

AM 1650: Midterm Exam I (Solution)

1. (6 pts) A fair six-sided die is thrown twice.

- (a) Identify the sample space. What is the total number of elements in the sample space?
- (b) Explicitly identify the event

$$E = \{\text{The sum of the face values of these two tosses is 5}\},$$

and calculate $P(E)$.

- (c) Given that the event E happens, what is the probability that the face value of the first toss is less than that of the second toss?

Solution.

- (a) The sample space

$$S = \{(i, j) : 1 \leq i \leq 6, 1 \leq j \leq 6\}.$$

In total $6 \times 6 = 36$ elements.

- (b) The event E is

$$E = \{(1, 4), (2, 3), (3, 2), (4, 1)\},$$

and

$$P(E) = \frac{4}{36} = \frac{1}{9}$$

- (c) This conditional probability is

$$\frac{2}{4} = \frac{1}{2}.$$

2. (4 pts) Among 35 students in a class, 17 earned “A” on the midterm, 14 earned “A” on final exam, and 11 did not earn “A” on either exam. What is the probability that a randomly selected student from this class earned “A” on both exams?

Solution. Define events

$$\begin{aligned} A &= \{\text{students who earned “A” on midterm}\}, \\ B &= \{\text{students who earned “A” on final}\}, \\ C &= \{\text{students who did not earn “A” on either exams}\}. \end{aligned}$$

The problem is asking for $P(A \cap B)$. Use formula

$$P(A \cap B) = P(A) + P(B) - P(A \cup B).$$

Note that $A \cup B = \bar{C}$, and

$$P(A) = \frac{17}{35}, \quad P(B) = \frac{14}{35}, \quad P(A \cup B) = 1 - P(C) = 1 - \frac{11}{35} = \frac{24}{35}.$$

Thus

$$P(A \cap B) = \frac{7}{35} = 0.2$$

3. (**3 pts**) Five cards are drawn randomly from a deck of 52 cards. What is the probability that at least one of the cards is a King?

Solution. There are in total

$$\binom{52}{5}$$

number of ways to choose five cards. Among these,

$$\binom{52-4}{5} = \binom{48}{5}$$

number of ways will NOT contain King. So the probability is

$$1 - \frac{\binom{48}{5}}{\binom{52}{5}}$$

4. (**6 pts**) The probability of a randomly selected person is suffering from a certain disease DIS is 0.001. The test to detect DIS has a correct detection rate of one, that is, for a person with DIS, the test will always detect the disease. But for a person without DIS, the test has probability 0.005 to give false detection of the disease. Given that you test positive for DIS, what is the probability that you actually have it?

Solution. Let “+” be the event of tested positive. Use Bayes formula (use tree method)

$$P(DIS|“+”) = \frac{P(DIS \cap “+”)}{P(“+”)} = \frac{0.001 \times 1}{0.999 \times 0.005 + 0.001 \times 1} \approx \frac{1}{6}.$$

5. (7 pts) A pair of coins are tossed simultaneously. Each coin has probability p to be heads (H).

(a) What is the probability that the outcomes of the two coins are different?

(b) Suppose this pair of coins are tossed n times, and let

X = The total number of tosses where the outcomes of the two coins are different.

What is the distribution of X ?

(c) Continue part (b). Suppose that you gain \$1 for each toss where the outcomes of the two coins are different, and lose \$1 for each toss where the outcomes of the two coins are the same. Let

Y = The total amount of gain.

Express Y in terms of X . And compute the expected value and variance of Y .

Solution.

(a) The probability is

$$P(\text{HT}) + P(\text{TH}) = 2p(1 - p)$$

(b) X is binomial $B(n, 2p(1 - p))$.

(c)

$$Y = X - (n - X) = 2X - n.$$

Therefore

$$E[Y] = E[2X - n] = 2E[X] - n = 4np(1 - p) - n$$

and

$$\text{Var}[Y] = \text{Var}[2X - n] = 4\text{Var}[X] = 4n \cdot 2p(1 - p) \cdot [1 - 2p(1 - p)].$$

6. (4 pts) Suppose that the proportion of colorblind people in a large population is 0.005. Use Poisson approximation to calculate the probability that there will be at most one colorblind person in a randomly chosen group of 600 people.

Solution. Let X = number of colorblind people in the sample. Then

X is binomial $B(600, 0.005)$, which is approximately $\text{Poisson}(600 \times 0.005) = \text{Poisson}(3)$.

The probability of interest is

$$P(X \leq 1) = P(X = 0) + P(X = 1) = e^{-3} + e^{-3} \cdot 3 = 4e^{-3}.$$

7. (6 pts) Consider a coin with probability p to be heads.

- (a) On average, how many tosses needed to obtain one heads?
- (b) On average, how many tosses needed to obtain two heads?
- (c) Determine the probability that the first heads will appear on the even-numbered tosses.

Solution. The number of tossed needed to obtain a heads, say X , is geometrically distribution with probability of success p .

- (a) $1/p$. This is the expected value of X .
- (b) $2/p$. Twice the expected value of X .
- (c) The probability is

$$\begin{aligned} & P(X = 2) + P(X = 4) + P(X = 6) + \dots \\ &= p(1-p) + p(1-p)^3 + p(1-p)^5 + \dots \\ &= p(1-p) [1 + (1-p)^2 + (1-p)^4 + \dots] \\ &= p(1-p) [1 + x + x^2 + \dots] \quad (\text{if you set } x = (1-p)^2) \\ &= p(1-p) \frac{1}{1-x} \\ &= p(1-p) \frac{1}{1-(1-p)^2} \\ &= \frac{1-p}{2-p} \end{aligned}$$