Chapters 3∼4. A Few Topics

Moment Generating Functions

Given a random variable X, its moment generating function $M(t)$ is defined as

$$M(t) \doteq E\left[e^{tX}\right].$$

Remark: It is possible that moment generating function takes infinite value for all $t \neq 0$. Say, Cauchy distribution with density

$$f(x) = \frac{1}{\pi}\frac{1}{1+x^2}, \qquad x \in \mathbb{R}.$$

Assumption: We always assume moment generating function exists in the sense that there is a number $b > 0$ such that $M(t)$ is finite for all $|t| < b$.

Why the name?

Use expansion

$$\exp\{x\} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots$$

We have

$$M(t) = E\left[1 + tX + \frac{t^2X^2}{2!} + \cdots + \frac{t^nX^n}{n!} + \cdots\right],$$

or

$$M(t) = 1 + tE[X] + \frac{t^2}{2!}E[X^2] + \cdots + \frac{t^n}{n!}E[X^n] + \cdots$$

Remark: $E[X^n]$ is called the n-th moment. Moreover,

$$\left.\frac{d^n M(t)}{dt^n}\right|_{t=0} = E[X^n]$$

Examples

- Discrete random variables: $q = 1 - p$.

Distribution	$M(t)$
Binomial $B(n; p)$	$\left[pe^t + q\right]^n$
Geometric with probability of success p	$\dfrac{pe^t}{1 - qe^t}$
Possion with parameter λ	$e^{\lambda(e^t - 1)}$

- Continuous random variables:

Distribution	$M(t)$
Uniform $[0, 1]$	$\dfrac{e^t - 1}{t}$
Exponential with rate λ	$\dfrac{\lambda}{\lambda - t}, \quad t < \lambda$
Normal $N(\mu, \sigma^2)$	$\exp\left\{\mu t + \dfrac{t^2\sigma^2}{2}\right\}$

Theorem: Moment generating function uniquely determines the distribution.

Proof. "Probability and Measure" by Patrick Billingsley, Theorem 22.2.

Example

1. Determine the distribution.

(a) $M(t) = [0.7e^t + 0.3]^4$.

(b) $M(t) = \dfrac{2e^t}{5 - 3e^t}$.

(c) $M(t) = \dfrac{1}{1 - 2t}$.

(d) $M(t) = 0.2e^{-t} + 0.3e^{2t} + 0.5$.

2. (This example is advanced). Suppose $Y_1, Y_2, \ldots$ is a sequence of iid (independent, identically ditributed) random variables such that

$$P(Y_i = 0) = P(Y_i = 1) = \frac{1}{2}.$$

Let

$$X = \sum_{i=1}^{\infty} \frac{Y_i}{2^i}.$$

What is the distribution of X?