10.7) a. Since it is necessary to test a claim that the average amount saved, μ is \$900, the hypothesis to be tested is two-tailed:

$$H_0: \mu = 900 \quad \text{vs.} \quad H_a: \mu \neq 900.$$

b. The rejection region with $\alpha = 0.01$ is determined by a critical value of z such that $P[|Z| > z_0] = 0.01$. This value is $z_0 = 2.58$ and the rejection region is $|z| > 2.58$.

c. The test statistic is

$$z = \frac{\bar{x} - \mu}{\frac{\sigma}{\sqrt{n}}} \doteq \frac{\bar{x} - \mu}{\frac{s}{\sqrt{n}}} = \frac{885 - 900}{\frac{50}{\sqrt{35}}} = -1.77$$

d. The observed value $z = -1.77$ does not fall in the rejection region, and H_0 is not rejected. We cannot conclude that the average savings is different than claimed.

10.11) $H_0: \mu_1-\mu_2=0$ $\quad H_a: \mu_2-\mu_1 \neq 0$ The test statistic and rejection region are

$$z = \frac{1.65-1.43}{\sqrt{\frac{(0.26)^2}{30}+\frac{(0.22)^2}{35}}} = 3.65$$

RR: Reject H_0 if $|z| > 2.575$

Conclusion: Reject H_0 at $\alpha=0.01$. The soils do appear to differ with respect to average shear strength, at the 1% significance level.

10.14) a. define p as the proportion of college students aged 30 years or more, then we test

$H_0: p=0.25$ vs. $H_a: p \neq 0.25$

The test statistic is

$$z = \frac{\hat{p}-p_0}{\sqrt{\frac{p_0q_0}{n}}} = \frac{\frac{98}{300}-0.25}{\sqrt{\frac{(0.25)(0.75)}{300}}} = 3.07$$

and the rejection region, with $\alpha=0.05$ is $|z|>1.96$. H_0 is rejected and we conclude that the 25% figure is not accurate.

b. Yes. the results do give evidence that the columnist's claim is too low.

10.23 Two binomial populations are involved.

To test $H_0: p_1 = p_2$ VS. $H_a: p_1 > p_2$ (one-tail)

The test statistic $Z = \dfrac{(\hat{p}_1 - \hat{p}_2) - (p_1 - p_2)}{\sqrt{\frac{p_1 q_1}{n_1} + \frac{p_2 q_2}{n_2}}}$

To evaluate the denominator, we must estimate p_1, p_2.

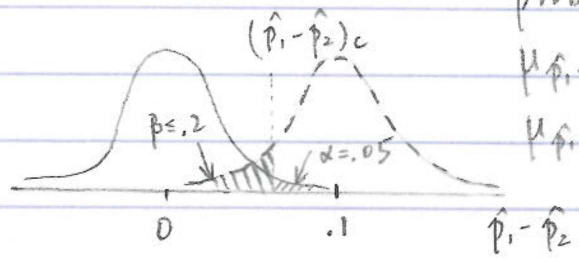
Under H_0: $p_1 = p_2$, the best estimate for this common value is $\hat{p} = \dfrac{y_1 + y_2}{n_1 + n_2} = \dfrac{46 + 34}{200 + 200} = 0.2$

$$\Rightarrow Z = \frac{\hat{p}_1 - \hat{p}_2 - 0}{\sqrt{\hat{p}\hat{q}\left(\frac{1}{n} + \frac{1}{n}\right)}} = \frac{\frac{46}{200} - \frac{34}{200}}{\sqrt{(0.2)(0.8)\left(\frac{1}{100}\right)}} = 1.5$$

Rejection region: under $\alpha = .05$, H_0 will be rejected if $Z > 1.645$ (one-tail)

So, we fail to reject H_0. There is insufficient evidence to support the researcher's belief

10.30



The left figure represents the two probability distribution, one assuming $\mu_{\hat{p}_1 - \hat{p}_2} = p_1 - p_2 = 0$ & one assuming $\mu_{\hat{p}_1 - \hat{p}_2} = p_1 - p_2 = .1$.

The right curve is the true distribution of $\hat{p}_1 - \hat{p}_2$ & thus any probabilities we wish to calculate concerning the random variable should be calculated as areas under the curve to the right.

We need to find a common sample sise such that

$\alpha = P(\text{reject } H_0 \mid H_0 \text{ true}) = .05$ & $\beta = P(\text{accept } H_0 \mid H_0 \text{ false}) \le .2$

For $\alpha = .05$, the critical value $(p_1 - p_2)_c$ is 1.645 (recall Ex 10.23), then $1.645 = (\hat{p}_1 - \hat{p}_2)_c - 0 \Big/ \sqrt{\frac{p_1 q_1}{n_1} + \frac{p_2 q_2}{n_2}}$

For β = the area under the curve to the right from $-\infty$ to $(p_1-p_2)c$, since $\beta = .2$, we have z value $= -.84$.

$$\Rightarrow \quad -.84 = \frac{(\hat{p}_1-\hat{p}_2)c - .1}{\sqrt{\frac{p_1 q_1}{n_1}+\frac{p_2 q_2}{n_2}}} \qquad (2)$$

Combine (1) & (2), $\quad 2.485 = \dfrac{1}{\sqrt{\frac{p_1 q_1}{n_1}+\frac{p_2 q_2}{n_2}}}$

Note: (1) $n_1 = n_2 = n$

(2) max value of $pq = p(1-p)$ is $.25$ where $p=.5$. Since p_1 & p_2 are unknown, the use of $p=.5$ will provide a valid (although may be larger than necessary) sample size

Then, $\quad 2.485 = \dfrac{1}{\sqrt{(.5)(.5)\left(\frac{1}{n}+\frac{1}{n}\right)}} \quad \Rightarrow \sqrt{n} = 17.57$

$\Rightarrow n = 308.76$

So, the common sample size for the researcher's test should be 309.

10.33 (a) Let μ_1 be the average manual dexterity score for those that participated in sports & μ_2 for those that did not. Then $H_0: \mu_1 = \mu_2$ VS $H_a: \mu_1 > \mu_2$. Test statistic is

$$z = \frac{(\bar{y}_1-\bar{y}_2)-0}{\sqrt{\frac{s_1^2}{n_1}+\frac{s_2^2}{n_2}}} = \frac{32.19-31.68}{\sqrt{\frac{(4.34)^2+(4.56)^2}{37}}} = .49$$

The RR with $\alpha = .05$ is $z > 1.645$. So, H_0 is not rejected. There is insufficient evidence to indicate $\mu_1 > \mu_2$

(b) The RR, written in terms of $\bar{Y}_1 - \bar{Y}_2$, is

$$Z = \frac{\bar{Y}_1 - \bar{Y}_2}{\hat{\sigma}_{\bar{Y}_1 - \bar{Y}_2}} > 1.645$$

or $\bar{Y}_1 - \bar{Y}_2 > 1.645 \sqrt{\frac{(4.34)^2 + (4.56)^2}{37}} = 1.702$

Then $\beta = P(\text{accept } H_0 \mid \mu_1 - \mu_2 = 3) = P(\bar{Y}_1 - \bar{Y}_2 < 1.702 \mid \mu_1 - \mu_2 = 3)$

$$= P\left(Z < \frac{1.702 - 3}{\hat{\sigma}_{\bar{Y}_1 - \bar{Y}_2}}\right) = P(Z < -1.25) = .1056$$

10.34 Using the procedure discussed following Example 10.8, we can write $\alpha = P(\bar{Y}_1 - \bar{Y}_2 > k \text{ when } \mu_1 - \mu_2 = 0) = P\left(Z > \frac{k - 0}{\sqrt{\frac{\sigma_1^2 + \sigma_2^2}{n}}}\right)$

$$\Rightarrow Z_\alpha = \frac{k\sqrt{n}}{\sqrt{\sigma_1^2 + \sigma_2^2}} \qquad ①$$

Also $\beta = P(\bar{Y}_1 - \bar{Y}_2 \le k \mid \mu_1 - \mu_2 = 3) = P\left(Z \le \frac{k - 3}{\sqrt{\frac{\sigma_1^2 + \sigma_2^2}{n}}}\right)$

$$\Rightarrow -Z_\beta = \frac{(k-3)\sqrt{n}}{\sqrt{\sigma_1^2 + \sigma_2^2}} \qquad ②$$

Combine ① & ② to eliminate k, we get

$$Z_\alpha \sqrt{\frac{\sigma_1^2 + \sigma_2^2}{n}} = 3 - Z_\beta \sqrt{\frac{\sigma_1^2 + \sigma_2^2}{n}}$$

Solving for n, we have $n = \frac{[2(1.645)]^2 [(4.34)^2 + (4.56)^2]}{3^2} = 47.66$

or $n = 48$ to provide the given levels of α and β.

10.42 (a) Let p_1 & p_2 be the proportions (attending vs. not attending) who were using safety seats 4 to 6 weeks after birth.

From the study, $n_1 = 78$, $\hat{p}_1 = .96$; $n_2 = 136 - 78 = 58$, $\hat{p}_2 = .78$

Then $\hat{p} = \frac{78(.96) + 58(.78)}{136} = .883$

The hypothesis to be tested is $H_0: p_1 = p_2$ vs $H_a: p_1 > p_2$

The test statistic $z = \dfrac{\hat{p}_1 - \hat{p}_2}{\sqrt{\hat{p}\hat{q}\left(\frac{1}{n_1} + \frac{1}{n_2}\right)}} = \dfrac{.96 - .78}{\sqrt{(.883)(.117)\left(\frac{1}{78} + \frac{1}{58}\right)}}$

$= 3.23 > 1.645$

So, reject H_0. There is evidence that the lecture is effective

1b) $p\text{-value} = P(z > 3.23) < 0.00135$

Homework 10, Part 3

(46) Does the data indicate that the use of Vitamin C reduces mean time to recover? Find P value. Do we reject at $\alpha = .05$?

	No Vit C (1)	Vit C (2)
n	35	35
μ	6.9	5.8
σ	2.9	1.2

Null: $H_0: \mu_1 = \mu_2$ or $\mu_1 - \mu_2 = 0$

$H_A: \mu_1 > \mu_2$

Test Statistic: $z = \dfrac{(6.9-5.8)-(0)}{\sqrt{\frac{1}{35}\left((2.9)^2+(1.2)^2\right)}} = 2.074$

$$\frac{(\mu_1-\mu_2)-0}{\sqrt{\frac{\sigma_1^2}{n_1}+\frac{\sigma_2^2}{n_2}}}$$

P-Value: $P(z > 2.074) = .0192$

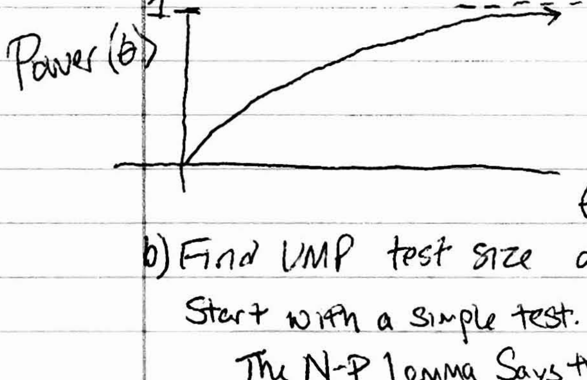
So at $\alpha = .05$ We reject the null Hypothesis which means the data supports Vitamin C reducing recovery time.

(84) Suppose $f_{Y|\theta} = \begin{cases} \theta y^{\theta-1} & 0 \le y \le 1 \\ 0 & \text{elsewhere} \end{cases}$ where $\theta > 0$

a) Sketch the Power function RR $Y > .5$

defn: $\text{Power}(\theta) = \Pr(W \text{ in } RR \mid H_a: \theta)$ where W is your test Statistic $(= Y)$

$$\int_{.5}^{1} \theta y^{\theta-1}\,dy = y^{\theta}\Big|_{.5}^{1} = 1-(.5)^{\theta}$$



b) Find UMP test size α for $H_0: \theta_0 = 1$ vs $H_a: \theta > 1$

Start with a simple test. Let $\theta_a > 1$.

The N-P Lemma Says the RR for a level α test that maximizes power at θ_a is determined by:

$$\frac{L(\theta_0)}{L(\theta_a)} \le K$$

K chosen to give you α

In our case $L(\theta_0 = 1) = 1$

$L(\theta_a) = \theta_a\, y^{\theta_a - 1}$

so RR has the form $\frac{1}{\theta_a y^{\theta_a - 1}} < K$ or $\frac{1}{K\theta_a} < y^{\theta_a - 1}$

equivalently $\left(\frac{1}{K\theta_a}\right)^{\frac{1}{\theta_a - 1}} < Y$

B/c θ_a is a known constant (we fixed it by assumption), the LHS of the inequality is a constant, call it K^*

So M.P. test of $H: \theta = 1$ vs $A: \theta = \theta_a$ has RR $\{Y > K^*\}$ where the value of K^* is determined by α

$$\alpha = P(Y \text{ in } RR \mid H_0: \theta_0 = 1) = P(Y > K^* \mid \theta_0 = 1)$$
$$= \int_{K^*}^{1} 1\, dy = 1 - K^*$$

so $K^* = 1 - \alpha$

Test statistic Y and RR $\{Y > K^*\}$ of the Level α test do not depend on a particular ~~Level~~ value of θ_a so long as it is larger than 1

i.e. Any value of θ_a Leads to the same RR (change the original K)

Thus this test is UMP Level α for $H: \theta = 1$ vs $A: \theta > 1$

(86) Let $Y_1 \dots Y_n \sim f_{Y|\theta} = \begin{cases} \frac{1}{\theta} m y^{m-1} e^{-y^m/\theta} & y > 0 \\ 0 & \text{elsewhere} \end{cases}$

where m is a known constant

a) Find UMP test $H_0: \theta = \theta_0$ vs $H_A: \theta > \theta_0$

Let $\theta_a > \theta_0$, start with a simple test so N-P applies

$$L(\theta_0) = \prod \frac{1}{\theta_0} m y^{m-1} e^{-y^m/\theta_0}$$
$$= \left(\frac{1}{\theta_0}\right)^n \left(\prod m y^{m-1}\right) e^{-\frac{1}{\theta_0}\sum_i y_i^m}$$

Similarly for $L(\theta_a)$

so

$$\frac{L(\theta_0)}{L(\theta_a)} = \frac{\left(\frac{1}{\theta_0}\right)^n e^{-\frac{1}{\theta_0}\sum_i y_i^m}}{\left(\frac{1}{\theta_a}\right)^n e^{-\frac{1}{\theta_a}\sum_i y_i^m}} = \left(\frac{\theta_a}{\theta_0}\right)^n e^{\underbrace{\left(\frac{1}{\theta_a} - \frac{1}{\theta_0}\right)}_{\text{Negative Quantity}}\underbrace{\left(\sum Y_i^m\right)}_{\text{sufficient statistic}}} < K$$

Take Logs and solve for $\sum_i Y_i^m$ $\quad \left(\frac{1}{\theta_a} - \frac{1}{\theta_0}\right)\left(\sum_i Y_i^m\right) < \ln\left(K\left(\frac{\theta_0}{\theta_a}\right)^n\right)$

$$\Rightarrow \sum_i Y_i^m > \frac{\ln\left(K \cdot \left(\frac{\theta_0}{\theta_a}\right)^n\right)}{\frac{1}{\theta_a} - \frac{1}{\theta_0}}$$

sign flipped b/c divide by negative number.

Call the RHS K^*, a constant.

<u>Note</u>: I would expect you all could get here... past this point, I am more or less copying the answer booklet. It's a distribution you were not told to know...

Consider the distribution of $Z = Y^m$ since $\frac{dz}{dY} = mY^{m-1}$, if H_0 is true

$g(z) = \frac{1}{\theta_0} e^{-z/\theta_0}$ for $z > 0$

That is Y^m has a gamma distribution $\beta = \theta_0$ $\alpha = 1$

So $\frac{2Y^m}{\theta_0}$ has χ^2 dist. with 2 degrees of freedom

$(\alpha = \nu/2,\ \alpha = 1$ so $\nu = 2 =$ d.f.$)$

Also $\frac{2\sum Y_i^m}{\theta_0}$ is χ^2, $2n$ d.f.

Thus the critical region $\sum Y_i^m > K^* \Rightarrow \frac{2\sum Y_i^m}{\theta_0} > \frac{2K^*}{\theta_0} = K^{**}$

where K^{**} chosen so test has size α

Notice the critical region does not depend on θ_a but only that $\theta_a > \theta_0$

(why? so that $\frac{1}{\theta_a} - \frac{1}{\theta_0}$ is negative, look back to where we used this)

So the same region holds for all $\theta_a > \theta_0$

Thus this test is UMP level α

b) $\theta_0 = 100$ $\alpha = \beta = .05$ $\theta_a = 400$ Find appropriate sample size and critical region

if H_0 true $\frac{2\sum Y_i^m}{\theta_0}$ has χ^2 dist $2n$ d.f.

Statistic to calculate cutoff point for RR

100

so $\alpha = P_r\left(\frac{2\sum Y_i^m}{100} \geq \chi^2_{.05, 2n}\right) = .05$

Cutoff point for RR

Prob. of statistic in RR

$\chi^2_{a,b}$ $0 \leq a \leq 1$ and $P_r(X > \chi^2_{a,b}) = a$

if H_a true $\frac{2\sum Y_i^m}{400}$ has χ^2 dist $2n$ d.f.

$.05 = \beta = P\left(\frac{2\sum Y_i^m}{400} \leq \chi^2_{.95, 2n}\right) = P\left(\frac{1}{4}\frac{2\sum Y_i^m}{400} \leq \cancel{\tfrac{1}{4}}\chi^2_{.95, 2n}\right)$

$= P\left(\frac{2\sum Y_i^m}{100} \leq 4\chi^2_{.95, 2n}\right)$

Since cutoffs for RR are equal $\Rightarrow 4\chi^2_{.95, 2n} = \chi^2_{.05, 2n}$

so now find an n that satisfies $P(\chi^2 \le 4\chi^2_{.95,2n}) = .05$

(We defined the cutoff for our RR as $\chi^2_{.05,2n}$, equivalently as $4\chi^2_{.95,2n}$)

or $\frac{1}{4}\chi^2_{.05,2n} = \chi^2_{.95,2n}$

You get $2n = 12$ or $n = 6$

(Use tables in Back of Book)