HW5.

Using Table 4,

4.46. a) The area between $z=0$ and $z=1.2$ is $(0.5-0.1151)=0.3849$.

b) The area between $z=0$ and $z=-0.9$ is $(0.5-0.1841)=0.3159$.

c) The area between $z=0.3$ and $z=1.56$ is $(0.3821-0.0594)=0.3227$.

d) The area between $z=-0.2$ and $z=0.2$ is $(1-2\times0.4207)=0.1586$.

e) The area between $z=-1.56$ and $z=-0.2$ is $(0.4207-0.0594)=0.3613$.

4.47. a) $z_0=0$

b) $z_0=1.1$

c) $z_0=1.645$

d) $z_0=2.575$

Remark: If the exact probability cannot be found in the table, we may choose to search for the probability closest to the one desired and perform an interpolation (usually linear interpolation is okay!) that will determine the exact value of z_0.

4.50 $\mu=400 \quad \sigma=20$.

$z_1 = \frac{450-400}{20} = 2.5$. Then

$P(Y>450) = P(Z>2.5) \underset{\text{Table 4}}{=} 0.0062$.

note: Y denotes "exam score"

4.60 a) $P(Y>72) = P(Z>\frac{72-78}{6}) = P(Z>-1) = 1-P(Z>1)$

$= 0.8413$;

b) We seek c s.t. $P(Y>c)=0.1 \Leftrightarrow 0.1 = P(Y>c) = P(Z>\frac{c-78}{6}) = P(Z>z_0)$

$\Rightarrow z_0 = 1.28 = \frac{c-78}{6} \Rightarrow c = 85.68$;

c) We seek c s.t. $P(Y>c)=0.281 \Rightarrow$

$0.281 = P(Y>c) = P(Z>\frac{c-78}{6}) = P(Z>z_0)$

$\Rightarrow z_0 = 0.58 = \frac{c-78}{6} \Rightarrow c = 81.48$;

d) $P(Z<\frac{c-78}{6}) = 0.25$

Now $P(Z>0.67)=0.25 \Rightarrow P(Z<-0.67)=0.25$

$\Rightarrow \frac{c-78}{6} = -0.67 \Rightarrow c = 73.98$,

We must now find $P(Y>73.98+5) = P(Y>78.98)$

$= P(Z>\frac{78.98-78}{6}) = P(Z>0.16)$

$= 0.4364$;

e) $P(Y>84 \mid Y>72) = \frac{P(Y>84)}{P(Y>72)} = \frac{P(Z>\frac{84-78}{6})}{P(Z>\frac{72-78}{6})} = \frac{P(Z>1)}{P(Z>-1)}$

$= \frac{P(Z>1)}{1-P(Z>1)} = \frac{0.1587}{0.8413} = 0.1886$.

4.77 $f(y) = \frac{1}{4} e^{-\frac{y}{4}}$ for $y \geq 0$ Then $P(Y>4) = \int_4^\infty \frac{1}{4} e^{-\frac{y}{4}} dy$

$= -e^{-\frac{y}{4}} \Big|_4^\infty = e^{-1} \doteq 0.368$.

4.142) (a) The probability that the 8:15 am appt has to wait is the probability that the 8:00 am appt lasts more than 15 minutes.
Recall that the CDF of an exponential random variable is given by (for $y \ge 0$):

$$F_Y(y) = \Pr(Y \le y) = \int_0^y \frac{\exp(-x/\beta)}{\beta}\,dx = [-\exp(-x/\beta)]_{x=0}^{x=y} = 1 - \exp(-y/\beta)$$

Let Y = time that the 8:00 am appointment lasts. Told that: $Y \sim \exp\left(\beta = \frac{1}{2}\right)$

We know the CDF for Y: $\Pr(Y \le y) = F_Y(y) = 1 - \exp(-y/\beta) = 1 - \exp(-2y)$

Compute: $\Pr\left(Y > \frac{1}{4}\right) = 1 - \Pr\left(Y \le \frac{1}{4}\right) = 1 - F_Y\left(\frac{1}{4}\right) = \exp\left(-2 \cdot \frac{1}{4}\right) = \exp\left(-\frac{1}{2}\right)$

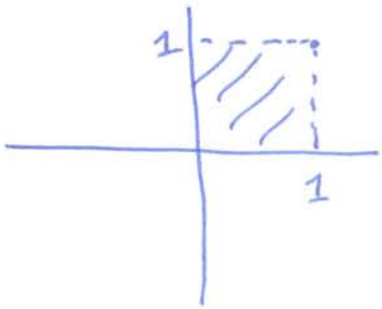
Part (b) asks for the probability that the waiting time is more than 20 minutes **GIVEN** that the 8:15 am appt has to wait. The memoryless property of the exponential distribution tells us that **if** the 8:00 am appointment has gone on for 15 minutes already (so that the 8:15 appt has to wait) the additional time it will last is exponentially distributed with the same parameter. That means that we can "restart the clock" on the first appointment at 8:15 am. So the wait time, $W \sim \exp\left(\beta = \frac{1}{2}\right)$.

Hence, $\Pr\left(W > \frac{1}{3}\right) = 1 - \Pr\left(W \le \frac{1}{3}\right) = 1 - F_W\left(\frac{1}{3}\right) = \exp\left(-2 \cdot \frac{1}{3}\right) = \exp\left(-\frac{2}{3}\right)$

Part (c) Since the expected value of the first appointment is half and hour, once we know it has lasted 15 minutes, it is expected to last an additional half an hour. So, the expected wait time is 30 min.

2D Integrals

1)



$$\int_0^1 \int_0^1 x+y+xy \, dy \, dx$$

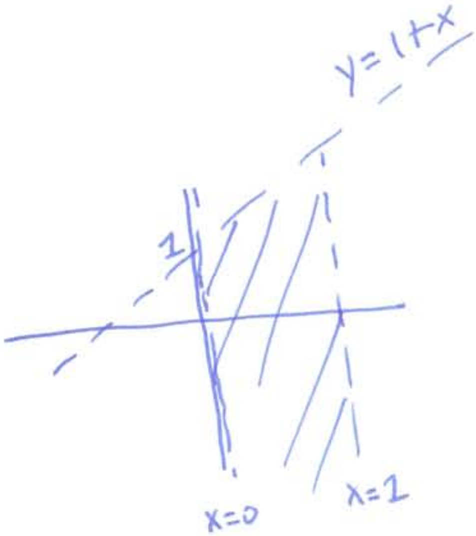
$$= \int_0^1 \left[yx + \frac{y^2}{2}(1+x) \right]_{y=0}^1 dx$$

$$= \int_0^1 \left(x + \frac{1}{2} + \frac{1}{2}x \right) dx$$

$$= \left[\frac{x^2}{2}\left(1+\frac{1}{2}\right) + \frac{1}{2}x \right]_0^1 = \frac{1}{2}\cdot\frac{3}{2} + \frac{1}{2}$$

$$= 5/4$$

2)



$$\int_{x=0}^1 \int_{y=-\infty}^{1+x} x e^y \, dy \, dx$$

$$= \int_0^1 x \left(e^y \Big|_{-\infty}^{1+x} \right) dx$$

$$= \int_0^1 x e^{1+x} dx$$

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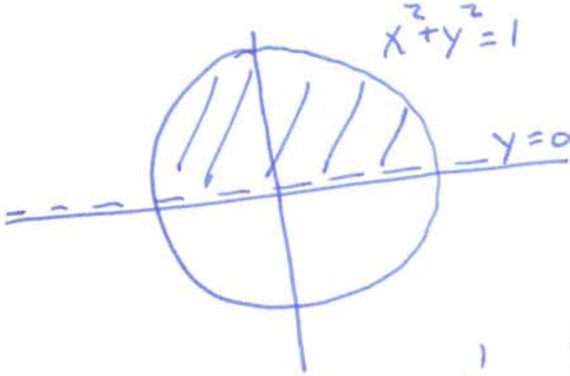
$$= e\left[x e^x \Big|_0^1 - \int_0^1 e^x dx \right]$$

$$= e\left[e - (e-1) \right] = e$$

3)

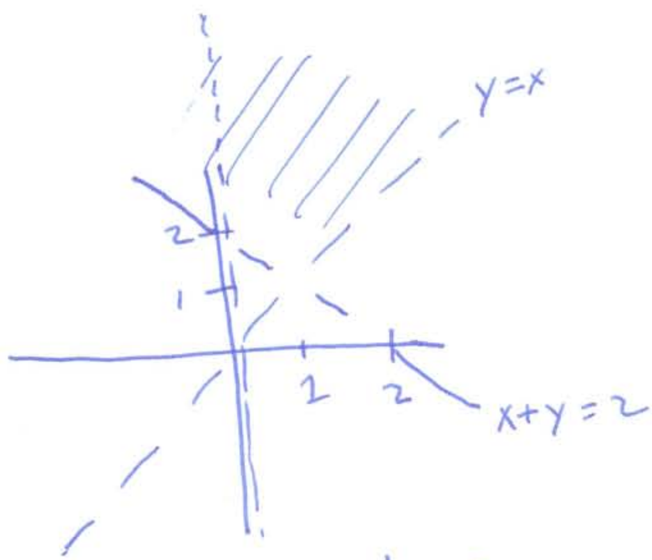


$$\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}} y\,dy\,dx = \int_{-1}^{1} \frac{1-x^2}{2}\,dx$$

$$= \left(\frac{x}{2} - \frac{x^3}{6}\right)\Big|_{-1}^{1}$$

$$= \frac{2}{3}$$

4)



$$\iint_D e^{-(x+y)}\,dx\,dy$$

$$= \int_{x=0}^{1}\int_{y=2-x}^{\infty} e^{-(x+y)}\,dy\,dx + \int_{x=1}^{\infty}\int_{x}^{\infty} e^{-(x+y)}\,dy\,dx$$

$$= \int_0^1 e^{-x}\left(\left[-e^{-y}\right]_{2-x}^{\infty}\right)dx + \int e^{-x}\left(\left[-e^{-y}\right]_{x}^{\infty}\right)dx$$

$$= \int_0^1 e^{-x} e^{x-2}\,dx + \int_1^{\infty} e^{-x}(e^{-x})\,dx$$

$$= e^{-2} + \frac{e^{-2x}}{-2}\Big|_1^{\infty} = \frac{3}{2}e^{-2}$$

5)

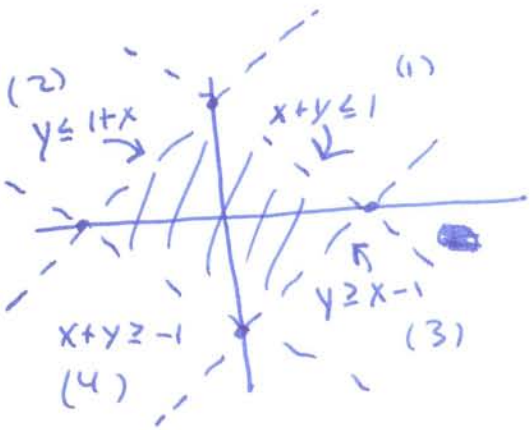
$$D = \{ (x,y) : |x| + |y| \le 1 \}$$

$|x| + |y| \le 1$

$|x| + y \le 1$ or $|x| + -y \le 1$

$x + y \le 1$ (1) or $-x + y \le 1$ (2)

$x + -y \le 1$ (3) or $-x - y \le 1$ (4) $= x + y \ge -1$



D is a square with side length $\sqrt{2}$.

$$\iint_D 2 \, dx\,dy = 2 \cdot \text{Area}(D) = 2 \cdot 2 = 4$$