Chapters 6. Functions of Random Variables

Question: Given a collection of random variables $X_1, X_2, \ldots, X_d$ and a function h, what is the distribution of the random variable

$$Y \doteq h(X_1, X_2, \ldots, X_d).$$

The method of distribution function:

1. When Y is continuous, compute $F(t) = P(Y \leq t)$ or $1 - F(t) = P(Y > t)$ for any t, and then taking derivatives of F to obtain the density of Y.
2. When Y is discrete, compute $P(Y = y_i)$ for each i.

The method of distribution function

1. Suppose X is uniform on $(0, 1)$.

   (a) What is the distribution of $-\ln(X)$?

   (b) What is the density of $\sqrt{X}$?

2.(a) Let X and Y be indepedent geometric random variables with rates λ and μ respectively. What is the distribution of $\min\{X, Y\}$?

(b) Same question, with X and Y be indepedent exponential random variables with rates λ and μ respectively.

3. Suppose X and Y are independent, identically distributed exponential random variables with rate λ. What is the distribution of $X/(X+Y)$?

4. Suppose X has cdf F and density f. Find the cdf and density for $aX + b$ where $a > 0$.

5. Suppose X is a continuous random variable with cumulative distribution function F. What is the distribution of $F(X)$?

Another approach is to use the moment generating function, and use the fact that moment generating function uniquely determines the distribution!

The moment generating function for $N(\mu, \sigma^2)$ is

$$M(t) = \exp\left\{\mu t + \frac{1}{2}\sigma^2 t^2\right\}$$

Proposition.

- X is $N(\mu, \sigma^2)$. Then

$$aX + b \text{ is } N(a\mu + b, a^2\sigma^2).$$

- Suppose X is $N(\mu, \sigma^2)$ and Y is $N(\nu, \theta^2)$. Assume X and Y are independent. Then

$$aX + bY \text{ is } N(a\mu + b\nu, a^2\sigma^2 + b^2\theta^2).$$

(This example is advanced). Suppose $Y_1, Y_2, \ldots$ is a sequence of iid (independent, identically ditributed) random variables such that

$$P(Y_i = 0) = P(Y_i = 1) = \frac{1}{2}.$$

Let

$$X = \sum_{i=1}^{\infty} \frac{Y_i}{2^i}.$$

What is the distribution of X?

Order Statistics

Let $X_1, X_2, \ldots, X_n$ be iid continuous random variables with common density $f(x)$. The order-statistics is denoted by $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$, where

$$X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(n)}$$

is the re-ordering of $\{X_1, X_2, \ldots, X_n\}$ in an increasing order.

1. What are the cdf and density for $X_{(1)}$? $X_{(n)}$?

2. Suppose $X_1, X_2, \ldots, X_n$ are iid uniform random variables on $[0, \theta]$. Find

$$E[X_{(n)}].$$

The joint density of order statistics

$$f_{(1)(2)\cdots(n)}(x_1, x_2, \ldots, x_n) = \begin{cases} n!f(x_1)f(x_2)\ldots f(x_n) & , \quad x_1 \leq x_2 \leq \cdots \leq x_n \\ 0 & , \quad \text{otherwise} \end{cases}$$