Chapters 7. Central Limit Theorem and Law of Large Numbers

Two most fundamental results in probability is Central Limit Theorem (CLT) and Law of Large Numbers (LLN)

Law of Large Numbers (LLN)

Suppose $X_1, X_2, \ldots$ is a sequence of independent identically distributed (iid) random variables with $E[X_i] = \mu$. Then

$$\lim_{n \to \infty} \frac{X_1 + X_2 + \cdots + X_n}{n} = \mu$$

with probability one.

Central Limit Theorem (CLT)

Suppose $X_1, X_2, \ldots$ is a sequence of independent identically distributed (iid) random variables with $E[X_i] = \mu$ and $\mathrm{Var}[X_i] = \sigma^2$. Then as $n \to \infty$,

$$\frac{X_1 + X_2 + \cdots + X_n - n\mu}{\sqrt{n}\sigma} \Rightarrow N(0,1).$$

The convergence "$\Rightarrow$" means

$$\lim_{n\to\infty} P\left(\frac{X_1 + X_2 + \cdots + X_n - n\mu}{\sqrt{n}\sigma} \le y\right) = \Phi(y)$$

for every $y \in \mathbb{R}$.

Remark: Formally speaking, CLT claims that the distribution of the "sample mean"

$$\bar{X} \doteq \frac{X_1 + X_2 + \cdots + X_n}{n}$$

is approximately

$$N\left(\mu, \frac{\sigma^2}{n}\right)$$

Remark: For a formal argument for LLN and CLT, use moment generating functions. See the textbook.

Normal Approximation

1. Suppose the IQ for college students has mean 110, and standard deviation 20. A random sample of $n = 100$ students are selected. Approxmiate the probability that the sample IQ average score is below 108?

2. Normal approximation to Binomial distributions.

Suppose X is $B(n; p)$. Write X as

$$X = Y_1 + Y_2 + \cdots + Y_n$$

where $\{Y_1, Y_2, \ldots, Y_n\}$ are iid Bernoulli random variables with

$$P(Y_i = 1) = p, \quad P(Y_i = 0) = q = 1 - p.$$

$E[Y_i] = p$, $\mathrm{Var}[Y_i] = pq$. By CLT,

$$\frac{X}{n}$$

is approximately

$$N\left(p, \frac{pq}{n}\right)$$

Remark: The normal approximation works well when

$$n > 9 \cdot \frac{\max\{p, q\}}{\min\{p, q\}}$$

3. Example: Suppose X is $B(100; 0.3)$. Compute $P(X > 35)$.

Solution:

$$P(X > 35) = P\left(\frac{X}{100} > 0.35\right)$$

Let

$$Z = \frac{\frac{X}{100} - 0.3}{\sqrt{\frac{0.3 \cdot 0.7}{100}}}.$$

Then

$$P(X > 35) \approx P\left(Z > \frac{0.35 - 0.3}{\sqrt{\frac{0.3 \cdot 0.7}{100}}}\right) = P(Z > 1.09) = 0.1379$$