AM 165 HMWK #7

Chapter 5

(75) Let Y_1 = # Contracts Awarded to A
Y_2 = # awarded to B

$y_2 \backslash y_1$	0	1	2
0	1	2	1
1	2	2	0
2	1	0	0

$\times \frac{1}{9}$

$$\text{cov}(Y_1 Y_2) = E(Y_1 Y_2) - \mu_1 \mu_2$$

$$\mu_1 = EY_1 = 0 \cdot 4/9 + 1 \cdot 4/9 + 2 \cdot \tfrac{1}{9} = 6/9 = 2/3$$
$$\mu_2 = EY_2 = 0 \cdot 4/9 + 1 \cdot 4/9 + 2 \cdot \tfrac{1}{9} = 2/3$$
$$E(Y_1Y_2) = \sum Y_1 Y_2 \, p(Y_1, Y_2) = 1 \cdot 1 \cdot 2/9 = 2/9$$ (all other terms are zero)

Thus $\text{cov}(Y_1 Y_2) = 2/9 - 2/3 \cdot 2/3 = -2/9$

No, this is not surprising since if we award more contracts to A then we necessarily award fewer to B

(80) Let $Y_1 Y_2$ be uncorrelated $(\Rightarrow \text{cov}(Y_1 Y_2) = EY_1Y_2 - EY_1EY_2 = 0)$
Find Covariance and Correlation between $U_1 = Y_1 + Y_2$ and $U_2 = Y_1 - Y_2$

$$\text{Cov}(U_1 U_2) = E(U_1 U_2) - E(U_1)E(U_2)$$
$$= E((Y_1+Y_2)(Y_1-Y_2)) - E(Y_1+Y_2)\underbrace{E(Y_1-Y_2)}_{EY_1 - EY_2 \text{ by Linearity}}$$
$$= E(Y_1^2 - Y_2^2) - [(EY_1)^2 - (EY_2)^2]$$
$$= EY_1^2 - (EY_1)^2 - EY_2^2 + (EY_2)^2$$
$$= \text{Var}\,Y_1 - \text{Var}\,Y_2$$

Correlation $\rho = \dfrac{\text{cov}(U_1 U_2)}{\sigma_{U_1}\sigma_{U_2}}$

$$\text{Var}(U_1) = \sigma_{U_1}^2 = EU_1^2 - (EU_1)^2 = E[(Y_1+Y_2)^2] - (EY_1 + EY_2)^2$$
$$= EY_1^2 + 2EY_1Y_2 + EY_2^2 - (EY_1)^2 - 2EY_1EY_2 - (EY_2$$

Since $Y_1 Y_2$ uncorrelated $EY_1Y_2 - EY_1EY_2 = 0$

so $= EY_1^2 - (EY_1)^2 + E(Y_2^2) - (EY_2)^2$

$$\sigma_{U_1}^2 = \text{Var}\,Y_1 + \text{Var}\,Y_2$$
$$\Rightarrow \sigma_{U_1} = \sqrt{\text{Var}\,Y_1 + \text{Var}\,Y_2}$$

Similarly $\text{Var}\, U_2 = \sigma_{U_2}^2 = \text{Var}\, Y_1 + \text{Var}\, Y_2$
& $\sigma_{U_2} = \sqrt{\text{Var}\, Y_1 + \text{Var}\, Y_2}$

Thus $\rho = \frac{\text{Cov}(U_1, U_2)}{\sigma_{U_1}\sigma_{U_2}} = \frac{V(Y_1) - V(Y_2)}{\sqrt{V(Y_1)+V(Y_2)} \cdot \sqrt{V(Y_1)+V(Y_2)}} = \frac{V(Y_1) - V(Y_2)}{V(Y_1) + V(Y_2)}$

(84) Let $Z \sim N(0,1)$ Standard Normal
$Y_1 = Z \qquad Y_2 = Z^2$

a) There are a few things to keep in mind for this problem.
① Normal distribution is symmetric about the mean (0)
② $\text{Var}(Z) = E((Z-\mu)^2)$ and since $\mu = 0$
$= E Z^2$

Thus $E(Y_1) = EZ = 0$ either because you are taking the integral of an odd function or because the mean is zero

$E(Y_2) = EZ^2 = \text{Var}(Z) = 1$

b) $E(Y_1 Y_2) = EZ^3 = \int_{-\infty}^{\infty} Z^3 \frac{1}{\sqrt{2\pi}} e^{-Z^2/2}\, dZ$

You can either do a change of variables with $u = Z^2$, $du = 2Z\, dZ$ or realize that the function is ODD so the integral is zero
(You could also use moment generating function - take 3rd derivative)

c) $\text{Cov}(Y_1 Y_2) = E(Y_1 Y_2) - E(Y_1)E(Y_2)$
$= 0 - 0 \cdot 1 = 0$

d) Notice $P(Y_2 > 1 \mid Y_1 > 1) = 1$
(since if $Y_1 = Z > 1$
$\Rightarrow Z^2 > 1$)

However Y_1 and Y_2 are NOT independent since $P(Y_2 > 1) \neq 1$
For two events to be independent $P(A|B) = P(A)$ which is not true in this case. This is an example of 0 covariance $\not\Rightarrow$ Independe[nce]

Two normally distributed R.V. are independent $\Leftrightarrow$ their covariance is zero
BUT Y_2 is NOT normally distributed.

(86) Chemicals:

	COST	Mean	Variance
Type I	\$3	40	4
Type II	\$5	65	8

Assume $Y_1 Y_2$ Independent

Find Mean and Variance of the total Amt. of Money spent per week

$U = 3Y_1 + 5Y_2$

$E(U) = 3EY_1 + 5EY_2 = 3(40) + 5(65) = \$445$

$V(U) = V(3Y_1 + 5Y_2) = 9V(Y_1) + 2\cdot 3\cdot 5\,\text{COV}(Y_1, Y_2) + 5^2 V(Y_2)$

but $\text{COV}(Y_1, Y_2) = 0$ since independent

$= 9\cdot 4 + 25\cdot 8$

$= \$236$

(87)

	Y_1	Y_2	Y_3
E	2	-1	4
V	4	6	8

$\text{COV}(Y_1 Y_2) = 1$

$\text{COV}(Y_2 Y_3) = 0$

$\text{COV}(Y_1 Y_3) = -1$

$E(3Y_1 + 4Y_2 - 6Y_3) = 3EY_1 + 4EY_2 - 6EY_3 = 6 - 4 - 24 = -22$

$V(3Y_1 + 4Y_2 - 6Y_3) = 3^2 V(Y_1) + 4^2 V(Y_2) + (-6)^2 V(Y_3) + 2\cdot 3\cdot 4\,\text{COV}(Y_1 Y_2)$
$+ 2\cdot 3\cdot(-6)\,\text{COV}(Y_1 Y_3) + 2\cdot 4(-6)\,\text{COV}(Y_2 Y_3)$

$= 9\cdot 4 + 16\cdot 6 + 36\cdot 8 + 24(1) + (-36)(-1) + -48(0)$

$= 480$

(103) Fires: 73% Family 20% APT 7% other

$P(2F, 1A, 1\,\text{other}) = \dfrac{4!}{2!\,1!\,1!} (.73)^2 (.20)^1 (.07)^1 = .08953$

(115) a) Randomly select 3 items, observe the # of defectives

Proportion p of defectives $p \sim U(0,1)$

Find expected numbers of defectives observed among 3 sampled items

Let Y_1 = # defectives $Y_2 = p$ probability of defectives

Long Way: $E[Y_1 | Y_2 = p] = \sum Y_1 P(Y_1 | Y_2 = p)$

Binomial

then $\int_0^1 \sum Y_1 P(Y_1 | Y_2 = p) \cdot P(p = p)\, dp$

Uniform.

Short way: $E(Y_1) = E\left[E[Y_1|Y_2]\right]$

and note for a binomial: $E[Y_1|Y_2=p] = 3\cdot p$

and if P uniform

$$E[3p] = 3EP = 3\cdot\tfrac{1}{2} = 3/2$$

(12) a) # eggs laid $\sim Poi(\lambda)$

$P(\text{Hatch}) = p$ Assume Eggs hatch independently of each other

Let N: # eggs laid Y: # eggs hatched

Given $N=n$ $Y \sim Bin(n,p)$

so $E(Y|N=n) = np$ (once again you could do this the long way by actually calculating the sum)

Recall: $E(Y) = E\left[E[Y|N=n]\right] = E[np]$

since $N \sim Poi(\lambda)$

$EN = \lambda$ (write it out if you want)

so $E[np] = pE(n) = p\lambda$

Chapter 6

(5) Waiting time $Y \sim U(1,5)$

cost of Delay $U = 2Y^2 + 3$. Find Probability density for U

$$P(U \le u) = P(2Y^2+3 \le u) = P\left(Y < \sqrt{\tfrac{u-3}{2}}\right)$$

Note that $f_Y(y) = \frac{1}{4}$ $\left(\frac{1}{5-1}\right)$ for $1 \le y \le 5$

so the cdf $F_Y(y) = \begin{cases} \frac{y-1}{4} & 1 \le y \le 5 \\ 0 & y < 1 \\ 1 & y > 5 \end{cases}$

So $F_u(u) = P(U \le u) = \begin{cases} \frac{\sqrt{\frac{u-3}{2}}}{4} - \frac{1}{4} & u \in (5,53) \\ 0 & u < 5 \\ 1 & u > 53 \end{cases}$

$2\cdot1^2+3=5$

$2\cdot5^2+3=53$

Differentiating $f_u(u) = \frac{1}{4}\cdot\frac{1}{2}\left(\frac{u-3}{2}\right)^{-\frac{1}{2}}\cdot\frac{1}{2} = \begin{cases} \frac{1}{16}\cdot\left(\frac{u-3}{2}\right)^{-\frac{1}{2}} & 5 \le u \le 53 \\ 0 & \text{else} \end{cases}$

(72) Y_1, Y_2 IID $N(\mu, \sigma^2)$ Find Probability Density function for $U = \frac{1}{2}Y_1 - \frac{3}{2}Y_2$

By Thm 6.3 (in the Moment generating Section)
U is a normally distributed RV with $EU = \sum a_i \mu_i$
and $V(U) = \sum a_i^2 \sigma_i^2$

In our Case $a_1 = \frac{1}{2}$ $a_2 = -3/2$
so $E(u) = \frac{1}{2}\mu - \frac{3}{2}\mu = -\mu$
$V(u) = (\frac{1}{2})^2 \sigma^2 + (\frac{3}{2})^2 \sigma^2 = \frac{10}{4}\sigma^2$

Thus $U \sim N(-\mu, 5/2\, \sigma^2)$

(73) I thought this one was really difficult...
Current $I \sim U(0,1)$
Resistance R: $f(r) = \begin{cases} 2r & 0 \le r \le 1 \\ 0 & \text{else} \end{cases}$ $W = I^2 R$

Find Probability density function for W (Assume Independence of I and R)

$P(W \le w) = P(I^2 R \le w)$ But you can't really divide by a RV and use the distribution of I

$\boxed{\text{Fix } r}$

so $P(W \le w \mid R = r) = P(I^2 r \le w) = P(I \le \sqrt{w/r})$
$= \sqrt{w/r}$ since I is Uniform

Thus $F(w \mid R = r) = \sqrt{w/r}$
differentiating $\Rightarrow f(w \mid R = r) = \frac{1}{2}(w/r)^{-\frac{1}{2}} \cdot \frac{1}{r} = \frac{1}{2r}(w/r)^{-\frac{1}{2}}$

Thus $f(w, r) = f(w \mid r) \cdot f(r) = \frac{1}{2r}(w/r)^{-\frac{1}{2}} \cdot 2r = (w/r)^{-\frac{1}{2}}$
$= (r/w)^{\frac{1}{2}}$ $0 \le w \le r \le 1$

To get $f(w)$, we integrate the Joint over r
$f(w) = \int_{-\infty}^{\infty} f(w, r)\, dr = \int_w^1 \left(\frac{r}{w}\right)^{\frac{1}{2}} dr$
$= \frac{1}{w^{\frac{1}{2}}} \frac{2}{3} r^{3/2} \Big|_w^1$
$= 2/3\left(\frac{1}{\sqrt{w}} - w\right)$

For the Limits, note that since $I \le 1$, $I^2 \le 1$ $\Rightarrow W = I^2 R$ has $W \le R$