9.51) with Y a poisson random variable with parameter λ, it is necessary to find the MVUE for

$$E(C) = 3E(Y^2) = 3(V(Y) + [E(Y)]^2) = 3(\lambda + \lambda^2)$$

In Exercise 9.31 it was determined that $\sum_{i=1}^{n} Y_i$ is sufficient for λ and thus for λ^2 and $3(\lambda+\lambda^2)$.
If a function of $\sum_{i=1}^{n} Y_i$ that is unbiased for $3(\lambda+\lambda^2)$ can be found, then this function will be MVUE. Note that as $\bar{Y}$ is distributed as poisson$(n\lambda)$ we can easily calculate

$$E(\bar{Y}^2) = V(\bar{Y}) + [E(\bar{Y})]^2 = \frac{\lambda}{n} + \lambda^2$$

and $$E\left(\frac{\bar{Y}}{n}\right) = \frac{1}{n}E(\bar{Y}) = \frac{\lambda}{n}$$

Then we have $\lambda^2 = E(\bar{Y}^2) - E\left(\frac{\bar{Y}}{n}\right)$
and $\lambda = E(\bar{Y})$.

$E(C) = 3E\left[\bar{Y}^2 - \left(\frac{\bar{Y}}{n}\right) + \bar{Y}\right]$ and the MVUE is
$3\left[\bar{Y}^2 + \bar{Y}\left(1 - \frac{1}{n}\right)\right]$.

9.55) a. First note that the distribution function corresponding to $f(y)$ is
$$F(y) = \begin{cases} 1 & \text{if } y > \theta \\ y^3/\theta^3 & \text{for } 0 \le y \le \theta \\ 0 & \text{for } y < 0 \end{cases}$$

Then the density of $Y_{(n)}$ is $f_{Y_{(n)}}(y) = n f(y)(F(y))^{n-1}$

$$= \begin{cases} n\,3\frac{y^2}{\theta^3}\left(\frac{y^3}{\theta^3}\right)^{n-1} = 3ny^{3n-1}/\theta^{3n} & \text{for } 0 \le y \le \theta \\ 0 & \text{otherwise} \end{cases}$$

b. We know that the UMVUE will be based on $Y_{(n)}$ as it is a complete sufficient statistic. All we need to do is properly scale it so that it is unbiased. Note

$$E(Y_{(n)}) = \int_0^\theta 3ny^{3n}/\theta^{3n}\,dy = \frac{3ny^{3n+1}}{(3n+1)\theta^{3n}}\Bigg|_0^\theta = \frac{3n}{3n+1}\theta$$

Thus $\frac{3n+1}{3n} Y_{(n)}$ is unbiased for θ. It is then UMVUE as $Y_{(n)}$ is complete sufficient.

9.72) a. The likelihood function is

$$L = \prod_{i=1}^{n} \frac{\lambda^{y_i} e^{-\lambda}}{y!} = \frac{\lambda^{\Sigma y_i} e^{-n\lambda}}{\prod_{i=1}^{n} y_i!}$$

and $\ln L = (\Sigma y_i)\ln\lambda - n\lambda - \Sigma \ln y_i!$

So that $(\frac{d}{d\lambda})[\ln L] = (\Sigma \frac{y_i}{\lambda}) - n$. Equating the derivative to 0, we obtain $\frac{\Sigma y_i}{\hat{\lambda}} - n = 0$

or $\hat{\lambda} = \frac{\Sigma Y_i}{n} = \bar{Y}$.

b. Recalling that $E(Y_i) = \lambda$ and $V(Y_i) = \lambda$, we obtain

$$E(\hat{\lambda}) = \frac{\sum_{i=1}^{n} E(Y_i)}{n} = \lambda$$

and

$$V(\hat{\lambda}) = \frac{\sum_{i=1}^{n} V(Y_i)}{n^2} = \frac{\lambda}{n}$$

c. Since $E(Y_i) = \lambda$ and $V(Y_i) = \lambda < \infty$, the law of large numbers applies and we conclude that $\hat{\lambda}$ converges in probability to λ. Hence $\hat{\lambda}$ is consistent for λ.

d. The MLE of λ was found in part a. to be $\hat{\lambda} = \bar{Y}$. Then, the MLE for $e^{-\lambda}$ is $e^{-\bar{Y}}$.

9.74) a. The likelihood function is

$$L = \prod_{i=1}^{n} \frac{r}{\theta} y_i^{r-1} e^{-y_i^r/\theta} = \frac{r^n}{\theta^n} \prod_{i=1}^{n} y^{r-1} e^{-\Sigma y_i^r/\theta}$$

$$= g(u,\theta)\, h(y_1, y_2, \cdots, y_n)$$

where

$$u = \sum_{i=1}^{n} y_i^r \qquad g(u,\theta) = \frac{r^n}{\theta^n} e^{-u/\theta}, \quad h(y_1, \cdots, y_n) = \prod_{i=1}^{n} y_i^{r-1}$$

Hence ΣY_i^r is a sufficient statistic for θ.

b. Consider $\ln L = n \ln r - n \ln \theta + (r-1) \sum_{i=1}^{n} \ln y_i - \sum_{i=1}^{n} \frac{y_i^r}{\theta}$

and $\frac{d}{d\theta} \ln L = \frac{-n}{\theta} + \frac{\Sigma y_i^r}{\theta^2}$

Equating the derivative to 0, the estimator is obtained.

$$\frac{-n}{\hat{\theta}} + \frac{\Sigma y_i^r}{\hat{\theta}^2} = 0 \quad \text{or} \quad -n\hat{\theta} + \Sigma y_i^r = 0$$

or $\hat{\theta} = \frac{\Sigma Y_i^r}{n}$.

c. The estimator $\hat{\theta}$ given in part b. is a function of the sufficient statistic. If it is unbiased, or could be adjusted to be unbiased, the MVUE of θ will be obtained.

Consider

$$E(Y_i^r) = \int_0^\infty \frac{r}{\theta} y^{2r-1} e^{-y^r/\theta}\, dy$$

let $x = y^r$; $dx = r y^{r-1} dy$, so that $E(Y_i^r) = \int_0^\infty (\frac{x}{\theta}) e^{-x/\theta} dx$
$= E(X)$

where X ~~was~~ has a gamma distribution with $\alpha = 1$, $\beta = \theta$

Then $E(Y_i^r) = \theta$ and

$$E(\hat{\theta}) = \frac{\sum_{i=1}^{n} E(Y_i^r)}{n} = \theta$$

Since $\hat{\theta}$ is unbiased for θ, it is the MVUE for θ.

9.79) Let p_1, p_2, p_3 be the proportions of voters in the population favoring candidates A, B, and C, respectively. Further, define the random variables n_1, n_2 and n_3 as the number of voters in a random sample of size n who favor candidates A, B, and C, respectively.

Note that

$$\sum_{i=1}^{3} p_i = 1 \quad \text{and} \quad \sum_{i=1}^{3} n_i = n$$

So that we may write $p_3 = 1 - p_1 - p_2$ and $n_3 = n - n_1 - n_2$, the random variables n_1, n_2, n_3 follow a multinomial probability distribution (see section 5.9 of the book), and the likelihood function is

$$L = \frac{n!}{n_1!n_2!n_3!} p_1^{n_1} p_2^{n_2} (1-p_1-p_2)^{n_3}$$

so that $\ln L = \ln k + n_1 \ln p_1 + n_2 \ln p_2 + n_3 \ln(1-p_1-p_2)$

Differentiating with respect to p_1 and p_2, we have

$$\frac{d\ln L}{dp_1} = \frac{n_1}{p_1} - \frac{n_3}{1-p_1-p_2} \quad \text{and} \quad \frac{d\ln L}{dp_2} = \frac{n_2}{p_2} - \frac{n_3}{1-p_1-p_2}$$

Set these two equations equal to 0 and solve simultaneously for $\hat{p}_1$ and $\hat{p}_2$.

$$(*) \quad n_1(1-\hat{p}_1-\hat{p}_2) - n_3\hat{p}_1 = 0$$

$$n_2(1-\hat{p}_1-\hat{p}_2) - n_3\hat{p}_2 = 0$$

Adding the two equations, we have

$$(n_1+n_2)(1-\hat{p}_1-\hat{p}_2) = (\hat{p}_1+\hat{p}_2)n_3 \quad \text{or} \quad \hat{p}_1+\hat{p}_2 = \frac{n_1+n_2}{n}$$

$$\Rightarrow \quad \hat{p}_1 = \frac{n_1}{n}, \quad \hat{p}_2 = \frac{n_2}{n} \quad \text{and thus } \hat{p}_3 = \frac{n_3}{n}$$

for the data given, $\hat{p}_1 = 0.30$ $\hat{p}_2 = 0.38$, $\hat{p}_3 = 0.32$.

9.81). $P(Y=y) = \binom{2}{y} p^y (1-p)^{2-y}$ Our estimator $\hat{p}$ must be either $1/4$ or $3/4$. We choose based on which has the larger likelihood value given the data, Y. It is important to remember in this problem that the likelihood is a function of the parameter p. Therefore we have three possible likelihood functions depending, one for each value of the data, Y.

$L(0, p) = P(Y=0) = (1-p)^2$ implying $\hat{p} = \frac{1}{4}$ as
$L(0, \frac{1}{4}) = (1-\frac{1}{4})^2 > (1-\frac{3}{4})^2 = L(0, \frac{3}{4})$

$L(1, p) = P(Y=1) = 2p(1-p)$ implying $\hat{p}$ can be either $\frac{1}{4}$ or $\frac{3}{4}$ as

$$L(1, \tfrac{1}{4}) = 2 * \tfrac{1}{4}(1-\tfrac{1}{4}) = 2 * \tfrac{3}{4}(1-\tfrac{3}{4}) = L(1, \tfrac{3}{4})$$

$L(2, p) = P(Y=2) = p^2$ implying

$\hat{p} = \frac{3}{4}$ as $(\frac{1}{4})^2 < (\frac{3}{4})^2$

Notice the case when $Y=1$ is an instance where the maximum likelihood estimator is not a single unique value!

#

10.2) The test statistic Y has a binomial distribution with $n=20$ and p.

a. A Type I error occurs if the experimenter concluded that the drug dosage level induces sleep in less than 80% of the people suffering from insomnia when, in fact, drug dosage level does induce sleep in 80% of insomniacs.

b. $\alpha = P(\text{reject } H_0 \mid H_0 \text{ true}) = P(Y \le 12 \mid p = 0.8)$
$= 0.032$ using Table 1, Appendix III.

c. A type II error would occur if the experimenter concluded that the drug dosage level induces sleep in 80% of the people suffering from insomnia when, in fact, fewer than 80% experience relief.

d. If $p = 0.6$,
$$\beta = P(\text{accept } H_0 \mid H_0 \text{ false}) = P(Y > 12 \mid p = 0.6)$$
$$= 1 - P(Y \le 12 \mid p = 0.6) = 1 - 0.584 = 0.416$$

e. If $p = 0.4$ then
$$\beta = P(Y > 12 \mid p = 0.4) = 1 - P(Y \le 12 \mid p = 0.4)$$
$$= 1 - 0.979 = 0.021$$

10.4) a. A Type I error occurs if we conclude that the proportion of ledger sheets with errors is larger than 0.05 when, in fact, the proportion is 0.05.

b. By the scheme being used, we will reject for the following situations:
(NOTE: NE = no error, E = error)

sheet 1	sheet 2	sheet 3
NE	NE	.
NE	E	NE
E	NE	NE
E	E	NE

Thus $\alpha = (0.95)^2 + 2(0.05)(0.95)^2 + (0.05)^2(0.95)$

$= 0.995125.$

c. A Type II error occurs if we conclude that the proportion of ledger sheets with errors is 0.05 when in fact the proportion is larger than 0.05.

d. $\beta = P(\text{accept } H_0 \text{ when } H_a \text{ is true}) = P(\text{accepting } H_0 \mid p = p_a) = 2p_a^2(1-p_a) + p_a^3$. Since we reject if we observe E, E, E or NE, E, E or E, NE, E.