Simplex Method for Linear Programming

Designed in 1947 by Dantzig, the *Simplex Algorithm* was the method of choice to solve linear programs for decades. Remarkably fast in practice, it had no contenders for "best practical linear programming algorithm" until the appearance of Karmarkar's Algorithm in 1984 and the more recent interior-point methods.

1 The intuitive idea behind Simplex Algorithm

Conceptually, the Simplex Algorithm could hardly be simpler. It is based on the following two facts:

(1) There always exists a vertex (i.e. corner point) of the feasible region that is an optimal solution to the LP.

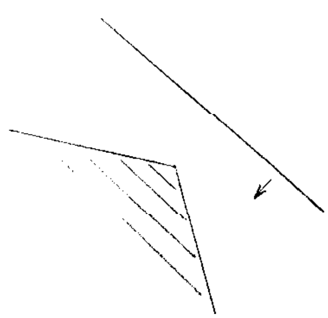
(2) A vertex is an optimal solution if there is no better *neighboring* vertex.

It follows from (1) that LP is a *finite* problem – a feasible region can only have finitely many vertices! Therefore, we need only scan these vertices of the feasible region in order to find an optimal solution.

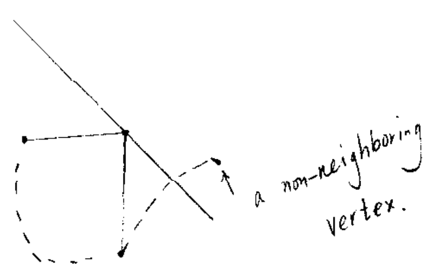
But *not* all of them! That would take too long. Now fact (2) comes to the rescue, and defines the algorithm – the Simplex Algorithm starts from an arbitrary vertex of the feasible region, and compare it with its neighboring vertices. If no neighboring vertex is better, fact (2) tells us that we already the optimality. Otherwise, the algorithm moves to a better neighboring vertex, and repeats until an optimal vertex is reached. In other words, Simplex Algorithm is nothing more than an *orderly* way of scanning vertices.

Remark: We should intuitively explain (1) and (2) by graphs. You might want to double-check these claims for all the previous examples.

(1)



(2)



Remark: Pathological and artificial examples have been constructed to show that it is possible for the Simplex Algorithm *not* to end, and get stuck in a vertex infinitely. Such a phenomenon is called *cycling*. However, one can slightly modify the algorithm so that the optimal solution can always be reached in finitely many steps. This only requires very little additional work.

In practice, this cycling problem is not even an issue. One reason is that it is so rare. The second reason is that even if potential cycling exists in a problem, the computer round-error usually makes the cycling impossible.

2 From geometry to algebra

It follows from the above discussion that the Simplex Algorithm takes the following steps:

(1) *Initialization:* Find a starting vertex in the feasible region.

(2) *Optimality test:* Compare the vertex with its neighboring vertices. If no neighboring vertex is better, the current vertex is optimal, and the algorithm stops. Otherwise,

(3) *Iteration:* find a better neighboring vertex and goes back to (2).

All these terminologies of "vertex", "neighboring vertex" have very clear geometric interpretations. However, since we want to solve the problem using computers, these words mean nothing unless we can associate the geometry with the algebra — computer cannot understand geometry, but can perform algebra operation fast.

We should try to do the translation for an LP problem in its canonical form:

$$\text{Maximize} \qquad Z = c^T x$$

subject to the constraints

$$Ax - b, \quad x \geq 0.$$

Here we use notation

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}, \quad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \quad c = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$$

We have n decision variables and m linear equations (assume $n \geq m$, otherwise the system is over-determined). Sometimes we call $n - m$ the *degree of freedom.*

2.1 Vertex vs. Basic Feasible Solution (BFS)

Arbitrarily choose $n - m$ decision variables (**non-basic variables**), and set each of these variables 0. The m linear equations now becomes m equations for the remaining m decision variables (**basic variables**). Solve for these m basic variables and we obtain a solution to the m linear solutions. It is called the **basic solution**. This solution is feasible if and only if its components are all non-negative, in which case it is said to be a **basic feasible solution** (i.e. BFS). The reason for this definition is due to the following result:

Theorem: A solution x is a vertex of the feasible region if and only if it is a BFS.

We should verify this theorem by computing some examples. Note the definition of BFS is independent of the objective function Z, hence Z is omitted in the following examples.

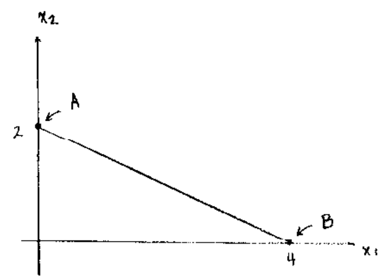
Example: Suppose the constraints are

$$x_1 + 2x_2 = 4, \quad x_1 \geq 0, \quad x_2 \geq 0.$$

Find all the basic feasible solutions (BFS).

Solution: In this case $n = 2$, $m = 1$, and the degree of freedom is $n - m = 1$.

Non-Basic Variables	Basic Variables	BFS
$x_1 = 0$	$x_2 = 2$	$A = (0, 2)$
$x_2 = 0$	$x_1 = 4$	$B = (4, 0)$



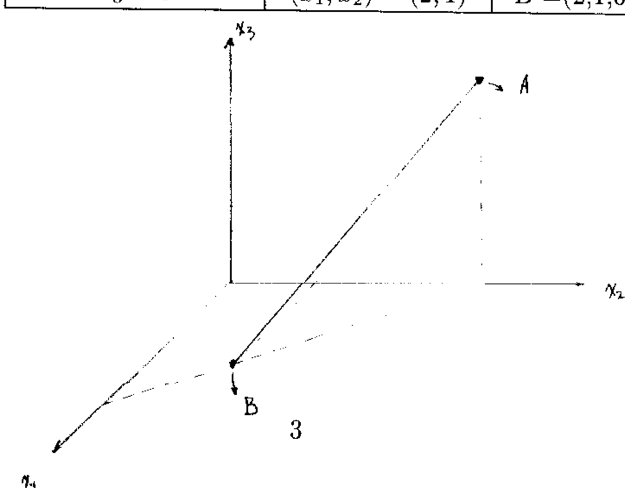
Example: Suppose the constraints are

$$\begin{aligned} x_1 + x_2 &= 3 \\ -x_2 + x_3 &= -1 \end{aligned}$$

and $x_1 \geq 0$, $x_2 \geq 0$. Find all the basic feasible solutions (BFS).

In this case $n = 3$, $m = 2$, and the degree of freedom is $n - m = 1$.

Non-Basic Variables	Basic Variables	BFS
$x_1 = 0$	$(x_2, x_3) = (3, 2)$	$A = (0, 3, 2)$
$x_2 = 0$	$(x_1, x_3) = (3, -1)$	non-feasible
$x_3 = 0$	$(x_1, x_2) = (2, 1)$	B =(2,1,0)



Remark: Sometimes when we solve for basic solutions, the resulting m linear equations might have infinitely many solutions, or no solution. Either way, there is *no* corresponding BFS. See the following examples:

Example: Suppose the constraints are

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 2 \\ 2x_1 + 4x_2 + 6x_3 &= b \end{aligned}$$

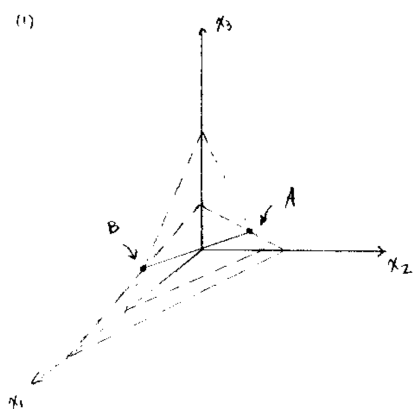
and $x_1 \geq 0$, $x_2 \geq 0$. Find all the basic feasible solutions (BFS) when $b = 6$ and $b = 4$. In this case $n = 3$, $m = 2$, and the degree of freedom is $n - m = 1$.

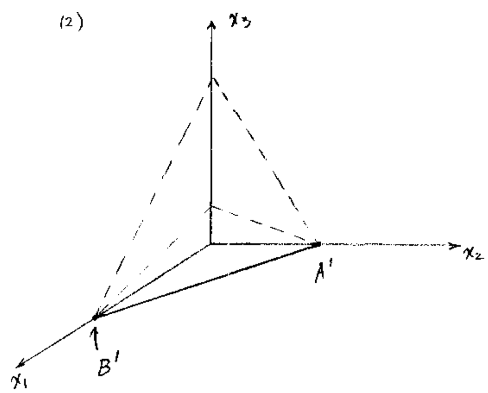
(1) $b = 6$:

Non-Basic Variables	Basic Variables	BFS
$x_1 = 0$	$(x_2, x_3) = (\frac{3}{4}, \frac{1}{2})$	$A = (0, \frac{3}{4}, \frac{1}{2})$
$x_2 = 0$	$(x_1, x_3) = (\frac{3}{2}, \frac{1}{2})$	$B = (\frac{3}{2}, 0, \frac{1}{2})$
$x_3 = 0$	(x_1, x_2) has no solution	NA

(2) $b = 4$:

Non-Basic Variables	Basic Variables	BFS
$x_1 = 0$	$(x_2, x_3) = (1, 0)$	$A' = (0, 1, 0)$
$x_2 = 0$	$(x_1, x_3) = (2, 0)$	$B' = (2, 0, 0)$
$x_3 = 0$	(x_1, x_2) has infinitely many solutions	NA





2.2 Adjacent Vertices vs. Basic Variables

We know from the above discussion that a vertex is essentially a BFS. The adjacency of vertices can be describe by basic variable.

Theorem: Two vertices are adjacent if and only if their corresponding BFS share all but one basic variables, or equivalently, all but one non-basic variables.

For example, if one BFS uses (x_1, x_3, x_5) as the basic variables, and another use (x_3, x_5, x_6) as its basic variables, then these two BFS (or, vertices) are adjacent. Note in this case, the common variables (x_3, x_5) might take different numeric value for these two BFS.

We should give examples later to verify this theorem.

2.3 Why we need simplex algorithm

¿From the above discussion, an LP in canonical form with m linear constraints and n decision variables, may have a basic feasible solution for every choice of $n-m$ non-basic variables (or equivalently, m basic variables). Therefore, the number of vertices (or, BFS) of the feasible region might as well be the same as that of the choices of $n-m$ non-basic variables. How many such choices? From combinatorics, the number of choices are

$$\binom{n}{n-m} = \binom{n}{m} = \frac{n!}{m!(n-m)!}.$$

Even though this is a finite number, it could be really large, even if the n, m are relatively small. For example, take $n = 20$, $m = 10$, we have

$$\binom{n}{n-m} = \binom{20}{10} = 184,756.$$

Imagine an LP with hundreds of constraints. In principle, we can scan all these BFS and obtain the optimum. But as we have already seen, for a medium-sized LP, the number of BFS are already overwhelmingly large so that it would not be efficient to do so.

However, the Simplex Algorithm turns out to be much more efficient. In practice, the Simplex Algorithm usually finds the optimal solution after scanning some $4m$ to $6m$ BFS, and very rarely beyond $10m$ BFS. As n grows, the number of BFS scanned grows very slowly, perhaps logarithmically in n.

Remark: Pathological examples can be constructed such that the Simplex Algorithm will not be efficient. But it is so artificial and rare, we should not be bothered in practice.

2.4 LP in standard form: slack variables

Suppose that the LP is in standard form:

$$\text{Maximize} \qquad Z = c^T x$$

subject to the constraints

$$Ax \le b, \quad x \ge 0.$$

In order to compute the vertices, we just need to add in the slack variables so as to turn the LP into canonical form, whose vertices (or, BFS) can be computed as before. Then we can ignore the slack variables to obtain the vertices for the original problem. See the example below for the illustration.

Example: Suppose the constraints are

$$\begin{aligned} 2x_1 + x_2 &\le 8 \\ x_3 &\le 10 \end{aligned}$$

and $x_i \geq 0$, $i = 1, 2, 3$. Find all the basic feasible solutions (BFS).

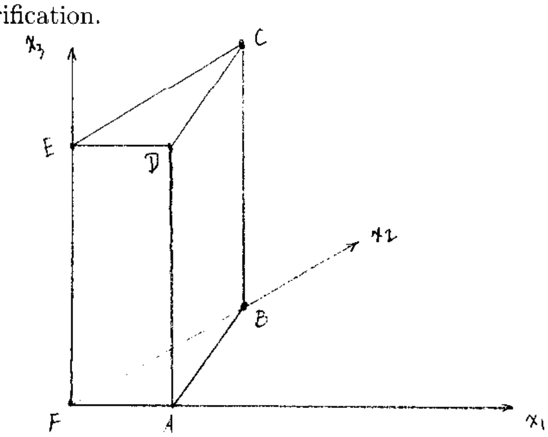
This is an LP in standard form, and we should add slack variable s_1, s_2 so as to change the constraints into canonical form

$$\begin{aligned} 2x_1 + x_2 + s_1 &= 8 \\ x_3 + s_2 &= 10, \end{aligned}$$

in which case, $n = 5$, $m = 2$ and the degree of freedom $n - m = 3$.

Non-Basic Variables	Basic Variables	BFS
$(x_1, x_2, x_3) = 0$	$(s_1, s_2) = (8, 10)$	$A = (0, 0, 0)$
$(x_1, x_2, s_1) = 0$	(x_3, s_1)	NA
$(x_1, x_2, s_2) = 0$	$(x_3, s_1) = (10, 8)$	$B = (0, 0, 10)$
$(x_1, s_1, x_3) = 0$	$(x_2, s_2) = (8, 10)$	$C = (0, 8, 0)$
$(x_1, s_2, x_3) = 0$	(x_2, s_1)	NA
$(s_1, x_2, x_3) = 0$	$(x_1, s_2) = (4, 10)$	$D = (4, 0, 0)$
$(s_2, x_2, x_3) = 0$	(x_1, s_1)	NA
$(x_1, s_1, s_2) = 0$	$(x_2, x_3) = (8, 10)$	$E = (0, 8, 10)$
$(s_1, x_2, s_2) = 0$	$(x_1, x_3) = (4, 10)$	$F = (4, 0, 10)$
$(s_1, s_2, x_3) = 0$	(x_1, x_2)	NA

Below is the graphic verification.



We can also verify the claim that two vertices (BFS) are adjacent: for example, point A and D are adjacent since they share the same basic variable s_2, and point A and F are not adjacent since they share no common basic variables.

3 Simplex algorithm: examples

In this section, we should give two detailed examples of simplex algorithm. Each step will be explained in detail.

Example: Solve the LP in standard form:

$$\text{Maximize} \qquad Z = 3x_1 + 2x_2$$

subject to constraints

$$\begin{aligned} 2x_1 + x_2 &\leq 10 \\ x_1 + x_2 &\leq 8 \\ x_1 &\leq 4 \end{aligned}$$

and $x_1 \geq 0, x_2 \geq 0$.

Solution: We should first write the problem in its canonical form by introducing slack variables.

$$\begin{aligned} 2x_1 + x_2 + s_1 &= 10 \\ x_1 + x_2 + s_2 &= 8 \\ x_1 + s_3 &= 4 \end{aligned}$$

In this case, $n = 5$ and $m = 3$ and degree of freedom $n - m = 2$.

Initialization (finding a starting BFS or a starting vertex): It is easy in this case – just set

$$NBV = (x_1, x_2) = (0, 0), \qquad BV = (s_1, s_2, s_3) = (10, 8, 4).$$

Optimality test (is the current BFS or vertex optimal): The current BFS will be optimal if and only if it is better than every neighboring vertex (or every BFS share all but one basic variables). To do this, we try to determine whether there is any way Z can be increased by increasing one of the non-basic variables from its current value zero while all other non-basic variables remain zero (while the values of the basic variables are adjusted to continue satisfying the system of equations).

In this case, the objective function is $Z = 3x_1 + 2x_2$, and Z take value 0 at $(0, 0)$. It is easy to see that no matter we increase x_1 (while holding $x_2 = 0$) or increase x_2 (while holding $x_1 = 0$), we are going to increase Z since all the coefficients are positive. We conclude that the current BFS is not optimal.

Moving to a neighboring BFS (or vertex): Two neighboring BFS share all but one basic variables. In other words, one of the variable (x_1, x_2) is going to become a basic variable (**entering basic variable**), and one of (s_1, s_2, s_3) is going to become a non-basic variable (**leaving basic variable**).

(a) *Determining the entering basic variable:* Choosing an entering basic variable amounts to choosing a non-basic variable to increase from zero. Note $Z = 3x_1 + 2x_2$. Z is increased by 3 if we increase x_1 by 1, and by 2 if we increase x_2 by 1. Therefore, we choose x_1 as the entering basic variable.

(b) *Determining how large the entering basic variable can be:* We cannot increase the entering variable x_1 arbitrarily, since it may cause some variables to become negative. What is the largest possible value that x_1 can attain? Note x_2 is held at zero. Hence

$$\begin{aligned} s_1 = 10 - 2x_1 \geq 0 &\quad\Rightarrow\quad x_1 \text{ cannot exceed } \tfrac{10}{2} = 5 \\ s_2 = 8 - x_1 \geq 0 &\quad\Rightarrow\quad x_1 \text{ cannot exceed } \tfrac{8}{1} = 8 \\ s_3 = 4 - x_1 \geq 0 &\quad\Rightarrow\quad x_1 \text{ cannot exceed } \tfrac{4}{1} = 4\ . \end{aligned}$$

It follows that the largest x_1 can be is the 4.

(c) *Determining the leaving basic variable:* When x_1 takes value 4, s_3 become 0. Therefore s_3 is the leaving basic variable.

Therefore, the neighboring vertex we select is

$$NBV = (s_3, x_2), \qquad BV = (s_1, s_2, x_1).$$

Pivoting (solving for the new BFS): Recall that we have

$$\begin{array}{cccccccccl} Z & -3x_1 & -2x_2 & & & & = & 0 & (0) \\ & 2x_1 & +x_2 & +s_1 & & & = & 10 & (1) \\ & x_1 & +x_2 & & +s_2 & & = & 8 & (2) \\ & x_1 & & & & +s_3 & = & 4 & (3) \end{array}$$

The goal is to solve for the BFS, and it is going to be achieved by *Gaussian elimination.* We end up with

$$\begin{array}{cccccccccl} Z & & -2x_2 & & & +3s_3 & = & 12 & (0) \\ & & x_2 & +s_1 & & -2s_3 & = & 2 & (1) \\ & & x_2 & & +s_2 & -s_3 & = & 4 & (2) \\ & x_1 & & & & +s_3 & = & 4 & (3) \end{array}$$

In other words, **each basic variable has been eliminated from all but one row (*its row*) and has coefficient +1 in that row**. The Gaussian elimination always starts with the row of the leaving basic variable (or the row that achieve the minimal ratio in the preceding step), or the entering basic variable's row is always the row of the leaving basic variable, or the entering basic variable's row is always the row that achieves the minimal ratio in the preceding step.

What we have is that the BFS is

$$NBV = (s_3, x_2) = (0, 0), \qquad BV = (s_1, s_2, x_1) = (2, 4, 4),$$

and

$$Z = 12 + 2x_2 - 3s_3,$$

while taking value 12 at this BFS.

Iteration: The above BFS is not optimal, since we can increase x_2, which increases Z. We do not want to increase s_3, which decreases the value of Z. So the entering basic variable is x_2. How large can x_2 be? Note $s_3 = 0$, we have

$$\begin{array}{rrrcl}
 & x_2 + s_1 & = 2 & \Rightarrow & x_2 \le \frac{2}{1} = 2 \\
 & x_2 \quad + s_2 & = 4 & \Rightarrow & x_2 \le \frac{4}{1} = 4 \\
x_1 & & = 4 & \Rightarrow & \text{no upperbound for } x_2
\end{array}$$

The maximum of x_2 is therefore 2 achieved at row (1), and the leaving basic variable is the (original) basic variable in row (1), i.e. s_1. In other words

$$BV = (s_1, s_3), \qquad NBV - (s_2, x_1, x_2).$$

Gaussian elimination yields, starting from row (1) yields,

$$\begin{array}{cccccccccc}
Z & & & +2s_1 & & -s_3 & = & 16 & (0) \\
 & & x_2 & +s_1 & & -2s_3 & = & 2 & (1) \\
 & & & -s_1 & +s_2 & +s_3 & = & 2 & (2) \\
 & x_1 & & & & +s_3 & = & 4 & (3)
\end{array}$$

or the new BFS is

$$BV = (s_1, s_3) = (0, 0), \qquad NBV = (s_2, x_1, x_2) = (2, 4, 2).$$

The value of Z is

$$Z = 16 - 2s_1 + s_3,$$

and it attains value 16 at this BFS.

This BFS is still not optimal, and clearly s_3 will be the entering basic variable. Note $s_1 = 0$, we have

$$\begin{array}{rrcl}
x_2 \quad -2s_3 = 2 & \Rightarrow & \text{no upper bound for } s_3 \\
s_2 \quad +s_3 = 2 & \Rightarrow & s_3 \le \frac{2}{1} = 2 \\
x_1 \quad +s_3 = 4 & \Rightarrow & s_3 \le \frac{4}{1} = 4.
\end{array}$$

The maximum of s_3 is therefore 2 achieved at row (2), and the leaving basic variable is the (original) basic variable in row (1), i.e. s_2. In other words

$$BV = (s_1, s_2), \qquad NBV = (x_1, x_2, s_3).$$

Gaussian elimination yields, starting from row (1) yields,

$$\begin{array}{cccccccccc}
Z & & & +s_1 & +s_2 & & = & 18 & (0) \\
 & & x_2 & -s_1 & +2s_2 & & = & 6 & (1) \\
 & & & -s_1 & +s_2 & +s_3 & = & 2 & (2) \\
 & x_1 & & +s_1 & -s_2 & & = & 2 & (3)
\end{array}$$

or the new BFS is

$$BV = (s_1, s_2) = (0, 0), \qquad NBV = (x_1, x_2, s_3) = (2, 6, 2).$$

The value of Z is

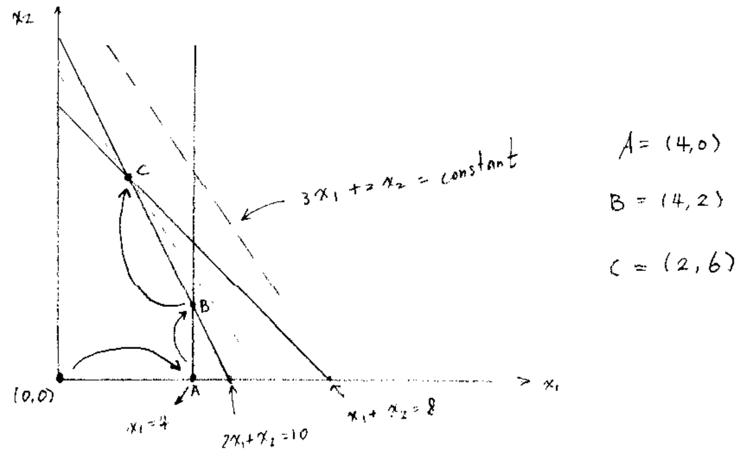
$$Z = 18 - s_1 - s_3$$

and it attains value 18 at this BFS.

This BFS turns out to be optimal; indeed, any increment in the non-basic variable will decrease the value of Z. Hence

$$\max Z = 18, \qquad \text{achieved at} \quad (x_1^*, x_2^*) = (2, 6).$$

Below is the graphical description of solution.



Example: Solve the LP in standard form:

$$\text{Maximize} \qquad Z = 60x_1 + 30x_2 + 20x_3$$

subject to constraints

$$\begin{array}{rcrcrcl} 8x_1 & + & 6x_2 & + & x_3 & \leq & 48 \\ 4x_1 & + & 2x_2 & + & 1.5x_3 & \leq & 20 \\ 2x_1 & + & 1.5x_2 & + & 0.5x_3 & \leq & 8 \\ & & x_2 & & & \leq & 5 \end{array}$$

and $x_i \geq 0$, $i = 1, 2, 3$.

Solution: We should first introduce slack variables and convert the LP into canonical form.

$$\begin{array}{ccrcrcrcccccccccl} Z & - & 60x_1 & - & 30x_2 & - & 20x_3 & & & & & & & & & = & 0 \qquad (0) \\ & & 8x_1 & + & 6x_2 & + & x_3 & + & s_1 & & & & & & & = & 48 \qquad (1) \\ & & 4x_1 & + & 2x_2 & + & 1.5x_3 & & & + & s_2 & & & & & = & 20 \qquad (2) \\ & & 2x_1 & + & 1.5x_2 & + & 0.5x_3 & & & & & + & s_3 & & & = & 8 \qquad (3) \\ & & & & x_2 & & & & & & & & & + & s_4 & = & 5 \qquad (4) \end{array}$$

and $x_i \geq 0$, $s_j \geq 0$. Here $n - 7, m - 4$ and degree of freedom $n - m = 3$.

<u>Iteration 1:</u> The starting BFS is

$$NBV = (x_1, x_2, x_3) = (0, 0, 0), \qquad BV = (s_1, s_2, s_3, s_4) = (48, 20, 8, 5)$$

and Z attain value 0 is not optimal.

- *Entering basic variable:* x_1, the variable with the largest negative coefficient in Row (0).
- *Leaving basic variable:* Computing ratios

$$\text{Row (1): } \frac{48}{8} = 6; \quad \text{Row (2): } \frac{20}{4} = 5; \quad \text{Row (3): } \frac{8}{2} = 4;$$

$$\text{Row (4): no upper bound, since the coefficient of } x_1 \text{ is } 0.$$

  So Row (3) has the minimal ratio, the basic variable of Row (3), which is s_3, is the leaving basic variable.

After Gaussian elimination (starting with Row (3), and eliminate all the entering basic variable x_1 in other rows), we obtain

$$\begin{array}{lllllllllllll}
Z & & + & 15x_2 & - & 5x_3 & & & & + & 30s_3 & & = & 240 & (0) \\
 & & & & - & x_3 & + & s_1 & & - & 4s_3 & & = & 16 & (1) \\
 & & - & x_2 & + & 0.5x_3 & & & + \; s_2 & - & 2s_3 & & = & 4 & (2) \\
 & x_1 & + & 0.75x_2 & + & 0.25x_3 & & & & + & 0.5s_3 & & = & 4 & (3) \\
 & & & x_2 & & & & & & & & + \; s_4 & = & 5 & (4)
\end{array}$$

Iteration 2: We obtain a new BFS

$$NBV = (x_2, x_3, s_3) = (0, 0, 0), \qquad BV = (s_1, s_2, s_4, x_1) = (16, 4, 5, 4).$$

and Z attain value 240, which is not optimal (why?)

- *Entering basic variable:* x_3, the variable with the largest negative coefficient in Row (0).
- *Leaving basic variable:* Computing ratios

$$\text{Row (1): no upper bound, since the coefficient of } x_3 \text{ is negative}$$

$$\text{Row (2): } \frac{4}{0.5} = 8; \quad \text{Row (3): } \frac{4}{0.25} = 16;$$

$$\text{Row (4): no upper bound, since the coefficient of } x_1 \text{ is } 0.$$

  So Row (2) has the minimal ratio, and s_2 is the leaving basic variable.

After Gaussian elimination (starting with Row (2), and eliminate all x_3 in other rows), we obtain

$$\begin{array}{llllllllllllll}
Z & & + & 5x_2 & & & & & + & 10s_2 & + & 10s_3 & & = & 280 & (0) \\
 & & - & 2x_2 & & & + & s_1 & + & 2s_2 & - & 8s_3 & & = & 24 & (1) \\
 & & - & 2x_2 & + & x_3 & & & + & 2s_2 & - & 4s_3 & & = & 8 & (2) \\
 & x_1 & + & 1.25x_2 & + & & & & - & 0.5s_2 & + & 1.5s_3 & & = & 2 & (3) \\
 & & & x_2 & & & & & & & & & + \; s_4 & = & 5 & (4)
\end{array}$$

Iteration 3: We obtain a new BFS

$$NBV = (x_2, s_3, s_2) = (0, 0, 0), \qquad BV = (s_1, s_4, x_1, x_3) = (24, 5, 2, 8).$$

and Z attains value 280, which is optimal since

$$Z = 280 - 5x_2 - 10s_2 - 10s_3,$$

all the coefficients are negative (or all the coefficients in Row (0) are positive).

4 Simplex algorithm: the tabular form

The simplex algorithm can be better presented in the *tabular form*. Recall all the steps in the preceding two examples – they are all concerned with the coefficients of the variables. We first revisit the following example

Example: Solve the LP in standard form:

$$\text{Maximize} \qquad Z = 3x_1 + 2x_2$$

subject to constraints

$$\begin{aligned} 2x_1 + x_2 &\leq 10 \\ x_1 + x_2 &\leq 8 \\ x_1 &\leq 4 \end{aligned}$$

and $x_1 \geq 0, x_2 \geq 0$.

Solution: The first table is the result after initialization:

Basic Variable	Row	Z	x_1	x_2	s_1	s_2	s_3	RHS	Ratios
Z	(0)	1	-3	-2	0	0	0	0	
s_1	(1)	0	2	1	1	0	0	10	
s_2	(2)	0	1	1	0	1	0	8	
s_3	(3)	0	1	0	0	0	1	4	

The next step is finding the entering and leaving basic variables – get the variable with the largest negative coefficient in row (0) and perform the ratio test.

Basic Variable	Row	Z	x_1	x_2	s_1	s_2	s_3	RHS	Ratios
Z	(0)	1	-3	-2	0	0	0	0	
s_1	(1)	0	2	1	1	0	0	10	$10/2 = 5$
s_2	(2)	0	1	1	0	1	0	8	$8/1 = 8$
s_3	(3)	0	$1*$	0	0	0	1	4	$4/1 = 4 \leftarrow$ min

The next step is the Gaussian elimination – eliminate all the entering basic variable in other rows and make the coefficient of the entering basic variable 1 in its own row. Do not forget to change the basic variables.

Basic Variable	Row	Z	x_1	x_2	s_1	s_2	s_3	RHS	Ratios
Z	(0)	1	0	-2	0	0	3	12	
s_1	(1)	0	0	1	1	0	-2	2	
s_2	(2)	0	0	1	0	1	-1	4	
x_1	(3)	0	1	0	0	0	1	4	

Repeat.

Basic Variable	Row	Z	x_1	x_2	s_1	s_2	s_3	RHS	Ratios
Z	(0)	1	0	-2	0	0	3	12	
s_1	(1)	0	0	1*	1	0	-2	2	$2/1 = 2$ ← min
s_2	(2)	0	0	1	0	1	-1	4	$4/1 = 4$
x_1	(3)	0	1	0	0	0	1	4	

Basic Variable	Row	Z	x_1	x_2	s_1	s_2	s_3	RHS	Ratios
Z	(0)	1	0	0	2	0	-1	16	
x_2	(1)	0	0	1	1	0	-2	2	
s_2	(2)	0	0	0	-1	1	1*	2	$2/1 = 2$ ←min
x_1	(3)	0	1	0	0	0	1	4	$4/1 = 4$

Basic Variable	Row	Z	x_1	x_2	s_1	s_2	s_3	RHS	Ratios
Z	(0)	1	0	0	1	1	0	18	
x_2	(1)	0	0	1	-1	2	0	6	
s_3	(2)	0	0	0	-1	1	1*	2	
x_1	(3)	0	1	0	1	-1	0	2	

Example: Solve the LP in standard form:

$$\text{Maximize} \qquad Z = 2x_1 - x_2 + x_3$$

subject to constraints

$$\begin{aligned} 3x_1 + x_2 + x_3 &\leq 60 \\ x_1 - x_2 + 2x_3 &\leq 10 \\ x_1 + x_2 - x_3 &\leq 20 \end{aligned}$$

and $x_i \geq 0$.

Solution: We will just present the tables.

Basic Variable	Row	Z	x_1	x_2	x_3	s_1	s_2	s_3	RHS	Ratios
Z	(0)	1	-2	1	-1	0	0	0	0	
s_1	(1)	0	3	1	1	1	0	0	60	$60/3 = 20$
s_2	(2)	0	1*	-1	2	0	1	0	10	$10/1 = 10$ ← min
s_3	(3)	0	1	1	-1	0	0	1	20	$20/1 = 20$

	Basic Variable	Row	Z	x_1	x_2	x_3	s_1	s_2	s_3	RHS	Ratios
	Z	(0)	1	0	-1	3	0	2	0	20	
	s_1	(1)	0	0	4	-5	1	-3	0	30	$30/4 = 7.5$
Did you forget →	x_1	(2)	0	1	-1	2	0	1	0	10	
	s_3	(3)	0	0	2*	-3	0	-1	1	10	$10/2 = 5$ ← min

Basic Variable	Row	Z	x_1	x_2	x_3	s_1	s_2	s_3	RHS	Ratios
Z	(0)	1	0	0	1.5	0	1.5	0.5	25	
s_1	(1)	0	0	0	1	1	-1	-2	10	
x_1	(2)	0	1	0	0.5	0	0.5	0.5	15	
x_2	(3)	0	0	1	-1.5	0	-0.5	0.5	5	

So the maximum is $Z = 25$ attained at $(x_1, x_2, x_3) = (15, 5, 0)$.

Remark: In each table, the coefficient of the basic variables in Row (0) are all zero. In the final table (or, the *optimal tableau*), all the coefficients in Row (0) are non-negative.

Remark: If there are multiple variables tie with the largest negative coefficient in row (0), when selecting the entering basic variable, the choice may be made arbitrarily among them.

Remark: If there are multiple variables tie with the minimal ratio, when selecting the leaving basic variable, the choice may be made arbitrarily among them in practice.

5 Potential break-down of the simplex algorithm

5.1 Multiple optimal solutions

Consider the following example.

Example: Consider the LP in standard form:

$$\text{Maximize} \qquad Z = -3x_1 + 6x_2$$

subject to constraints

$$\begin{aligned} 2x_1 + x_2 &\leq 6 \\ -x_1 + 2x_2 &\leq 2 \end{aligned}$$

and $x_i \geq 0$.

Solution: By introducing the slack variables and simplex algorithm, we have the following two tables.

Basic Variable	Row	Z	x_1	x_2	s_1	s_2	RHS	Ratios
Z	(0)	1	3	-6	0	0	0	
s_1	(1)	0	2	1	1	0	6	$6/1 = 6$
s_2	(2)	0	-1	2*	0	1	2	$2/2 = 1$ ←min

Basic Variable	Row	Z	x_1	x_2	s_1	s_2	RHS	Ratios
Z	(0)	1	0	0	0	3	6	
s_1	(1)	0	2.5*	0	1	-0.5	5	
x_2	(2)	0	-0.5	1	0	0.5	1	

Since the coefficients in Row (0) are all non-negative, we arrive at an optimal solution:

$$\max Z = 6, \quad \text{at } (x_1^*, x_2^*) = (0, 1).$$

However, in this optimal tableau, one of the non-basic variable, x_1, has 0 coefficient in Row (0). If we choose x_2 as the entering basic variable, we obtain another *optimal* tableau:

Basic Variable	Row	Z	x_1	x_2	s_1	s_2	RHS	Ratios
Z	(0)	1	0	0	0	3	6	
x_1	(1)	0	1	0	0.2	-0.1	2	
x_2	(2)	0	0	1	0.2	0.4	2	

Note, Row (0) will remain the same, since the coefficient of the entering basic variable in Row (0) is zero. We have another optimal solution:

$$\max Z = 6, \quad \text{at } (x_1^*, x_2^*) = (2, 2).$$

Exercise: Graphically solve this LP and explain why the non-uniqueness happens. From the graph, identify all the optimal solutions.

Exercise (no leaving basic variable): This exercise shows that even though one non-basic variable has zero coefficient in the optimal tableau, the optimal solution could still be unique. Also verify this graphically.

$$\text{Maximize} \qquad Z = -3x_1 + 6x_2$$

subject to constraints

$$\begin{aligned} -2x_1 + 2x_2 &\leq 1 \\ -x_1 + 2x_2 &\leq 2 \end{aligned}$$

and $x_i \geq 0$.

Conclusion: In the optimal tableau, if all the coefficient of the non-basic variable are strictly positive in Row (0), then the optimal solution is unique. If in the optimal tableau, there is a non-basic variable has 0 coefficient in Row (0), then another iteration (if possible) with this non-basic variable serving as the entering basic variable, will also lead to an optimal solution.

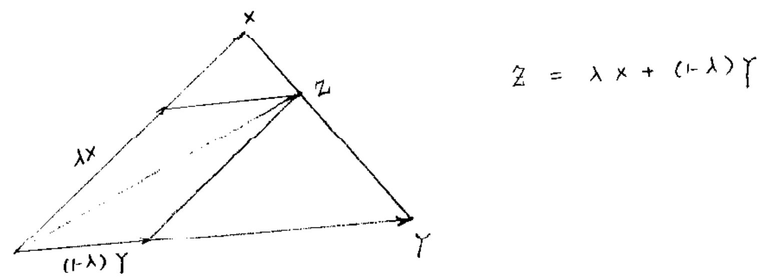
Digression to convex set

Definition: A set $A \subseteq \mathsf{R}^d$ is said to be a **convex** set, if $\forall\, X, Y \in A$, we have

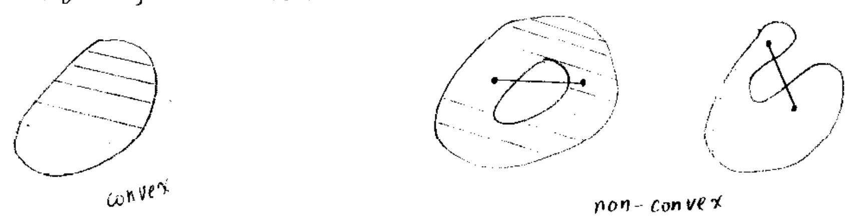
$$\lambda X + (1 - \lambda) Y \in A, \qquad \forall\, \lambda \in [0, 1].$$

Sometimes $\lambda X + (1 - \lambda) Y$ is called the **convex combination** of X and Y.

Geometric interpretation of convexity: A set $A \in \mathsf{R}^d$ is a convex set if and only if for any $X, Y \in A$, the line segment connecting X and Y also belong to the set A. Indeed, any point of the line segment connecting X and Y is a convex combination of them, and vice versa; see the following graph for illustration.



Example: Any interval $[a, b]$ on the real line is convex. The set $\{(x, y);\ x^2 + y^2 \leq 1\}$ is convex. The set $\{(x, y);\ x^2 + y^2 > 1\}$ is *not* convex.



Proposition: Suppose A is a convex set and $X_1, X_2, \cdots, X_n \in A$. Then for any $\lambda_1, \lambda_2, \cdots, \lambda_n \geq 0$ such that $\lambda_1 + \lambda_2 + \cdots + \lambda_n = 1$, we have

$$\lambda_1 X_1 + \lambda_2 X_2 + \cdots + \lambda_n X_n \in A.$$

Sometimes $\lambda_1 X_1 + \lambda_2 X_2 + \cdots + \lambda_n X_n$ is said to be the **convex combination** of $\{X_1, \cdots, X_n\}$.
Proof: We will prove by induction. The claim is true for $n = 2$, and assume it is true for $n - 1$. Without loss of generality, assume that $\lambda_n < 1$. Let

$$Y \doteq \frac{\lambda_1}{1 - \lambda_n} X_1 + \frac{\lambda_2}{1 - \lambda_n} X_2 + \cdots + \frac{\lambda_{n-1}}{1 - \lambda_n} X_{n-1} := \mu_1 X_1 + \mu_2 X_2 + \cdots + \mu_{n-1} X_{n-1}.$$

Clearly μ_i are all non-negative and $\mu_1 + \cdots + \mu_{n-1} = 1$. We have $Y \in A$, which in turn implies

$$\lambda_1 X_1 + \lambda_2 X_2 + \cdots + \lambda_n X_n = (1 - \lambda_n) Y + \lambda_n X_n \in A.$$

We complete the proof. □

Proposition: Suppose A and B are both convex sets. Then $A \cap B$ is also a convex set.

Proof: The proof is trivial, and left as an exercise. □

Corollary: The feasible region for any LP is a convex set.

Proof: Each constraint of an LP takes from

$$a_1 x_1 + \cdots + a_n x_n \geq (\leq, =) b.$$

All the vectors $X = (x_1, x_2, \cdots, x_n)$ that satisfy this constraint is convex (why?). The feasible region, which is the intersection of such convex sets, is convex from the preceding proposition. □

Corollary: Suppose X and Y are both optimal solutions to an LP. Then any convex combination of X and Y is also an optimal solution to the LP.

Proof: Suppose $X^* = (x_1^*, \cdots, x_n^*)$ and $Y^* = (y_1^*, \cdots, y_n^*)$ are both optimal solution to the LP problem. In other words, X, Y are both in the feasible region, and

$$z^* = \max Z = c_1 x_1^* + \cdot + c_n x_n^* = c_1 y_1^* + \cdots c_n y_n^*.$$

Any convex combination of X^* and Y^*, say

$$W^* = \lambda X^* + (1 - \lambda) Y^* = (\lambda x_1^* + (1 - \lambda) y_1^*, \cdots, \lambda x_n^* + (1 - \lambda) y_n^*),$$

is also in the feasible region. Furthermore, we have

$$c_1(\lambda x_1^* + (1 - \lambda) y_1^*) + \cdots + c_n(\lambda x_n^* + (1 - \lambda) y_n^*) = \lambda z^* + (1 - \lambda) z^* = z^* = \max Z.$$

This completes the proof. □

5.2 Unbounded LP

For some LPs, there exist points in the feasible region for which Z assumes arbitrarily large values. When this situation occurs, we say that the LP is *unbounded.*

Example: Consider the LP in standard form:

$$\text{Maximize} \qquad Z = 3x_1 + 2x_2$$

subject to constraints

$$\begin{aligned} x_1 - x_2 &\leq 3 \\ x_1 &\leq 2 \end{aligned}$$

and $x_i \geq 0$.

Solution: The simplex algorithm will yield

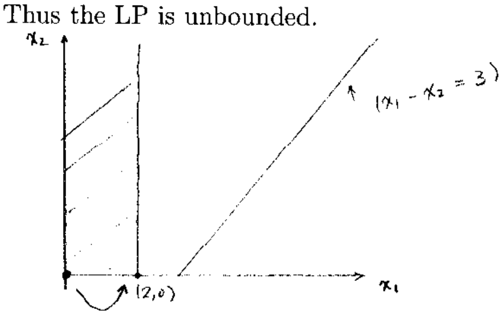
Basic Variable	Row	Z	x_1	x_2	s_1	s_2	RHS	Ratios
Z	(0)	1	-3	-2	0	0	0	
s_1	(1)	0	1	-1	1	0	3	$3/1 = 3$
s_2	(2)	0	1*	0	0	1	2	$2/1 = 2$ ←min

Basic Variable	Row	Z	x_1	x_2	s_1	s_2	RHS	Ratios
Z	(0)	1	0	-2	0	3	6	
s_1	(1)	0	0	-1	1	-1	1	
x_1	(2)	0	1	0	0	1	2	

In this last tableau, the entering basic variable should be x_2. However, since the coefficients of x_2 in row (1) and (2) are either negative or zero, the ratio test fails to indicate which variable should be the leaving basic variable. In this case, Z can take arbitrarily large values. Indeed, holding the other non-basic variable s_2 zero, we have

$$\begin{aligned} Z &= 6 + 2x_2 \\ -x_2 + s_1 &= 1 \\ x_1 &= 2 \end{aligned}$$

If we increase x_1, s_1 will increase accordingly. However, there is no limit how big x_1 can be. For example, can we find a feasible $Z \geq 1000$? Just choose $x_2 = (1000 - 6)/2 = 497$ and $s_1 = x_2 + 1 = 498$ (while holding $x_1 = 2$, $s_2 = 0$). In this fashion, we can find feasible solution so that Z is as large as we want it to be. Thus the LP is unbounded.



Conclusion: An LP is unbounded if all the coefficients for the entering basic variable are either negative or zero in all the rows (whence no leaving basic variable).

Exercise: Show that the following LP is unbounded.

$$\text{Maximize} \qquad Z = 36x_1 + 30x_2 - 3x_3 - 4x_4$$

subject to constraints

$$\begin{aligned} x_1 + x_2 - x_3 &\leq 5 \\ 6x_1 + 5x_2 - x_4 &\leq 10 \end{aligned}$$

and $x_i \geq 0$. Find a feasible solution for which Z takes value 1000. How about 10000?

5.3 Cycling: degeneracy

Theoretically, the simplex algorithm could fail to terminate, and thus fail to find an optimal solution to an LP. To investigate this phenomenon, let us take a closer look of the simplex algorithm. Assume that in some iteration the table is as follows,

Basic Variable	Row	Z	$\cdots$ x_i $\cdots$	RHS	Ratios
Z	(0)	1	$\cdots$ c_i $\cdots$	Z_{old}	
$\vdots$	$\vdots$	0	$\vdots$ $\vdots$ $\vdots$	$\vdots$	
x_j		0	$\cdots$ a_{ij} $\cdots$	b_j	
$\vdots$	$\vdots$	0	$\vdots$ $\vdots$ $\vdots$	$\vdots$	

and the entering basic variable and the leaving basic variable are x_i and x_j respectively. The next table will therefore take form

Basic Variable	Row	Z	$\cdots$ x_i $\cdots$	RHS	Ratios
Z	(0)	1	$\cdots$ 0 $\cdots$	$Z_{\text{new}} = Z_{\text{old}} - c_i b_j / a_{ij}$	
$\vdots$	$\vdots$	0	$\vdots$ $\vdots$ $\vdots$	$\vdots$	
x_i		0	$\cdots$ 1 $\cdots$	b_j / a_{ij}	
$\vdots$	$\vdots$	0	$\vdots$ $\vdots$ $\vdots$	$\vdots$	

In other words, the entering basic variable x_i will take value b_j/a_{ij} in the new BFS, and the improvement for the value of Z is

$$\triangle Z = -c_i b_j / a_{ij} = -c_i \cdot (\text{the value of entering basic variable in the new BFS}).$$

Since $c_i < 0, a_{ij} > 0, b_j \geq 0$, the increment $\triangle Z \geq 0$. We have the following important observation

Proposition: In each iteration of the simplex algorithm, the value of Z is always non-decreasing. It is strictly increasing if and only if the value of the entering basic variable in the new BFS is strictly positive.

This result inspires the following definition.

Definition: If in each of the LP's BFS, all the basic variables are strictly positive, then we say the LP os **non-degenerate**.

Corollary: For a non-degenerate LP, the simplex always terminate in finitely many steps.

Proof: Since in each iteration the value Z is strictly increasing, the simplex algorithm will not scan the same vertex twice. But there are only finitely many vertices. The claim follows readily. □

It is now clear how could a simplex algorithm fail to terminate. If the LP is **degenerate**, i.e. some basic variables take value zero in a BFS, then it is possible that the simplex algorithm could scan the same BFS twice (whence infinitely many times). This occurence is called **cycling**.

Example (*degeneracy does not lead to cycling*)**:** Solve the following LP

$$\text{Maximize} \qquad Z = 5x_1 + 2x_2$$

subject to constraints

$$\begin{aligned} x_1 + x_2 &\leq 6 \\ x_1 - x_2 &\leq 0 \end{aligned}$$

and $x_i \geq 0$.

Solution: After adding the slack variables, we obtain the initial table. In this BFS, the basic variable $s_2 = 0$, hence the LP is degenerate.

Basic Variable	Row	Z	x_1	x_2	s_1	s_2	RHS	Ratios
Z	(0)	1	-5	-2	0	0	0	
s_1	(1)	0	1	1	1	0	6	$6/1 = 6$
s_2	(2)	0	1*	-1	0	1	0	$0/1 = 0 \leftarrow$min

Basic Variable	Row	Z	x_1	x_2	s_1	s_2	RHS	Ratios
Z	(0)	1	-5	-7	0	5	0	
s_1	(1)	0	0	2*	1	-1	6	$6/2 = 3 \leftarrow$min
x_1	(2)	0	1	-1	0	1	0	

Basic Variable	Row	Z	x_1	x_2	s_1	s_2	RHS	Ratios
Z	(0)	1	0	0	3.5	1.5	21	
x_2	(1)	0	0	1	0.5	-0.5	3	
x_1	(2)	0	1	0	0.5	0.5	3	

The optimal solution is therefore

$$\max Z = 21, \qquad \text{at} \ \ (x_1^*, x_2^*) = (3, 3).$$

Example (*degeneracy does lead to cycling*)**:** This is a very artificial example.

$$\text{Maximize} \qquad Z = \frac{3}{4}x_1 - 20x_2 + \frac{1}{2}x_3 - 6x_2$$

subject to constraints

$$\begin{aligned} \frac{1}{4}x_1 - 8x_2 - x_3 + 9x_4 &\leq 0 \\ \frac{1}{2}x_1 - 12x_2 - \frac{1}{2}x_3 + 3x_4 &\leq 0 \\ x_3 &\leq 1 \end{aligned}$$

and $x_i \geq 0$.

Solution: We have the following tables. Note when we have a tie in the ratio test, we pick the leaving basic variable as indicated.

Basic Variable	Row	Z	x_1	x_2	x_3	x_4	s_1	s_2	s_3	RHS	Ratios
Z	(0)	1	$-\frac{3}{4}$	20	$-\frac{1}{2}$	6	0	0	0	0	
s_1	(1)	0	$\frac{1}{4}$*	-8	-1	9	1	0	0	0	$0 \cdot 4 = 0 \leftarrow$leaving
s_2	(2)	0	$\frac{1}{2}$	-12	$-\frac{1}{2}$	3	0	1	0	0	$0 \cdot 2 = 0$
s_3	(3)	0	0	0	1	0	0	0	1	1	

Basic Variable	Row	Z	x_1	x_2	x_3	x_4	s_1	s_2	s_3	RHS	Ratios
Z	(0)	1	0	-4	$-\frac{7}{2}$	33	3	0	0	0	
x_1	(1)	0	1	-32	-4	36	4	0	0	0	
s_2	(2)	0	0	4*	$\frac{3}{2}$	-15	-2	1	0	0	$0/4 = 0 \leftarrow$min
s_3	(3)	0	0	0	1	0	0	0	1	1	

Basic Variable	Row	Z	x_1	x_2	x_3	x_4	s_1	s_2	s_3	RHS	Ratios
Z	(0)	1	0	0	-2	18	1	1	0	0	
x_1	(1)	0	1	0	8*	-84	-12	8	0	0	$0/8 = 0$ ←leaving
x_2	(2)	0	0	1	$\frac{3}{8}$	$-\frac{15}{4}$	$-\frac{1}{2}$	$\frac{1}{4}$	0	0	$0 \cdot 8/3 = 0$
s_3	(3)	0	0	0	1	0	0	0	1	1	

Basic Variable	Row	Z	x_1	x_2	x_3	x_4	s_1	s_2	s_3	RHS	Ratios
Z	(0)	1	$\frac{1}{4}$	0	0	-3	-2	3	0	0	
x_3	(1)	0	$\frac{1}{8}$	0	1	$-\frac{21}{2}$	$-\frac{3}{2}$	1	0	0	
x_2	(2)	0	$-\frac{3}{64}$	1	0	$\frac{3}{16}$*	$\frac{1}{16}$	$-\frac{1}{8}$	0	0	$0 \cdot 16/3 = 0$ ←leaving
s_3	(3)	0	$-\frac{1}{8}$	0	0	$\frac{21}{2}$	$\frac{3}{2}$	-1	1	1	$1 \cdot 2/21 = 2/21$

Basic Variable	Row	Z	x_1	x_2	x_3	x_4	s_1	s_2	s_3	RHS	Ratios
Z	(0)	1	$-\frac{1}{2}$	16	0	0	-1	1	0	0	
x_3	(1)	0	$-\frac{5}{2}$	56	1	0	2	-6	0	0	$0/2$ ←leaving
x_4	(2)	0	$-\frac{1}{4}$	$\frac{16}{3}$	0	1	$\frac{1}{3}$	$-\frac{2}{3}$	0	0	$0 \cdot 3 = 0$
s_3	(3)	0	$\frac{5}{2}$	-56	0	0	-2	6	1	1	

Basic Variable	Row	Z	x_1	x_2	x_3	x_4	s_1	s_2	s_3	RHS	Ratios
Z	(0)	1	$-\frac{7}{4}$	44	$\frac{1}{2}$	0	0	-2	0	0	
s_1	(1)	0	$-\frac{5}{4}$	28	$\frac{1}{2}$	0	1	-3	0	0	
x_4	(2)	0	$\frac{1}{6}$	-4	$-\frac{1}{6}$	1	0	$\frac{1}{3}$	0	0	$0 \cdot 3 = 0$ ←min
s_3	(3)	0	0	0	1	0	0	0	1	1	

Basic Variable	Row	Z	x_1	x_2	x_3	x_4	s_1	s_2	s_3	RHS	Ratios
Z	(0)	1	$-\frac{3}{4}$	20	$-\frac{1}{2}$	6	0	0	0	0	
s_1	(1)	0	$\frac{1}{4}$	-8	-1	9	1	0	0	0	
s_2	(2)	0	$\frac{1}{2}$	-12	$-\frac{1}{2}$	3	0	1	0	0	
s_3	(3)	0	0	0	1	0	0	0	1	1	

Observe that the last table here is exactly the initial table.

The solution to this probelm is indeed,

$$\max Z = \frac{5}{4}, \qquad \text{at } (x_1^*, x_2^*, x_3^*, x_4^*) = (1,0,1,0),\ (s_1^*, s_2^*, s_3^*) = (\tfrac{3}{4}, 0, 0);$$

here $BV = (x_1, x_3, s_1), NBV = (x_2, x_4, s_2, s_3)$, and

$$Z = \frac{5}{4} - 2x_2 - \frac{21}{2}x_4 - \frac{3}{2}s_2 - \frac{5}{4}s_3$$

□

Remark: The cycling can occur in simplex algorithm in theory. However, no cycling has ever been found for any practical problems.

Remark: If some additional rule is given for the selection of the leaving basic variable whenever there is a tie between ratios, we can eliminate cycling even *in theory*. For example, the so-called lexico-minimum ratio test (not easy to implement) or the Bland's rule (not computationally efficient) can both serve the purpose. Most computer programs in existence for solving LP, however, do not include these rules to guard against cycling. The reason is that the occurance is so rare – even if we have one, the computer round-off error will usually change it to a non-degenerate one.