1. (3 pts.) Suppose $X$ and $Y$ are random variables such that

$$E[X] = 1, \ Var[X] = 1, \ E[Y] = 2, \ Var[Y] = 2, \ \text{Cov}[X,Y] = 1.$$ 

Compute the following quantities:

(a) $E[X + 2Y]$.
(b) $E[XY]$.
(c) $\text{Var}[X - 2Y + 1]$.

Solution:

(a) $E[X + 2Y] = E[X] + 2E[Y] = 1 + 2 \times 2 = 5$.
(b) Recall $\text{Cov}[X,Y] = E[XY] - E[X]E[Y]$, we have


(c) $\text{Var}[X - 2Y + 1] = \text{Var}[X - 2Y] = \text{Var}[X] + (-2)^2 \text{Var}[Y] - 4 \text{Cov}[X,Y] = 1 + 4 \times 2 - 4 = 5$. 
2. (5 pts.) Suppose $X$ and $Y$ are independent standard normal random variables [i.e., $N(0, 1)$].

(a) Determine the value of Cov[$X, Y$].
(b) Determine the value of $E[X^2Y^2]$.
(c) Find the probability $P(-3 \leq 3X - 4Y \leq 5)$.

The normal table is attached.

Solution:

(a) By the independence of $X$ and $Y$, Cov[$X, Y$] = 0.
(b) By independence again,

$$E[X^2Y^2] = E[X^2]E[Y^2].$$

But $E[X^2] = \text{Var}[X] + (E[X])^2 = 1 + 0 = 1$, and similarly $E[Y^2] = 1$. We have

$$E[X^2Y^2] = 1.$$

(c) Note that $3X - 4Y$ is again a normal random variable with mean 0 and variance $(-3)^2 + 4^2 = 5^2$. Thus

$$Z \equiv \frac{3X - 4Y}{5}$$

is standard normal, i.e., $N(0, 1)$. It follows that

$$P(-3 \leq 3X - 4Y \leq 5) = P\left(\frac{-3}{5} \leq \frac{3X - 4Y}{5} \leq \frac{5}{5}\right)$$

$$= P(-0.6 \leq Z \leq 1)$$

$$= P(Z \geq -0.6) - P(Z > 1)$$

$$= 1 - P(Z \geq 0.6) - P(Z > 1)$$

$$= 1 - 0.2743 - 0.1587$$

$$= 0.5670.$$
3. (4 pts.) Suppose a certain test is to be taken by 3 students independently, and the time required by any student to complete the test has an exponential distribution with mean 1 hour. Suppose that all the three students start the test at 9 A.M., and the first student to complete the test finishes at 10 A.M. Determine the probability that at least one other student will complete the test by 11 A.M.

Solution: Let $X$ and $Y$ denote the extra time (that is, the time from 10 A.M.) needed for the remaining two students to finish the test, respectively. By assumption $X$ and $Y$ are independent. Moreover, by memoryless property of exponential distributions, $X$ and $Y$ are both exponential distribution with mean 1 hours (i.e., $\lambda = 1$). The question is asking for $P(X \leq 1 \text{ or } Y \leq 1)$. Therefore

$$P(X \leq 1 \text{ or } Y \leq 1) = 1 - P(X > 1, Y > 1)$$
$$= 1 - P(X > 1)P(Y > 1)$$
$$= 1 - e^{-1} \cdot e^{-1}$$
$$= 1 - e^{-2}.$$
4. (6 pts.) John and Betty each randomly and independently selects a point from interval $[0, 1]$. What is the probability that the distance between these two points are no more than 0.5?

Solution: Let $X$ be the point John chooses and $Y$ the point Betty chooses. $X$ and $Y$ are independent, and are both uniformly distributed on $[0, 1]$. Therefore the joint distribution for $X$ and $Y$ is uniform on square $[0, 1] \times [0, 1]$. The problem is asking for $P(|X - Y| \leq 0.5)$. See the figure below, the shaded region is the event of interest.

Therefore, the probability we are looking for is

$$\frac{\text{area of the shaded region}}{\text{total area}} = \frac{1 - \frac{1}{8} - \frac{1}{8}}{1} = \frac{3}{4}.$$
5. (4 pts.) Suppose \( X \) is an exponential random variable with rate \( \lambda = 1 \). Find the distribution of \( Y = e^{-X} \).

*Solution:* The range for \( Y \) is from 0 to 1. Now fix an arbitrary \( t \in [0, 1] \). We have

\[
P(Y \leq t) = P(e^{-X} \leq t) = P(X \geq -\ln t) = e^{-\lambda(-\ln t)} = t.
\]

Taking derivative of \( P(Y \leq t) \), we can see that \( Y \) has density 1 on interval \([0, 1]\). Clearly the density of \( Y \) is 0 outside interval \([0, 1]\). Therefore, \( Y \) is uniform on interval \([0, 1]\).
6. (10 pts.) Suppose \((X, Y)\) has the following joint density:

\[
f(x, y) = \begin{cases} 
    cxy & \text{if } 0 \leq y \leq x \leq 1 \\
    0 & \text{otherwise}
\end{cases}
\]

(a) Determine the value of \(c\).
(b) Are \(X\) and \(Y\) independent? [Just a simple explanation. Calculation is not necessary]
(c) Find the marginal density of \(X\).
(d) Find the conditional density of \(Y\) given \(X = x\), that is, \(f(y|x)\).
(e) Compute \(P(Y \leq 0.25|X = 0.5)\).
(f) Find the conditional expectation \(E[Y|X = x]\).
(g) Identify \(E[Y|X]\).

Solution:

(a) Since \(f\) is a density, we have

\[
\int_{\mathbb{R}^2} f(x, y) \, dx \, dy = 1.
\]

In other words,

\[
1 = \int_0^1 dx \left[ \int_0^x cxy \, dy \right] = \int_0^1 \frac{c}{2} x^3 \, dx = \frac{c}{8}.
\]

Therefore \(c = 8\).

(b) Since the region on which \(f(x, y) \neq 0\) is not a rectangle, the joint density function \(f(x, y)\) cannot be written as \(g(x)h(y)\). Therefore, \(X\) and \(Y\) are not independent.

(c) The marginal density of \(X\) is

\[
f_X(x) = \int_{\mathbb{R}} f(x, y) \, dy.
\]

For \(x \in [0, 1]\),

\[
f_X(x) = \int_0^x cxy \, dy = \frac{c}{2} x^3 = 4x^3.
\]

and for \(x \notin [0, 1]\) clearly \(f_X(x) = 0\).

(d) For all \(x \in [0, 1]\), the conditional density is

\[
f(y|x) = \frac{f(x, y)}{f_X(x)} = \frac{8xy}{4x^3} = \frac{2y}{x^2}
\]

for all \(y \in [0, x]\), and \(f(y|x) = 0\) if \(y \notin [0, x]\). (Note that for \(x \notin [0, 1]\), the conditional density can be defined in any way you want.)
(e) Given $X = 0.5$, the condition density of $Y$ is (from previous problem)

$$f(y|X = 0.5) = \begin{cases} 
\frac{2y}{0.5^2} & \text{if } 0 \leq y \leq 0.5 \\
0 & \text{otherwise} 
\end{cases} = \begin{cases} 
8y & \text{if } 0 \leq y \leq 0.5 \\
0 & \text{otherwise} 
\end{cases}$$

Therefore,

$$P(Y \leq 0.25|X = 0.5) = \int_0^{0.25} 8y dy = \frac{1}{4}$$

(f) Thanks to part (d),

$$E[Y|X = x] = \int_\mathbb{R} yf(y|x)dy = \int_0^x y \cdot \frac{2y}{x^2} dy = \frac{2}{3}x$$

(g) From part (f),

$$E[Y|X] = \frac{2}{3}X.$$