5.2 a) The sample space for the toss of three balanced coins, the values for Y_1 and Y_2 at each outcome and the probability of each outcome are given below:

outcomes	(Y_1, Y_2)	probability
HHH	$(3, 1)$	$\frac{1}{8}$
HHT	$(2, 1)$	$\frac{1}{8}$
HTH	$(2, 1)$	$\frac{1}{8}$
HTT	$(1, 1)$	$\frac{1}{8}$
THH	$(2, 2)$	$\frac{1}{8}$
THT	$(1, 2)$	$\frac{1}{8}$
TTH	$(1, 3)$	$\frac{1}{8}$
TTT	$(0, -1)$	$\frac{1}{8}$

y_2 \ y_1	0	1	2	3
-1	$\frac{1}{8}$	0	0	0
1	0	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{8}$
2	0	$\frac{1}{8}$	$\frac{1}{8}$	0
3	0	$\frac{1}{8}$	0	0

b). $F(2,1) = P(Y_1 \le 2, Y_2 \le 1) = P(0,-1) + P(1,1) + P(2,1)$

$$= \frac{1}{8} + \frac{1}{8} + \frac{1}{4} = \frac{1}{2} \text{。}$$

5.6 a). We must have $F(\infty,\infty) = \int_0^1 \int_0^1 k y_1 y_2 \, dy_1 dy_2 = 1$

Then

$$\int_0^1 \int_0^1 k y_1 y_2 \, dy_1 dy_2 = k \int_0^1 y_2 \cdot \left(\frac{y_1^2}{2}\right)\Big|_0^1 dy_2 = \frac{k}{4} = 1$$

$\Rightarrow k = 4$;

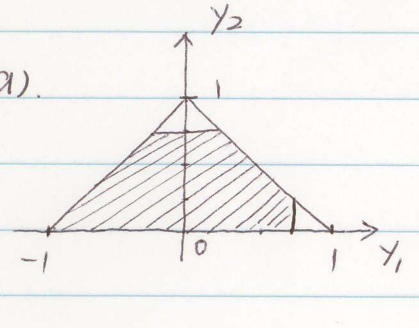
b). $F(y_1, y_2) = \int_0^{y_2} \int_0^{y_1} 4 t_1 t_2 \, dt_1 dt_2 = y_1^2 y_2^2$

for $0 \le y_1 \le 1$ and $0 \le y_2 \le 1$。 Recall that

$$F(y_1, y_2) = \begin{cases} 0 & \text{for } y_1 \le 0 \text{ or } y_2 \le 0 \\ 1 & \text{for } y_1 \ge 1 \text{ and } y_2 \ge 1 \\ y_2^2 & \text{for } y_1 \ge 1 \text{ and } 0 \le y_2 \le 1 \\ y_1^2 & \text{for } 0 \le y_1 \le 1 \text{ and } y_2 \ge 1 \end{cases};$$

c) $P(Y_1 \le \frac{1}{2}, Y_2 \le \frac{3}{4}) = F(\frac{1}{2}, \frac{3}{4}) = (\frac{1}{2})^2 (\frac{3}{4})^2 = \frac{9}{64}$ 。

5.9 a).

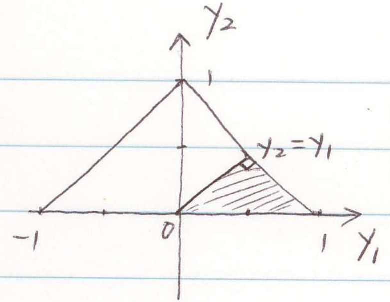


$P(Y_1 \le \frac{3}{4}, Y_2 \le \frac{3}{4})$ = the shaded area

$$= 1 - \frac{1}{4} \times \frac{1}{4} \times \frac{1}{2} - \frac{1}{4} \times \frac{1}{4} \times \frac{1}{2} \times 2$$

$$= 1 - \frac{1}{32} - \frac{2}{32} = \frac{29}{32};$$

5.9 b)



$P(Y_1 - Y_2 \geq 0)$ = the shaded area

$$= \int_0^{\frac{1}{2}} \int_{y_2}^{1-y_2} dy_1\, dy_2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$

5.20 a) The marginal distribution for Y_1 is $P(Y_1=0)=0.76$ and $P(Y_1=1)=0.24$; and the marginal distribution for Y_2 is given by $P(Y_2=0)=0.55$, $P(Y_2=1)=0.16$, and $P(Y_2=2)=0.29$;

b) $P(Y_2=0 \mid Y_1=0) = \frac{0.38}{0.76} = 0.5$; $P(Y_2=1 \mid Y_1=0) = \frac{0.14}{0.76} = 0.18$

$P(Y_2=2 \mid Y_1=0) = \frac{0.24}{0.76} = 0.32$;

c) $P(Y_1=0 \mid Y_2=2) = \frac{0.24}{0.29} = 0.83$.

5.23 a) $f_1(y_1) = \int_{y_1}^{1} 6(1-y_2)\,dy_2 = 6(1-y_1) - 3 + 3y_1^2 = 3(1-y_1)^2$, for $0 \le y_1 \le 1$

$f_2(y_2) = \int_0^{y_2} 6(1-y_2)\,dy_1 = 6y_2(1-y_2)$ for $0 \le y_2 \le 1$.

b) $P(Y_2 \le 1/2 \mid Y_1 \le \frac{3}{4}) = \frac{F(\frac{3}{4}, \frac{1}{2})}{\int_0^{\frac{3}{4}} f_1(y_1)\,dy_1} = \frac{\int_0^{\frac{1}{2}} \int_0^{y_2} 6(1-y_2)\,dy_1\,dy_2}{\int_0^{\frac{3}{4}} 3(1-y_1)^2\,dy_1}$

$= \frac{\frac{1}{2}}{\frac{63}{64}} = \frac{32}{63}$;

c) $f(y_1 \mid y_2) = \frac{f(y_1, y_2)}{f_2(y_2)} = \frac{6(1-y_2)}{6y_2(1-y_2)} = \frac{1}{y_2}$ if $0 < y_2 < 1$ for $0 \le y_1 \le y_2$;

d) $f(y_2 \mid y_1) = \frac{f(y_1, y_2)}{f_1(y_1)} = \frac{6(1-y_2)}{3(1-y_1)^2} = \frac{2(1-y_2)}{\cancel{8}(1-y_1)^2}$ if $0 \le y_1 < 1$ for $y_1 \le y_2 \le 1$;

e) $P(Y_2 \ge \frac{3}{4} \mid Y_1 = \frac{1}{2}) = \int_{\frac{3}{4}}^{1} f(y_2 \mid y_1 = \frac{1}{2})\,dy_2$

$= \int_{\frac{3}{4}}^{1} 8(1-y_2)\,dy_2 = 8 \times \frac{1}{4} - 4 \times \frac{7}{16} = \frac{1}{4}$.

5.32 a) By definition $f(y_1, y_2) = f_1(y_1) f(y_2 | y_1)$

now $f_1(y_1) = 1$ for $0 \le y_1 \le 1$ where Y_1 = amount stocked,

and $f(y_2 | y_1) = \frac{1}{y_1}$ for $0 \le y_2 \le y_1$ where Y_2 = amount sold.

$\Rightarrow$

$$f(y_1, y_2) = \begin{cases} \frac{1}{y_1} & \text{for } 0 \le y_2 \le y_1 \le 1 \\ 0 & \text{elsewhere} \end{cases} \quad ;$$

b) $P(Y_2 > \frac{1}{4} | Y_1 = \frac{1}{2}) = \int_{\frac{1}{4}}^{\frac{1}{2}} f(y_2 | y_1 = \frac{1}{2})\, dy_2 = 2 \times \frac{1}{4} = \frac{1}{2}$;

c) $P(Y_1 > \frac{1}{2} | Y_2 = \frac{1}{4}) = \int_{\frac{1}{2}}^{1} f(y_1 | y_2 = \frac{1}{4})\, dy_1$

$f(y_1 | y_2) = \frac{f(y_1, y_2)}{f_2(y_2)}$ and $f_2(y_2) = \int_{y_2}^{1} \frac{1}{y_1}\, dy_1 = -\ln y_2$

for $0 \le y_2 \le 1$

so $P(Y_1 > \frac{1}{2} | Y_2 = \frac{1}{4}) = \int_{\frac{1}{2}}^{1} \frac{1}{\ln 4} \cdot \frac{1}{y_1}\, dy_1 = \frac{\ln 2}{\ln 4} = \frac{1}{2}$.

5.45 Dependent! Because the range of y_1 values on which $f(y_1, y_2)$ is defined is dependent on y_2. (and vice versa).

5.53 Independent! Because $f(y_1, y_2)$ can be factored.

5.54 The desired probability is

$P(X=1, Y=1) + P(X=2, Y=2) + \cdots + \cdots$

$$= \sum_{n=0}^{\infty} (q^n p) \cdot (q^n p) = p^2 \cdot \sum_{n=0}^{\infty} (q^2)^n = \frac{p^2}{1-q^2} .$$

5.59 a) $$f(y_1, y_2) = \begin{cases} \frac{1}{9} e^{-(y_1+y_2)/3} & y_1 > 0 \quad y_2 > 0 \\ 0 & \text{elsewhere} \end{cases} \quad ;$$

b) $P(Y_1+Y_2 \le 1) = \int_0^1 dy_2 \int_0^{1-y_2} \frac{1}{9} e^{-(y_1+y_2)/3}\, dy_1$

$= \int_0^1 \frac{1}{3} e^{-\frac{y_2}{3}} \int_0^{1-y_2} \frac{1}{3} e^{-\frac{y_1}{3}}\, dy_1$

$= \int_0^1 \frac{1}{3} e^{-\frac{y_2}{3}} \cdot \left(1-e^{\frac{-(1-y_2)}{3}}\right) dy_2$

$= -\frac{1}{3} e^{-\frac{1}{3}} + 1 - e^{-\frac{1}{3}} = 1-\frac{4}{3} e^{-\frac{1}{3}}$

5.60 Y_1 denotes the bus arrival time and Y_2 the passenger arrival time, the joint density of Y_1 and Y_2 is

$f(y_1,y_2) = f_1(y_1)\, f_2(y_2) = \begin{cases} 1 & 0 \le y_1 \le 1,\ 0 \le y_2 \le 1 \\ 0 & \text{otherwise} \end{cases}$

$P(Y_2 \le Y_1 \le Y_2 + \frac{1}{4}) = \int_0^{\frac{1}{4}} dy_1 \int_0^{y_1} f(y_1,y_2)\, dy_2 +$

$\int_{\frac{1}{4}}^1 dy_1 \int_{y_1-\frac{1}{4}}^{y_1} f(y_1,y_2)\, dy_2 = \frac{1}{32} + \frac{3}{16} = \frac{7}{32}$.

5.68 . $E(Y_1) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} y_1 f(y_1,y_2)\, dy_1 dy_2 = \int_0^1 \int_0^1 y_1(y_1+y_2)\, dy_1 dy_2$

$= \frac{1}{3} + \frac{1}{4} = \frac{7}{12}$

by symmetry, $E(Y_2) = \frac{7}{12}$, so $E(30Y_1+25Y_2) = (30+25)\times\frac{7}{12} = 32.08$ $\left(\frac{385}{12}\right)$.

5.69 $f(y_1,y_2) = f_1(y_1)\cdot f_2(y_2)$, $f_1(y_1) = \frac{1}{4} y_1 e^{-y_1/2}$ and $f_2(y_2) = \frac{1}{2} e^{-y_2/2}$,

$E\left(\frac{Y_2}{Y_1}\right) = \int_0^\infty \int_0^\infty \frac{y_2}{y_1} f_1(y_1)\cdot f_2(y_2)\, dy_1 dy_2 = \int_0^\infty f_1(y_1)\cdot\frac{1}{y_1}\, dy_1 \cdot \int_0^\infty y_2 f_2(y_2)\, dy_2$

$= E\left(\frac{1}{Y_1}\right)\cdot E(Y_2) = \int_0^\infty \frac{1}{4} e^{-\frac{y_1}{2}} dy_1 \cdot \int_0^\infty \frac{1}{2} y_2 \cdot e^{-\frac{y_2}{2}} dy_2$

$= \left[-\frac{1}{2} e^{-y_1/2}\right]\Big|_0^\infty \cdot \left[\left(-y_2 e^{-\frac{y_2}{2}}\right)\Big|_0^\infty + \left(-2e^{-\frac{y_2}{2}}\right)\Big|_0^\infty\right]$

$= \frac{1}{2}\cdot[0+2] = 1$.