Chapter 4. Continuous Random Variables

Review

- Continuous random variable: A random variable that can take any value on an interval of $\mathbb{R}$.
- Distribution: A density function $f : \mathbb{R} \to \mathbb{R}_+$ such that

  1. non-negative, i.e., $f(x) \geq 0$ for all x.
  2. for every subset $I \subset \mathbb{R}$,

  $$P(X \in I) = \int_I f(x)\, dx$$

  3.

  $$\int_{\mathbb{R}} f(x)\, dx = 1.$$

- Graphic description: (1) density f, (2) cdf F.
- Relation between density f and cdf F:

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(y)\, dy$$

$$f(x) = F'(x).$$

Remark: Given a *continuous* random variable X,

$$P(X = a) = 0$$

for any real number a. Therefore,

$$P(X < a) = P(X \leq a), \quad P(X > a) = P(X \geq a)$$

$$P(a < X < b) = P(a \leq X < b) = P(a < X \leq b) = P(a \leq X \leq b).$$

- Example: Suppose X has density f that takes the following form:

$$f(x) = \begin{cases} cx(2-x) & , \quad 0 \le x \le 2, \\ 0 & , \quad \text{otherwise.} \end{cases}$$

  Determine c, cdf F, and find $P(X < 1)$.

Expected Value, Variance, and Standard Deviation

Continuous random variable X with density f.

- Expectation (expected value, mean, "μ"):

$$EX \doteq \int_{-\infty}^{\infty} x f(x)\, dx$$

THEOREM: Consider a function h and random variable $h(X)$. Then

$$E[h(X)] = \int_{-\infty}^{\infty} h(x)f(x)\,dx$$

Corollary: Let a, b be real numbers. Then $E[aX + b] = aEX + b$.

Corollary: Consider functions $h_1, \ldots, h_k$. Then

$$E[h_1(X) + \cdots + h_k(X)] = E[h_1(X)] + \cdots + E[h_k(X)].$$

- Variance ("σ^2") and Standard deviation ("σ"):

$$\mathrm{Var}[X] \doteq E\left[(X - EX)^2\right], \qquad \mathrm{Std}[X] \doteq \sqrt{\mathrm{Var} X}.$$

Proposition: $\mathrm{Var}[X] = 0$ if and only if $P(X = c) = 1$ for some constant c.

Proposition:

$$\mathrm{Var}[X] = E[X^2] - (EX)^2.$$

Proposition: Let a, b be real numbers. Then $\mathrm{Var}[aX + b] = a^2 \mathrm{Var}[X]$.

Uniform Distribution

- Uniform distribution on $[0,1]$. A random variable X with density

$$f(x) = \begin{cases} 1 & , \quad 0 \leq x \leq 1 \\ 0 & , \quad \text{otherwise.} \end{cases}$$

Expected value and variance.

- Uniform distribution on $[a, b]$. A random variable X with density

$$f(x) = \begin{cases} \dfrac{1}{b-a} & , \quad a \le x \le b \\ 0 & , \quad \text{otherwise.} \end{cases}$$

- Remark: If X is uniformly distributed on $[0, 1]$, then $Y \doteq (b-a)X + a$ is uniformly distributed on $[a, b]$. Expected value and variance.

Example

1.(a) A point is randomly chosen from interval $[-1, 1]$. Find the probability that the point is non-negative.

(b) 1000 points are randomly chosen from interval $[-1, 1]$. What do you think their average should be close to?

2. Suppose customers arrive at post office randomly such that

- The number of arrivals on any time interval with length s is Poisson distributed with parameter λs.
- The numbers of arrivals on disjoint time intervals are independent.

Define event

$$A \doteq \{\text{on time interval } [0, T], \text{ there is in total one customer}\}.$$

Let

$$X \doteq \text{the arrival time of the first customer.}$$

(a) Compute $P(X \leq t | A)$ for all t.

(b) What is the distribution of X given that event A happens?

Exponential Distribution

- Exponential distribution with rate λ. A random variable X with density

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & , \quad 0 \le x < \infty \\ 0 & , \quad \text{otherwise.} \end{cases}$$

$$EX = \frac{1}{\lambda}, \qquad \text{Var}[X] = \frac{1}{\lambda^2}.$$

Remark: "Exponential distribution with rate λ" is called in the textbook "Exponential distribution with parameter β" where $\beta = 1/\lambda$.

Example

1. Suppose X is exponentially distributed with rate λ. Find $P(X > t)$.

2. Suppose customers arrive at post office randomly such that

- The number of arrivals on any time interval with length s is Poisson distributed with parameter λs.
- The numbers of arrivals on disjoint time intervals are independent.

Let

$$X \doteq \text{the arrival time of the first customer.}$$

Compute $P(X \leq t)$. What is the distribution of X?

Memoryless property

Suppose X is exponentially distributed with rate λ. Then

$$P(X > t + s | X > t) = P(X > s).$$

Comment: Exponential distribution is the only continuous distribution that has memoryless property.

Example: The life length of a computer is exponentially distributed with mean 5 years. You bought an old (working) computer for 10 dollars. What is the probability that it would still work for more than 3 years?

Normal Distribution

Normal distribution $N(\mu, \sigma^2)$. A random variable X is normally distributed with mean μ and variance σ^2 if it has density

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}, \qquad x \in \mathbb{R}.$$

1. f defines a probability density function.
2. $EX = \mu$.
3. $\mathrm{Var}[X] = \sigma^2$.

Standard Normal Distribution

The standard normal means $N(0,1)$. Its cdf is often denoted by Φ, that is,

$$\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{y^2}{2}\right\} dy$$

- Suppose Z is $N(0,1)$, then

$$P(Z > x) = P(Z < -x) = \Phi(-x).$$

  In particular,

$$\Phi(x) + \Phi(-x) = 1.$$

Standardization

Theorem: Suppose X has normal distribution $N(\mu, \sigma^2)$. Then its linear transform $Y \doteq aX + b$ is also normally distributed with distribution $N(a\mu + b, a^2\sigma^2)$.

Corollary: Suppose X has distribution $N(\mu, \sigma^2)$. Then its standardization

$$Z \doteq \frac{X - \mu}{\sigma}$$

is standard normal.

Examples

1. Suppose X has distribution $N(40, 100)$. Find $P(50 < X < 60)$.

2. Suppose X has distribution $N(\mu, \sigma^2)$. Then

$$P(|X - \mu| > k\sigma) = 2\Phi(-k).$$

In particular,

k	1	2	3
$2\Phi(-k)$	32%	5%	0.3%

Remark: Normal distribution is often used as approximation of non-negative distributions. For example, SAT scores, IQ, GRE scores