• Suppose that $X$ and $Y$ has joint density
\[
f(x, y) = \begin{cases} 
2 & \text{if } x \geq 0, y \geq 0, x + y \leq 1 \\
0 & \text{otherwise}
\end{cases}
\]

Determine the conditional expectation $E[Y|X]$.

\textbf{Solution:} First we need to compute $E[Y|X = x]$. To this end, we need to calculate $f(y|x)$. Note that the marginal density of $X$ is, for an arbitrarily fixed $0 \leq x \leq 1$,
\[
f_X(x) = \int_{-\infty}^{\infty} f(x, y)dy = \int_0^{1-x} 2dy = 2(1 - x)
\]
The conditional density is
\[
f(y|x) = \frac{f(x, y)}{f_X(x)} = \frac{2}{2(1 - x)} = \frac{1}{1 - x}
\]
for all $0 \leq y \leq 1 - x$, and $f(y|x) = 0$ otherwise. Therefore, for every $0 \leq x \leq 1$,
\[
E[Y|X = x] = \int_{-\infty}^{\infty} yf(y|x)dy = \int_0^{1-x} y \frac{1}{1 - x} dy = \frac{1}{1 - x} \int_0^{1-x} ydy = \frac{1 - x}{2}
\]
Thus, replace $x$ by $X$ we arrive at
\[
E[Y|X] = \frac{1 - X}{2}
\]