AM 165 - HMWK #4

$\boxed{3.53}$ 73% express dissatisfaction. You conduct telephone poll.
What is prob. dist. for Y, the # of calls until the 1st person is found who IS satisfied

Soln: "Success" = satisfied person

$$P(Y=n) = Pr(1^{st} \text{ success on } n\text{th trial}) = (.27)(.73)^{n-1}$$

or simply $Y \sim Geo(.27)$

$\boxed{3.55}$ a) $Y \sim Geo(p)$

Show for $a > 0$ $\quad P(Y > a) = q^a$

Soln $P(Y > a) = \sum_{i=a+1}^{\infty} P(Y=i)$ where $P(Y=i) = pq^{i-1}$

$$= \sum_{i=a+1}^{\infty} pq^{i-1} \qquad \text{Let } J = i-a \quad \text{so} \quad J+a = i$$

$$= \sum_{J=\phi}^{\infty} p \cdot q^{J+a-1}$$

$$= q^a \sum_{J=1}^{\infty} pq^{J-1}$$

$$= q^a \qquad \text{since } \sum_{J=1}^{\infty} pq^{J-1} = 1$$

(its a geometric distribution so Sum=1

$\boxed{3.61}$ Find the Prob. dist. of Y = # people you need to question until a "yes" response

80% truthfully answer no

of the 20% who should answer Yes, 70% lie

Soln: $P(\text{Yes}) = P(\text{Yes} \mid \text{should answer Yes})\, P(\text{should answer Yes})$

$$= .3 \cdot .2$$

$$= .06$$

so $Y \sim Geo(.06)$ P(NO) found similarly

or $Pr(Y=n) = (.06)(.94)^{n-1}$

$\boxed{3.62}$ Given that B threw the first 6, what is Prob. B throws a 6 on her 2^{nd} toss (4^{th} overall)

Soln: $P(B \text{ gets } 6 \text{ on } 2^{nd} \text{ toss} \mid B \text{ throws } 6 \text{ first})$

$$= \frac{P(B \text{ gets } 6 \text{ on } 2^{nd} \text{ toss And } B \text{ throws } 6 \text{ first})}{P(B \text{ throws } 6 \text{ first})}$$

Numerator $= \left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^3$ 3 failures, then success

$P(B \text{ throws } 6 \text{ first}) = P(Y=2,4,6,\dots) = P(2) + P(4) + P(6) + \dots$

$$= \sum_{i=1}^{\infty} p(2i) = \sum_{i=1}^{\infty} pq^{2i-1} = pq^{-1} \sum_{i=1}^{\infty} (q^2)^i$$

$$= pq^{-1} \cdot \left(\frac{1}{1-q^2} - 1\right)$$

from $\sum_{i=1} (q^2)^i$ $\sum_{i=0} (q^2)^i - (q^2)^0$ $= \frac{1}{1-q^2}$, $= 1$

$$= \frac{p}{q} \cdot \frac{q^2}{1-q^2}$$

$$= \frac{pq}{1-q^2}$$

In our case $p = 1/6$ $q = 5/6$

$$= \frac{5/36}{11/36} = 5/11$$

Hence $P(B \text{ gets } 6 \text{ on } 2^{nd} \text{ toss} \mid B \text{ throws } 6 \text{ first}) = \frac{\left(\frac{1}{6}\right)\left(5/6\right)^3}{5/11} = \frac{275}{1296}$

$\boxed{3.98}$ $Y \sim Poi(7)$

a) Find $P(\text{No more than } 3 \text{ customers}) = p(0) + p(1) + p(2) + p(3)$

$$= e^{-7}\left(\frac{7^0}{0!} + \frac{7^1}{1!} + \frac{7^2}{2!} + \frac{7^3}{3!}\right) = .0818$$

b) At Least 2 customers Arrive

$$P(C \geq 2) = 1 - p(0) - p(1) = 1 - \frac{e^{-7}7^0}{0!} - \frac{e^{-7}7^1}{1!} = .9927$$

c) Exactly 5

$$P(C=5) = \frac{e^{-7}7^5}{5!} = .12772$$

$\boxed{3.102}$ Two entrances $X_1 \sim Poi(3)$ $X_2 \sim Poi(4)$

What is Pr(Total of 3 Cars in One hour)?

Possibilities

Entrance	1	2
number:	3	0
	2	1
	1	2
	0	3

P(3 Cars in 1, 0 Cars in 2) = P(3 Cars in 1) $\cdot$ P(0 Cars in 2)

$$= \frac{e^{-3}3^3}{3!} \cdot \frac{e^{-4}4^0}{0!}$$ Since Independent

Do all 4 cases this way

Answer: $$e^{-3} \cdot e^{-4}\left(\frac{3^3}{3!}\frac{4^0}{0!} + \frac{3^2 4^1}{2!1!} + \frac{3^1 4^2}{1!2!} + \frac{3^0 4^3}{0!3!}\right)$$

$$= e^{-7}\left(9/2 + 18 + 24 + 32/3\right) = \boxed{.0521}$$

$\boxed{3.107}$ $P(\text{Sale}) = .03$

Contact 100 prospects $(=n)$

Pr(At least 1 sale) $= 1 - Pr$(0 sales)

Let X = # Sales. Notice that $X \sim Poi(\lambda = n \cdot p = 3)$

or $X \sim Bin(100, .03)$

Binomial: $1 - P(0) = 1 - \binom{100}{0}(.03)^0(.97)^{100} = .9524$

Poisson $1 - P(0) = 1 - \frac{e^{-3}3^0}{0!} = 1 - e^{-3} \doteq .9502$

$\boxed{4.5}$ Suppose $f(y) = \begin{cases} cy & 0 \le y \le 2 \\ 0 & \text{elsewhere} \end{cases}$

a) Find the value of c: We know $\int_{-\infty}^{\infty} f(y)\,dy = 1$

so $\int_0^2 cy\,dy = 1$

$\Rightarrow$ $cy^2/2 \Big|_0^2 = 1$

$\Rightarrow$ $2c = 1$

$c = 1/2$

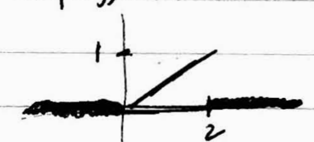
b) $F(y) = \int_{-\infty}^{y} f(x)\,dx$

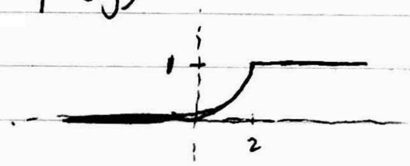
$= \int_0^y x/2\,dx$

$= x^2/4 \Big|_0^y = y^2/4$

$= \begin{cases} 0 & y < 0 \\ y^2/4 & 0 \le y \le 2 \\ 1 & y > 2 \end{cases}$

c) $f(y)$ $F(y)$



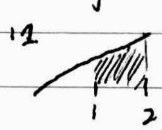


d) $Pr(1 \le Y \le 2) = Pr(Y \le 2) - Pr(Y \le 1) + Pr(Y = 1)$

$= F(2) - F(1) + 0$

$= 1 - 1/4 = 3/4$

e) Use geometry



Area(shaded) = 1 − Area(small triangle)

$= 1 - \frac{1}{2}(1)(1/2)$

$= 3/4$

$\boxed{4.9}$ Mouse in a Maze. Y = time to complete

b = minimum time to traverse maze

$f(y) = \begin{cases} b/y^2 & y \ge b \\ 0 & \text{elsewhere} \end{cases}$

a) Show $f(y)$ has properties of density function

① $f(y) \ge 0$ We know $b > 0$ (time is always positive)

and $y^2 > 0$

$\Rightarrow f(y) \ge 0$

② $\int_{-\infty}^{\infty} f(y)\,dy = 1$ $\int_{-\infty}^{\infty} f(y) = \int_b^{\infty} b/y^2\,dy = -\frac{b}{y}\Big|_b^{\infty} = 0 + \frac{+b}{b}$

$= 1$

b) Find $F(y)$: $F(y) = \int_{-\infty}^{y} f(x)\,dx$

$= \int_b^y b/x^2\,dx = \frac{-b}{x}\Big|_b^y$

$= \frac{-b}{y} + \frac{+b}{b}$

$= \frac{y-b}{y}$

$F(y) = \begin{cases} 0 & y < b \\ \frac{y-b}{y} & y \ge b \end{cases}$

c) Find $P(Y > b+c)$ for some constant C

$$= 1 - P(Y \le b+c) = 1 - F(b+c)$$

$$= 1 - \frac{(b+c)-b}{b+c} = 1 - \frac{c}{b+c} = \frac{b}{b+c}$$

d) if $d > c > 0$ Find $P(Y > b+d \mid Y > b+c)$

$$= \frac{P(Y > b+d \text{ and } Y > b+c)}{P(Y > B+c)} = \frac{P(Y > b+d)}{P(Y > b+c)} = \frac{\frac{b}{b+d}}{\frac{b}{b+c}}$$

$$= \boxed{\frac{b+c}{b+d}}$$

$\boxed{4.24}$ $f(y) = \begin{cases} 2y & 0 \le y \le 1 \\ 0 & \text{else} \end{cases}$

a) $EY = \int_0^1 y \cdot f(y)\,dy = \int_0^1 2y^2\,dy = \frac{2}{3}y^3\Big|_0^1 = 2/3$

$$V(Y) = \underbrace{\int_0^1 y^2 f(y)\,dy}_{EY^2} - \underbrace{(2/3)^2}_{(EY)^2}$$

$$= \int_0^1 2y^3\,dy - (2/3)^2$$

$$= y^4/2\Big|_0^1 - (2/3)^2 = \frac{1}{2} - 4/9 = 1/18$$

b) Profit given by $X = 200Y - 60$

$EX = E[200Y - 60] = 200\,EY - 60 = 200(2/3) - 60 = \$73.33$

$V(X) = V[200Y - 60] = 200^2\,V(Y) + 0 = 40{,}000 \cdot \frac{1}{18} = \$2222.22$

$\boxed{4.37}$ Call time $Y \sim U(0,1)$ Uniform so $f(y) = \begin{cases} 1 & 0 \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$

$$P\left(\tfrac{1}{4} < Y < 1\right) = \int_{1/4}^1 1\,dy = 3/4$$

$\boxed{4.39}$ Cycle time $\sim U(50, 70)$

Find $Pr(\text{time} > 65 \mid \text{time} > 55)$

$f(y) = \frac{1}{70-50} = \frac{1}{20}$ for $50 \le y \le 70$

$\quad\quad 0$ else

$$P(Y>65 \mid Y>55) = \frac{Pr(Y>65 \text{ and } Y>55)}{P(Y>55)} = \frac{Pr(Y>65)}{Pr(Y>55)}$$

$$= \frac{1/4}{3/4} = \boxed{\frac{1}{3}}$$

Think: $P(Y>55) = 1 - Pr(Y<55)$
$= 1 - F(55)$
and $F(y) = \frac{Y-50}{20}$ for $50 \le y \le 70$

$\boxed{4.45}$ Spherical particles Diameters $\sim U(.01, .05)$
Find mean and variance of Volume of particles
Volume $= 4/3 \pi r^3$

if Diameter $\sim U(.01, .05) \Rightarrow$ Radius $\sim U(.005, .025)$

$$\Rightarrow f(r) = \frac{1}{.025-.005} = \frac{1}{.02} = 50 \text{ for } .005 < r < .025$$

$$E(\text{Volume}) = E(4/3 \pi r^3) = 4/3\pi \, E r^3$$

$$= 4/3\pi \int_{.005}^{.025} r^3 \cdot 50 \, dr = \frac{4\pi}{3} \cdot 50 \cdot \frac{r^4}{4}\Big|_{.005}^{.025}$$

$$= \boxed{6.5 \times 10^{-6} \cdot \pi}$$

$$Var(\text{Volume}) = Var(4/3 \pi r^3) = \frac{16\pi^2}{9} Var(r^3) = \frac{16\pi^2}{9}\left[E(r^3)^2 - (Er^3)^2\right]$$

$$E(r^6) = \int_{.005}^{.025} r^6 \cdot 50 \, dr$$

$$50 \cdot \frac{r^7}{7}\Big|_{.005}^{.025} = 4.36 \times 10^{-11}$$

$$(Er^3)^2 = \left[50 \cdot \frac{r^4}{4}\Big|_{.005}^{.025}\right]^2$$

$$= 2.377 \times 10^{-11}$$

$$\Rightarrow Var(\text{Volume}) = \frac{16\pi^2}{9}\left(4.36 \times 10^{-11} - 2.377 \times 10^{-11}\right)$$

$$= \boxed{3.525 \times 10^{-11} \cdot \pi^2}$$