Chapters 3~4. A Few Topics
**Moment Generating Functions**

Given a random variable $X$, its moment generating function $M(t)$ is defined as

$$M(t) = E \left[ e^{tX} \right].$$

**Remark:** It is possible that moment generating function takes infinite value for all $t \neq 0$. Say, Cauchy distribution with density

$$f(x) = \frac{1}{\pi} \frac{1}{1 + x^2}, \quad x \in \mathbb{R}.$$  

**Assumption:** We always assume moment generating function exists in the sense that there is a number $b > 0$ such that $M(t)$ is finite for all $|t| < b$. 
WHY THE NAME?

Use expansion
\[
\exp\{x\} = 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!} + \cdots
\]

We have
\[
M(t) = E \left[ 1 + tX + \frac{t^2 X^2}{2!} + \cdots + \frac{t^n X^n}{n!} + \cdots \right],
\]
or
\[
M(t) = 1 + tE[X] + \frac{t^2}{2!}E[X^2] + \cdots + \frac{t^n}{n!}E[X^n] + \cdots
\]

Remark: \(E[X^n]\) is called the \(n\)-th moment. Moreover,
\[
\left. \frac{d^n M(t)}{dt^n} \right|_{t=0} = E[X^n]
\]
Examples

- Discrete random variables: \( q = 1 - p \).

<table>
<thead>
<tr>
<th>Distribution</th>
<th>( M(t) )</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Binomial</strong> ( B(n; p) )</td>
<td>([pe^t + q]^n)</td>
</tr>
<tr>
<td><strong>Geometric</strong> with probability of success ( p )</td>
<td>(\frac{pe^t}{1 - qe^t})</td>
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<tr>
<td><strong>Possion</strong> with parameter ( \lambda )</td>
<td>(e^{\lambda(e^t - 1)})</td>
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</table>
- Continuous random variables:

<table>
<thead>
<tr>
<th>Distribution</th>
<th>$M(t)$</th>
</tr>
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<tbody>
<tr>
<td><strong>Uniform</strong> [0, 1]</td>
<td>$\frac{e^t - 1}{t}$</td>
</tr>
<tr>
<td><strong>Exponential</strong> with rate $\lambda$</td>
<td>$\frac{\lambda}{\lambda - t}, \ t &lt; \lambda$</td>
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<tr>
<td><strong>Normal</strong> $N(\mu, \sigma^2)$</td>
<td>$\exp \left{ \mu t + \frac{t^2 \sigma^2}{2} \right}$</td>
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Theorem: Moment generating function uniquely determines the distribution.

Proof. “Probability and Measure” by Patrick Billingsley, Theorem 22.2.
Example

1. Determine the distribution.
   (a) \( M(t) = [0.7e^t + 0.3]^4 \).
   (b) \( M(t) = \frac{2e^t}{5 - 3e^t} \).
   (c) \( M(t) = \frac{1}{1 - 2t} \).
   (d) \( M(t) = 0.2e^{-t} + 0.3e^{2t} + 0.5 \).
2. (This example is advanced). Suppose $Y_1, Y_2, \ldots$ is a sequence of iid (independent, identically distributed) random variables such that

$$P(Y_i = 0) = P(Y_i = 1) = \frac{1}{2}.$$ 

Let

$$X = \sum_{i=1}^{\infty} \frac{Y_i}{2^i}.$$ 

What is the distribution of $X$?