Review of Multivariable Calculus

Two-dimensional integral takes form

$$\iint_D f(x, y)dxdy$$

where D is a subset of $\mathbb{R}^2$.

D: the domain on which the integral takes place.

f: integrand.

x, y: dummy variables.

To compute this integral, the *key* step is to express the domain D in the following form.

$$D = \{(x, y) : a \leq x \leq b,\ h_1(x) \leq y \leq h_2(x)\}.$$

Always draw a graph of domain D if possible.

Examples

1. $D = \{(x,y) : 0 \le x \le 1,\ 0 \le y \le 1\}$.
2. $D = \{(x,y) : 0 \le x \le 1,\ 0 \le x + y \le 1\}$.
3. $D = \{(x,y) : 0 \le y \le x\}$.
4. $D = \{(x,y) : x + y \le 0\}$.
5. $D = \{(x,y) : 0 \le x \le y, x + y \le 2\}$.
6. $D = \{(x,y) : |x| + y \le 1, y \ge 0\}$.
7. $D = \{(x,y) : x^2 + y^2 \le 1\}$. Unit disk.

Then the integral can be written as

$$\iint_D f(x,y)dxdy = \int_a^b dx \left[\int_{h_1(x)}^{h_2(x)} f(x,y)\, dy \right].$$

The calculation of this integral goes as follows.

Step 1. For each x, calculate

$$\int_{h_1(x)}^{h_2(x)} f(x,y)\, dy.$$

In this calculation, regarding x as if it were a fixed number. This integral will yield a function of x alone.

Step 2. Integrate from a to b the function of x you obtained from Step 1.

Examples

1. Compute integral

$$\iint_D (x + xy)dxdy$$

on region

$$D = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 2\}.$$

2. Compute integral

$$\iint_D e^{x-y} dx dy$$

on region

$$D = \{(x, y) : x \leq 1, y \geq 1\}.$$

3. Compute integral

$$\iint_D (x^2 y + y^2)\, dx dy$$

on region

$$D = \{(x, y) : 0 \le x \le y \le 1\}.$$

4. Compute the integral

$$\iint_D 2e^{-(x+2y)}\,dxdy$$

on region

$$D = \{(x,y) : 0 \le x \le y\}.$$

and region

$$D = \{(x,y) : 0 \le y \le x\}.$$

5. Compute the integral

$$\iint_D x dx dy$$

on region

$$D = \{(x, y) : x \geq 0, y \geq 0,\ x^2 + y^2 \leq 1\}.$$

Useful Relations

- $$\iint_D cf = c\iint_D f$$

- $$\iint_D [f+g] = \iint_D f + \iint_D g$$

- $$\iint_D f = \iint_{D_1} f + \iint_{D_2} f,$$

  provided $D_1 \cap D_2 = \emptyset$, $D_1 \cup D_2 = D$.

Special Cases

1.

$$\iint_D dxdy = \text{Area of region D.}$$

2. When $D = [a, b] \times [c, d]$ and $f(x, y) = g(x) \cdot h(y)$,

$$\iint_D f(x, y)dxdy = \left[\int_a^b g(x)dx\right] \cdot \left[\int_c^d h(y)dy\right].$$