AM 1650: Midterm Exam 2

NAME: **ID:**

1. (**3 pts.**) Suppose X and Y are random variables such that

$$E[X] = 1, \quad \mathrm{Var}[X] = 1, \quad E[Y] = 2, \quad \mathrm{Var}[Y] = 2, \quad \mathrm{Cov}[X, Y] = 1.$$

Compute the following quantities:

(a) $E[X + 2Y]$.

(b) $E[XY]$.

(c) $\mathrm{Var}[X - 2Y + 1]$.

Solution:

(a) $E[X + 2Y] = E[X] + 2E[Y] = 1 + 2 \times 2 = 5$.

(b) Recall $\mathrm{Cov}[X, Y] = E[XY] - E[X]E[Y]$, we have

$$E[XY] = \mathrm{Cov}[X, Y] + E[X]E[Y] = 1 + 1 \times 2 = 3.$$

(c) $\mathrm{Var}[X - 2Y + 1] = \mathrm{Var}[X - 2Y] = \mathrm{Var}[X] + (-2)^2\mathrm{Var}[Y] - 4\mathrm{Cov}[X, Y] = 1 + 4 \times 2 - 4 = 5.$

2. (**5 pts.**) Suppose X and Y are independent standard normal random variables [i.e., $N(0,1)$].

(a) Determine the value of $\mathrm{Cov}[X,Y]$.

(b) Determine the value of $E[X^2Y^2]$.

(c) Find the probability

$$P(-3 \le 3X - 4Y \le 5).$$

The normal table is attached.

Solution:

(a) By the independence of X and Y, $\mathrm{Cov}[X,Y] = 0$.

(b) By independence again,

$$E[X^2Y^2] = E[X^2]E[Y^2].$$

But $E[X^2] = \mathrm{Var}[X] + (E[X])^2 = 1 + 0 = 1$, and similarly $E[Y^2] = 1$. We have

$$E[X^2Y^2] = 1.$$

(c) Note that $3X - 4Y$ is again a normal random variable with mean 0 and variance $(-3)^2 + 4^2 = 5^2$. Thus

$$Z \doteq \frac{3X - 4Y}{5}$$

is standard normal, i.e., $N(0,1)$. It follows that

$$\begin{aligned}
P(-3 \le 3X - 4Y \le 5) &= P\left(\frac{-3}{5} \le \frac{3X - 4Y}{5} \le \frac{5}{5}\right) \\
&= P(-0.6 \le Z \le 1) \\
&= P(Z \ge -0.6) - P(Z > 1) \\
&= 1 - P(Z \ge 0.6) - P(Z > 1) \\
&= 1 - 0.2743 - 0.1587 \\
&= 0.5670.
\end{aligned}$$

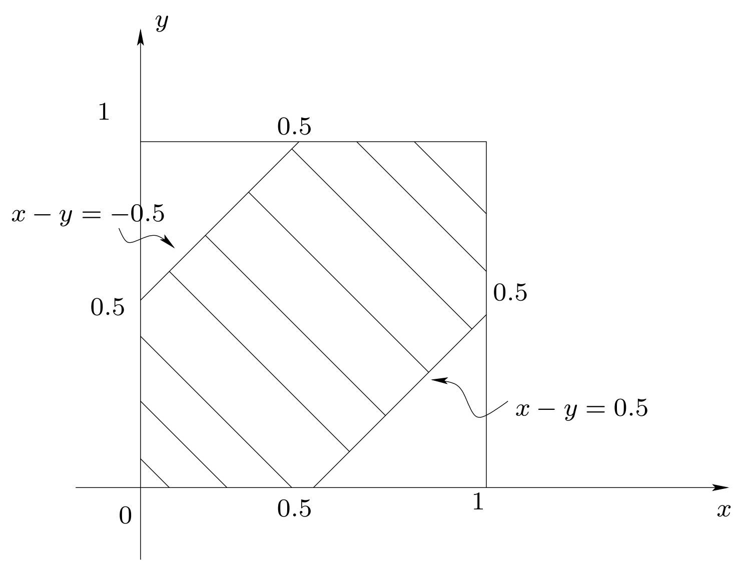
3. (**4 pts.**) Suppose a certain test is to be taken by 3 students independently, and the time required by any student to complete the test has an exponential distribution with mean 1 hour. Suppose that all the three students start the test at 9 A.M., and the first student to complete the test finishes at 10 A.M.. Determine the probability that at least one other student will complete the test by 11 A.M..

Solution: Let X and Y denote the extra time (that is, the time from 10 A.M.) needed for the remaining two students to finish the test, respectively. By assumption X and Y are independent. Moreover, by memoryless property of exponential distributions, X and Y are both exponential distribution with mean 1 hours (i.e., $\lambda = 1$). The question is asking for $P(X \leq 1 \text{ or } Y \leq 1)$. Therefore

$$\begin{aligned} P(X \leq 1 \text{ or } Y \leq 1) &= 1 - P(X > 1, Y > 1) \\ &= 1 - P(X > 1)P(Y > 1) \\ &= 1 - e^{-1} \cdot e^{-1} \\ &= 1 - e^{-2}. \end{aligned}$$

4. (**6 pts.**) John and Betty each randomly and independently selects a point from interval $[0,1]$. What is the probability that the distance between these two points are no more than 0.5?

Solution: Let X be the point John chooses and Y the point Betty chooses. X and Y are independent, and are both uniformly distributed on $[0,1]$. Therefore the joint distribution for X and Y is uniform on square $[0,1]\times[0,1]$. The problem is asking for $P(|X-Y|\le 0.5)$. See the figure below, the shaded region is the event of interest.



Therefore, the probability we are looking for is

$$\frac{\text{area of the shaded region}}{\text{total area}} = \frac{1-\frac{1}{8}-\frac{1}{8}}{1} = \frac{3}{4}.$$

5. (**4 pts.**) Suppose X is an exponential random variable with rate $\lambda = 1$. Find the distribution of $Y = e^{-X}$.

Solution: The range for Y is from 0 to 1. Now fix an arbitrary $t \in [0, 1]$. We have

$$P(Y \le t) = P(e^{-X} \le t) = P(X \ge -\ln t) = e^{-\lambda \cdot (-\ln t)} = t.$$

Taking derivative of $P(Y \le t)$, we can see that Y has density 1 on interval $[0, 1]$. Clearly the density of Y is 0 outside interval $[0, 1]$. Therefore, Y is uniform on interval $[0, 1]$.

6. (**10 pts.**) Suppose (X, Y) has the following joint density:

$$f(x,y) = \begin{cases} cxy & \text{if } 0 \le y \le x \le 1 \\ 0 & \text{otherwise} \end{cases} .$$

(a) Determine the value of c.

(b) Are X and Y independent? [Just a simple explanation. Calculation is not necessary]

(c) Find the marginal density of X.

(d) Find the conditional density of Y given $X = x$, that is, $f(y|x)$.

(e) Compute $P(Y \le 0.25 | X = 0.5)$.

(f) Find the conditional expectation $E[Y|X = x]$.

(g) Identify $E[Y|X]$.

Solution:

(a) Since f is a density, we have

$$\iint_{\mathbb{R}^2} f(x,y)dxdy = 1.$$

In other words,

$$1 = \int_0^1 dx \left[\int_0^x cxydy\right] = \int_0^1 \frac{c}{2}x^3\, dx = \frac{c}{8}.$$

Therefore $c = 8$.

(b) Since the region on which $f(x,y) \ne 0$ is not a rectangle, the joint density function $f(x,y)$ cannot be written as $g(x)h(y)$. Therefore, X and Y are not independent.

(c) The marginal density of X is

$$f_X(x) = \int_{\mathbb{R}} f(x,y)dy.$$

For $x \in [0,1]$,

$$f_X(x) = \int_0^x cxydy = \frac{c}{2}x^3 = 4x^3.$$

and for $x \notin [0,1]$ clearly $f_X(x) = 0$.

(d) For all $x \in [0,1]$, the conditional density is

$$f(y|x) = \frac{f(x,y)}{f_X(x)} = \frac{8xy}{4x^3} = \frac{2y}{x^2}.$$

for all $y \in [0,x]$, and $f(y|x) = 0$ if $y \notin [0,x]$. (Note that for $x \notin [0,1]$, the conditional density can be defined in any way you want.)

(e) Given $X = 0.5$, the condition density of Y is (from previous problem)

$$f(y|X = 0.5) = \begin{cases} \dfrac{2y}{0.5^2} & \text{if } 0 \le y \le 0.5 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 8y & \text{if } 0 \le y \le 0.5 \\ 0 & \text{otherwise} \end{cases}$$

Therefore,

$$P(Y \le 0.25 | X = 0.5) = \int_0^{0.25} 8y dy = \frac{1}{4}.$$

(f) Thanks to part (d),

$$E[Y|X = x] = \int_{\mathbb{R}} y f(y|x) dy = \int_0^x y \cdot \frac{2y}{x^2} dy = \frac{2}{3} x.$$

(g) From part (f),

$$E[Y|X] = \frac{2}{3} X.$$