t)\, dt = \int_0^1 5e^{-st}\, dt = \frac{5}{s}\left(1 - e^{-s}\right),$$

which implies that

$$Y(s) = \frac{5}{s(s^2 + 2s + 5)}\left(1 - e^{-s}\right) = H(s) - H(s)e^{-s} \quad \Rightarrow \quad y(t) = \mathcal{L}^{-1}\{H(s)\} - \mathcal{L}^{-1}\left\{H(s)e^{-s}\right\}.$$

We first compute $\mathcal{L}^{-1}\{H\}$, which is straightforward since

$$H(s) = \frac{5}{s(s^2 + 2s + 5)} = \frac{1}{s} - \frac{s + 2}{s^2 + 2s + 5} = \frac{1}{s} - \frac{(s + 1) + 1}{(s + 1)^2 + 4}$$

or

$$h(t) = \mathcal{L}^{-1}\{H\} = 1 - e^{-t}\left(\cos 2t + \frac{1}{2}\sin 2t\right).$$

It remains to compute $\mathcal{L}^{-1}\left\{e^{-s}H\right\}$ (to be continued).

In general, suppose $\mathcal{L}\{f\} = F$, or

$$\int_0^\infty e^{-st} f(t)\, dt = F(s).$$

For any constant c, assume

$$g(t) = \begin{cases} 0 & ; \quad 0 \le t < c \\ f(t - c) & ; \quad t \ge c \end{cases},$$

in other words, g is a translation of f; see the following graph.

The Laplace transform of g is

$$\mathcal{L}\{g\} = \int_0^\infty e^{-st} g(t)\, dt = \int_c^\infty e^{-st} f(t-c)\, dt = e^{-cs} F(s).$$

Notation: From now on, we will denote

$$u_c(t) \triangleq \begin{cases} 0 & ; \quad 0 \le t < c \\ 1 & ; \quad t \ge c \end{cases}$$

Furthermore, it follows from the preceding discussion that

$$\mathcal{L}\left\{u_c(t) f(t-c)\right\} = e^{-cs} F(s), \quad \text{or} \quad \mathcal{L}^{-1}\left\{e^{-cs} F(s)\right\} = u_c(t) f(t-c).$$

Example (continued): We have

$$\mathcal{L}^{-1}\left\{e^{-s} H\right\} = u_1(t) h(t-1),$$

and the solution

$$y(t) = h(t) - u_1(t) h(t-1); \qquad t \ge 0.$$

Example: Find the Laplace inverse of the following functions

$$(1).\ \ \frac{e^{-cs}}{s}; \qquad (2).\ \ \frac{e^{-\pi s}}{s^2+1}$$

Solution: (1) The Laplace inverse is $u_c(t)$.

(2) The Laplace inverse is

$$u_\pi(t) \sin(t-\pi) = -u_\pi(t) \sin t.$$

Example: Solve the initial value problem

$$y'' + 2y' + 5y = 5 \sum_{n=0}^\infty (-1)^n u_n(t); \quad t \ge 0$$

with initial condition $y(0) = 0$, $y'(0) = 0$.

Solution: The function $\sum_{n=0}^\infty (-1)^n u_n(t)$ looks like

Letting $Y = \mathcal{L}\{y\}$, we have

$$(s^2 + 2s + 5) Y = 5 \sum_{n=0}^\infty (-1)^n \frac{e^{-ns}}{s}.$$

or

$$Y(s) = \sum_{n=0}^{\infty} (-1)^n e^{-ns} \cdot \frac{5}{s(s^2+2s+5)}$$

or

$$y(t) = \sum_{n=0}^{\infty} (-1)^n u_n(t) h(t-n).$$

Exercise: Solve the above equation with a different initial condition $y(0) = 4$, $y'(0) = 2$.

Hint: The solution is

$$y(t) = e^{-t}(4\cos 2t + 3\sin 2t) + \sum_{n=0}^{\infty} (-1)^n u_n(t) h(t-n).$$

Example: Find the Laplace inverse of the following function

$$F(s) = \frac{1}{s^2+1} \cdot \frac{1}{1-e^{-\pi s}}; \qquad s > 0$$

and graph the inverse function (roughly).

Solution: Using the formula

$$\frac{1}{1-a} = 1 + a + a^2 + a^3 + \cdots, \qquad \forall\ |a| < 1,$$

we have

$$\frac{1}{1-e^{-\pi s}} = 1 + e^{-\pi s} + e^{-2\pi s} + \cdots = \sum_{n=0}^{\infty} e^{-n\pi s}.$$

Since $\mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = \sin t$, we have

$$f(t) = \mathcal{L}^{-1}\{F\} = \sum_{n=0}^{\infty} u_{n\pi}(t) \sin(t - n\pi) = \sin t \cdot \sum_{n=0}^{\infty} (-1)^n u_{n\pi}(t)$$

Below are the figures.

5.1 Impulse functions

The impulse function is *not* a function in the usual sense – indeed, it is a *generalized derivative* of step funcitons.

One way to define the *unit impluse function* $\delta(t)$ is as the intuitive limit of a sequence of functions

$$\delta(t) \triangleq \lim_{\epsilon\to 0} f_\epsilon(t) := \lim_{\epsilon\to 0} \begin{cases} \frac{1}{2\epsilon} & ; \quad |t| < \epsilon \\ 0 & ; \quad |t| \geq \epsilon \end{cases} .$$

The function δ has the following property:

1. $\delta(t) = 0$ as $t \neq 0$.
2. $\int_{-\infty}^{\infty} \delta(t)\, dt = 1$.
3. $\int_{-\infty}^{\infty} \delta(t) g(t)\, dt = g(0)$ for all functions $g(t)$ continuous at $t = 0$.

These properties are straightforward; e.g.,

$$\int_{-\infty}^{\infty} \delta(t) g(t)\, dt = \lim_{\epsilon\to 0} \frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} g(t)\, dt = g(0)$$

Intuitively, the impulse function $\delta(t)$ can be understood as a force of a large magnitude acting for a very short amount of time around $t = 0$, hence the name "impulse". In general, one can define impulse function $\delta(t - t_0)$ (sometimes denoted by $\delta_{t_0}(t)$), which satisfies

1. $\delta(t - t_0) = 0$ as $t \neq t_0$.
2. $\int_{-\infty}^{\infty} \delta(t - t_0)\, dt = 1$.
3. $\int_{-\infty}^{\infty} \delta(t - t_0) g(t)\, dt = g(t_0)$ for all functions $g(t)$ continuous at $t = t_0$.

It is easy to see that the Laplace transform of $\delta(t - t_0)$, $t_0 \geq 0$ is

$$\mathcal{L}\left\{\delta(t - t_0)\right\} = \int_0^{\infty} e^{-st} \delta(t - t_0)\, dt = \int_{-\infty}^{\infty} e^{-st} \delta(t - t_0)\, dt = e^{-st_0}.$$

Remark: The other way to define the impulse function $\delta(t)$ is to understand it as the *generalized derivative* of the step function

$$h(t) \triangleq \begin{cases} 0 & ; \quad t < 0 \\ 1 & ; \quad t \geq 0 \end{cases} .$$

It roughly goes as follows. For any continuously differentiable function f, its derivative f' can be uniquely characterized by the following identity

$$\int_{-\infty}^{\infty} f(t) g'(t)\, dt = -\int_{-\infty}^{\infty} f'(t) g(t), dt; \qquad \forall\; g \in \mathcal{C}_0^{\infty}(\mathbb{R}).$$

Here $\mathcal{C}_0^{\infty}(\mathbb{R})$ is the set of all smooth functions with compact support (i.e., $g(t) = 0$ for all t outside some interval). This definition of derivatives can be easily extended to obtained the generalized derivative of $h(t)$. In other words, $h'(t)$ is defined so that

$$\int_{-\infty}^{\infty} h(t) g'(t)\, dt = -\int_{-\infty}^{\infty} h'(t) g(t), dt; \qquad \forall\; g \in \mathcal{C}_0^{\infty}(\mathbb{R}).$$

But
$$\int_{-\infty}^{\infty} h(t)g'(t)\,dt = \int_{0}^{\infty} g'(t)\,dt = -g(0) \quad\Rightarrow\quad \int_{-\infty}^{\infty} h'(t)g(t)\,dt = g(0).$$
for all $g \in \mathcal{C}_0^\infty(\mathbb{R})$. It is easy to see that $h'(t)$ also satisfies the property 3 above. Indeed, $h'(t)$ here is the true meaning of $\delta(t)$.

Example: Solve the IVP

$$y'' + 2y' + 5y = \delta(t-1); \qquad y(0) = 0,\ y'(0) = 0.$$

Solution: Let $Y = \mathcal{L}\{y\}$. We have

$$(s^2 + 2s + 5)Y = e^{-s} \qquad\Rightarrow\qquad Y = \frac{1}{(s+1)^2 + 4}e^{-s}$$

Hence
$$y(t) = u_1(t)f(t-1), \qquad \text{where}\ \ f(t) = \frac{1}{2}\sin 2t \cdot e^{-t}.$$

Remark: There is another way to find the Laplace inverse of function of form $e^{-cs}F(s)$. Indeed, by Faltung theorem,

$$\mathcal{L}^{-1}\left\{e^{-cs}F(s)\right\} = \mathcal{L}^{-1}\left\{e^{-cs}\right\} * \mathcal{L}^{-1}\{F\} = \delta_c * f$$

However,

$$(\delta_c * f)(t) = \int_0^t f(t-s)\delta_c(s)\,ds = \int_0^t f(t-s)\delta(s-c)\,ds = \left\{\begin{array}{ccc} 0 & ; & t < c \\ f(t-c) & ; & t \geq c \end{array}\right\} = u_c(t)f(t-c).$$