Continuous time process and Brownian motion

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Consider a complete probability space $(\Omega, \mathcal{F}, \mathsf{P}; \mathsf{F})$ equipped with the *filtration* $\mathsf{F} = \{\mathcal{F}_t;\ 0 \le t < \infty\}$. A *stochastic process* is a collection of random variables $X = \{X_t;\ 0 \le t < \infty\}$ where, for every t, $X_t : \ \Omega \to \mathsf{R}^d$ is a random variable. We assume the space R^d is equipped with the usual Borel σ-algebra $\mathcal{B}(\mathsf{R}^d)$. Every fixed $\omega \in \Omega$ corresponds to a *sample path* (or, *trajectory*), that is, the function $t \mapsto X_t(\omega)$ for $t \ge 0$.

Definition: The process is said to be *continuous* if every sample path is continuous on $[0, \infty)$. The processes we discuss here are all continuous, even though almost all the results can be carried over to the more general *RCLL* processes (right continuous, with finite left-hand limit).

Definition: The stochastic process X is said to be *adapted* to filtration F, if X_t is $\mathcal{F}_t$-measurable for every $t \ge 0$.

Unless otherwise specified, the processes are always assumed to be continuous and the filtration F is assumed to satisfy the *usual conditions*, that is,

1. $\mathcal{F}_0$ contains all the P-negligible sets (hence so does every $\mathcal{F}_t$).
2. the filtration F is *right-continuous*, or, $\mathcal{F}_t = \mathcal{F}_{t+} \doteq \bigcap_{s>t} \mathcal{F}_s$.

1 Continuous Time Martingales

Consider a complete probability space $(\Omega, \mathcal{F}, \mathsf{P}; \mathsf{F})$ where the filtration F satisfies the usual conditions. We can similarly define continuous-time martingales, stopping times, etc.

Definition: An adapted process $X = (X_t, \mathcal{F}_t)$ is said to be a *martingale* (resp. *sub-*, *super-*) wrt to filtration F if it is integrable, and

$$\mathsf{E}\big[X_t \,|\, \mathcal{F}_s\big] = X_s \qquad \text{(resp. } \ge, \le)$$

almost surely, for all $0 \le s \le t$.

Basic convergence theorem: Let $X = (X_t, \mathcal{F}_t)$ be a supermartingale such that

$$\sup_{t \ge 0} \mathsf{E}(X_t^-) < \infty.$$

Then $X_\infty \doteq \lim_t X_t$ exists almost everywhere.

Definition: A *stopping time* wrt filtration $\mathsf{F} = (\mathcal{F}_t)$, is a random variable $T: \ \Omega \to [0, \infty]$ such that

$$\{T \le t\} \in \mathcal{F}_t, \qquad \forall\ t \ge 0.$$

The *filtration prior to the stopping time* T is defined as

$$\mathcal{F}_T \doteq \{A \in \mathcal{F};\ A \cap \{T \le t\} \in \mathcal{F}_t, \ \ \forall\ t \ge 0\}\,.$$

Exercise: Consider a continuous, adapted process $X = (X_t, \mathcal{F}_t)$ where the filtration F satisfies the usual conditions. Let Γ be a Borel set in R^d, and define the hitting time

$$H_\Gamma(\omega) \doteq \inf\{t \ge 0;\ \ X_t(\omega) \in \Gamma\}\,, \qquad \text{with convention} \quad \inf\{\emptyset\} = \infty.$$

Show H_Γ is a stopping time if Γ is either open or closed.

Optional sampling theorem: If $X = (X_t, \mathcal{F}_t)$ is a martingale (resp. sub-, super-) and T is an arbitrary stopping time, then the *stopped process* $X^T = (X_{t\wedge T}, \mathcal{F}_t)$ is also a martingale (resp. sub-, super-). In particular,

$$\mathsf{E}[X_{t\wedge T}] = \mathsf{E}[X_0] \qquad (\text{resp. } \ge, \le) \qquad \text{for all } t \ge 0.$$

Proof: Here we give a sketch of the proof. Assume X is a supermartingale. First, for a fixed $n \ge 1$, let

$$D_n \doteq \left\{\frac{k}{2^n}, \qquad k = 0, 1, 2, \cdots\right\} \ \subseteq\ D_{n+1} \ \subseteq \cdots$$

be the set of non-negative dyadic rationals of order no greater than n. It follows that

$$X = (X_t, \mathcal{F}_t;\ t \in D_n)$$

is a supermartingale (discrete time).

Second, we construct a stopping time T_n such that $T_n \ge T$ and T_n only take values in D_n. Indeed, let

$$T_n(\omega) \doteq \inf\{t \in D_n;\ t \ge T(\omega)\}\,.$$

Then $T_n \ge T_{n+1} \ge \cdots$ and T_n is a stopping time (exercise!).

Fix $0 \le s \le t$, we wish to show that

$$\mathsf{E}[X_{t\wedge T}\,|\,\mathcal{F}_s] \le X_s$$

almost surely. Similarly define

$$t_n \doteq \inf\{u \in D_n;\ u \ge t\} \ge t_{n+1} \ge \cdots \qquad \text{and} \qquad s_n \doteq \inf\{u \in D_n;\ u \ge s\} \ge s_{n+1} \ge \cdots\,.$$

It follows from the discrete time optional sampling theorem that

$$\mathsf{E}[X_{t_n\wedge T_n}\,|\,\mathcal{F}_{s_m}] \le X_{s_m\wedge T_n}$$

for any integers $m \ge n$. Letting $m \to \infty$, we have $s_m \downarrow s$ and $\mathcal{F}_{s_m} \downarrow \mathcal{F}_s$. By Lévy's Downward theorem and the continuity of process X, we have

$$\mathsf{E}[X_{t_n\wedge T_n}\,|\,\mathcal{F}_s] \le X_{s\wedge T_n}.$$

Observe that $(X_{t_n \wedge T_n}, \mathcal{F}_{t_n \wedge T_n})$ is a *backward martingale*, whence it is uniformly integrable. Letting $n \to \infty$, we arrive at

$$\mathsf{E}[X_{t\wedge T} \,|\, \mathcal{F}_s] \leq X_{s\wedge T}.$$

This completes the proof. $\square$

Optional sampling theorem: Assume that $X = (X_t, \mathcal{F}_t)$ is a martingale (resp. sub-, super-) such that $\{X_t\}$ (resp. $\{X_t^+\}$, $\{X_t^-\}$) are *uniformly integrable*. For any pair of stopping times $S \leq T$, we have

$$\mathsf{E}[X_T \;\big|\; \mathcal{F}_S] = X_S \qquad \text{(resp. } \geq, \leq).$$

almost surely. In particular,

$$\mathsf{E}[X_T] = \mathsf{E}[X_0] \qquad \text{(resp. } \geq, \leq)$$

for any stopping time T.

Theorem: Assume that $X = (X_t, \mathcal{F}_t)$ is a martingale (resp. sub-, super-). Then the following statements are equivalent

1. $\{X_t\}$ (resp. $\{X_t^+\}$, $\{X_t^-\}$) are *uniformly integrable*.
2. The martingale (resp. sub-, super-) $X = (X_t, \mathcal{F}_t)$ is *closed*.

Furthermore, if this is the case, $X_\infty = \lim_t X_t$ exists and serves as a last emelement.

A martingale (resp. sub-, super-) is said to be *closed* if there exists a $\mathcal{F}_\infty$-measurable random variable Y such that

$$\mathsf{E}[Y \,|\, \mathcal{F}_t] = X_t \quad \text{(resp. } \geq, \leq), \qquad \forall\; t \geq 0.$$

We call Y a *last element*.

Exercise: Assume $X = (X_t, \mathcal{F}_t)$ is a martingale (resp. sub-, super-), and $S \leq T$ are two bounded stopping times. That is $\mathsf{P}(S \leq T \leq a) = 1$ for some constant a. Then $\mathsf{E}\left[X_T \,|\, \mathcal{F}_S\right] \geq X_S$.

Exercise: Every uniformly integrable supermatingale martingale $X = (X_t, \mathcal{F}_t)$ admits a unique Riesz decomposition

$$X_t = M_t + Z_t$$

where M is a uniformly integrable martingale and Z is a non-negative supermatingale with $\lim_{t\to\infty} \mathsf{E} Z_t = 0$.

Exercise: Let $Z = (Z_t, \mathcal{F}_t)$ be a continuous, non-negative martingale with $Z_\infty = \lim_t Z_t = 0$ almost surely. Then for every $s \geq 0$, $b > 0$, we have

$$\mathsf{P}\left(\max_{t>s} Z_t \geq b \,\Big|\, \mathcal{F}_s\right) = \frac{1}{b} Z_s, \quad \text{a.s. on } \; \{Z_s < b\}.$$

2 Brownian motion

Definition: An adapted process $B = (B_t, \mathcal{F}_t)$ is said to be a (*standard*) *Brownian motion* in R if,

1. The process B is continuous.
2. $B_0 \equiv 0$ almost surely.
3. The increment $B_t - B_s$ is independent of $\mathcal{F}_s$ for all $0 \le s \le t$.
4. The increment $B_t - B_s$ has a normal distribution $N(0, t-s)$ for all $0 \le s \le t$.

Exercise: Suppose that a continuous process $X = (X_t)$ with $X_0 \equiv 0$ in probability space $(\Omega, \mathcal{F}, \mathsf{P})$ has independent, stationary increments with $X_t - X_s \sim N(0, t-s)$. Then $X = (X_t, \mathcal{F}_t^X)$ is a Brownian motion; here $(\mathcal{F}_t^X)$ is the natural filtration generated by X. (Hint: Use Dynkin system theorem to prove that $X_t - X_s$ is independent of $\mathcal{F}_s^X$.)

2.1 Construction of Brownian Motion

In the following we will discuss the construction of the standard Brownian motion on time interval $[0,1]$. The method can also be generalized to construct Brownian motion on the infinite time horizon. There are several ways to construct a standard Brownian motion. The one we are going to illustrate is the *weak convergence approach*.

A naive illustration: Let $(\xi_1, \xi_2, \cdots)$ be a sequence of iid random variables with distribution

$$\mathsf{P}(\xi = 1) = \mathsf{P}(\xi = -1) = \frac{1}{2}.$$

Consider the following symmetric random walk. Let time step be δ and step size h. At time t, the random walk is roughly at state

$$S_t = \sum_{i=1}^{[\frac{t}{\delta}]} h\xi_i.$$

Then S_t define a stochastic process (not strictly continuous). It follows that

$$\mathsf{E}[S_t] = 0, \qquad \mathrm{Var} S_t = h^2 \cdot \left[\frac{t}{\delta}\right] \approx \frac{h^2}{\delta} t.$$

To get a continuous version of this random walk, we let $\delta \to 0$, $h \to 0$ and $\frac{h^2}{\delta} = 1$. For example, let $h = \frac{1}{\sqrt{n}}$ and $\delta = \frac{1}{n}$. It follows that

$$S_t^{(n)} = \frac{1}{\sqrt{n}} \sum_{i=1}^{[nt]} \xi_i.$$

By the central limit theorem, we have $S_t^{(n)} \xrightarrow{\mathcal{D}} N(0,t)$.

However, the symmetric simple random walk has stationary, independent increment. We would expect that the limiting process still keep this property. It follows that the limiting process is a Brownian motion.

Space $C[0,1]$: We will give the space $C[0,1]$ the usual *uniform topology*. For every pair of $f, g \in C[0,1]$, we define distance

$$\rho(f,g) \doteq \max_{0 \le t \le 1} |f(t) - g(t)|.$$

- ρ define a metric on space $C[0,1]$.
- The metric space $(C[0,1], \rho)$ is a *Polish* space (i.e. complete and seperable).

We will associate with the space the Borel σ-algebra, denoted by $\mathcal{B}\big(C[0,1]\big)$ or sometimes simply $\mathcal{B}$ when no confusion is incurred. The *coordinate mapping process*, say $W = (W_t)$ is defined as follows:

$$W_t(\omega) \doteq \omega(t), \qquad \forall\, \omega \in C[0,1].$$

Clearly this is a continuous process. Now that we have a measurable space $(\Omega, \mathcal{B})$, under which the coordinate mapping process $W = (W_t, \mathcal{F}_t^W)$ is a continuous, adapted process, it remains to find a suitable probability measure so as to make W a Brownian motion.

Weak Convergence: We will consider a generic Polish space (S, ρ) (complete and seperable) equipped with the Borel σ-algebra, denoted by $\mathcal{B}$. A sequence of probability measures (P_n) on the space $(S, \mathcal{B})$ is said to *converges weakly* to a probability measure P on $(S, \mathcal{B})$ (write $\mathsf{P}_n \Rightarrow \mathsf{P}$), if for every continuous, bounded function $f:\ S \to \mathsf{R}$

$$\lim_{n\to\infty} \int_S f(s)dP_n(s) = \int_S f(s)dP(s).$$

Similarly, consider a sequence of random variables $(X_0, X_1, \cdots)$ and random variable X (could be defined on different probability space) that all take value in space (S, ρ). We say X_n *converges to X in distribution*, denoted by $X_n \xrightarrow{\mathcal{D}} X$, if for any bounded continuous function $f:\ S \to \mathsf{R}$,

$$\lim_{n\to\infty} \mathsf{E}f(X_n) = \mathsf{E}f(X).$$

Definition: A family of probability measures on space (S, ρ), say Π, is said to be *tight*, if for every $\epsilon > 0$, there exists a compact set K such that $\mathsf{P}(K) > 1 - \epsilon$ for every $\mathsf{P} \in \Pi$.

The following theorem is the key result toward the construction of Brownian motion.

Prohorov Theorem: A family of probability measures on Polish space (S, ρ), say Π, is tight if and only if every sequence in Π contains a weak convegent subsequence (*relative compactness*).

We will refer the reader to the book by Billingsley "Convergence of Probability Measures" for a proof of this theorem. The proof of the special case with $S = \mathsf{R}$ can be found in Chow and Teicher "Probability Theory".

Construction of Brownian motion: The main result here is the *Invariance Principle* of Donsker. A rough thread of the proof is the following:

1. Start with the general random walk to get a sequence of probability measures (P_n) on $C[0,1]$.
2. Show that (P_n) is tight.

3. By Prohorov theorem, there exists a subsequence of (P_n) that converges weakly to a probability measure P^*. Show that, under P^*, the coordinate mapping is a Brownian motion.
4. Prove the whole sequence P_n converges weakly to P^*.

Suppose that $\xi_1, \xi_2, \cdots$ is a sequence of iid random variables with mean zero and variance σ^2. Let $S_n = \xi_1 + \xi_2 + \cdots + \xi_n$ with convention $S_0 \equiv 0$. Define the following continuous process

$$X_t^{(n)} \doteq \frac{1}{\sigma\sqrt{n}} S_{[nt]} + (nt - [nt]) \frac{1}{\sigma\sqrt{n}} \xi_{[nt]+1} := \frac{1}{\sigma\sqrt{n}} S_{[nt]} + R_t^{(n)}, \qquad \forall\, 0 \le t \le 1.$$

This is just the linear interpolation of a radom walk with time size $\frac{1}{n}$ and step size $\frac{1}{\sigma\sqrt{n}}\xi$. We will let P_n denote the probability measure induced on $C[0,1]$ by $X^{(n)}$.

Invariance Principle of Donsker: The probability measure P_n converges weakly to a probaility measure P^* under which the coordinate mapping process $W = (W_t, \mathcal{F}_t^W)$ is a standard Brownian motion.

Remark: The standard Brownian motion starts at $B_0 \equiv 0$. It is not difficult to see that one can also construct a Brownian with B_0 has arbitrary distribution, say μ. In this case, we will say B is a Brownian motion with *initial distribution* μ. Sometimes, to specify the initial distribution, we will use notation P^μ or E^μ. If the initial distribution is just a Dirac distribution δ_x, we will use P^x or E^x.

2.2 Strong Markov property of Brownian motion and reflection principle

Let us start with an easy exercise, which shows that the Brownian motion starts afresh at time t.

Exercise: Suppose B is a Brownian motion with arbitrary intial distribution μ. For any $t \ge 0$, show that the process $(W_s \doteq B_{t+s} - B_t;\ s \ge 0)$ is a standard Brownian motion (wrt to F^W).

Theorem: Brownian motion is a *Markov process.* That is, for any $A \in \mathcal{B}(\mathsf{R})$,

$$\mathsf{P}\left[B_{t+h} \in A \,\middle|\, \mathcal{F}_t\right] = \mathsf{P}\left[B_{t+h} \in A \,\middle|\, B_t\right].$$

Proof: It follows from that the Brownian motion has independent increment. □

Exercise: For every bounded, measurable function $f:\ \mathsf{R} \to \mathsf{R}$, show that

$$\mathsf{E}\left[f(B_{t+h}) \,\middle|\, \mathcal{F}_t\right] = \mathsf{E}\left[f(B_{t+h}) \,\middle|\, B_t\right].$$

A discrete-time Markov chain always has the strong Markov property. The question is whether the Brownian motion has the strong Markov property. The answer is affirmative.

Strong markov property: Suppose $B = (B_t, \mathcal{F}_t)$ is a Brownian motion and T is an arbitrary finite stopping time. Then

- for any $A \in \mathcal{B}(\mathsf{R})$, we have $\mathsf{P}\left[B_{T+t} \in A \,\middle|\, \mathcal{F}_T\right] = \mathsf{P}\left[B_{T+t} \in A \,\middle|\, B_T\right]$.

- the Brownian motion starts afresh at stopping time T, that is, $W_t \doteq B_{T+t} - B_T$ is a standard Brownian motion and independent of $\mathcal{F}_T$.

Proof: All we need to show is that, for every $s, t \geq 0$, conditional on $\mathcal{F}_{T+s}$, the increment $W_{t+s} - W_s = B_{T+t+s} - B_{T+s}$ always has distribution $N(0,t)$. Hence $W_{t+s} - W_s \sim N(0,t)$ is independent of $\mathcal{F}_{T+s}$. It is well known that the characteristic function uniquely determines the distribution. Similar result is also true for conditional probabilities. Roughly speaking, conditional distribution is uniquely determined by the conditional characeristic function.

Let us consider the following complex-valued process

$$X_t \doteq e^{i\theta B_t + \frac{1}{2}\theta^2 t}; \qquad t \geq 0,\ \theta \in \mathsf{R}.$$

It is easy to show that $X = (X_t, \mathcal{F}_t)$ is a martingale. Let $S = T + s$, which is still a stopping time. By optional sampling theorem, we know that

$$\mathsf{E}\left[X_{S\wedge n+t} \,\middle|\, \mathcal{F}_{S\wedge n}\right] = X_{S\wedge n}, \qquad \forall\ n \geq 0,$$

which implies that, by rearranging terms,

$$\mathsf{E}\left[e^{i\theta(B_{S\wedge n+t} - B_{S\wedge n})} \,\middle|\, \mathcal{F}_{S\wedge n}\right] = e^{-\frac{1}{2}\theta^2 t}.$$

Letting $n \to \infty$, by Lévy upward theorem,

$$\mathsf{E}\left[e^{i\theta(B_{S+t} - B_S)} \,\middle|\, \mathcal{F}_{T+s}\right] = \mathsf{E}\left[e^{i\theta(B_{T+s+t} - B_{T+s})} \,\middle|\, \mathcal{F}_{T+s}\right] = \mathsf{E}\left[e^{i\theta(W_{t+s} - W_s)} \,\middle|\, \mathcal{F}_{T+s}\right] = e^{-\frac{1}{2}\theta^2 t}$$

This completes the proof. □

Remark: The restriction to finite-valued stopping time is not essential. It is included to avoid worrying about the definition at infinity.

Reflection Principle: Let $B = (B_t, \mathcal{F}_t)$ be a standard Brownian motion. For $b \in \mathsf{R}$, define the *hitting time* T_b as

$$T_b \doteq \inf\{t \geq 0;\ B_t = b\}.$$

Then the following *reflected* process

$$W_t \doteq \begin{cases} B_t & ; \quad t \leq T_b \\ 2b - B_t & ; \quad t \geq T_b \end{cases}$$

is still a standard Brownian motion.

Proof: This is an immediate result of strong Markov property and the fact that $-B$ is a Broanian motion for any Brownian motion B. □

Corollary (*Density of hitting time T_b*): It suffices to consider the case where $b > 0$. Let W be the reflected process defined above. We have

$$\mathsf{P}(T_b \le t) = \mathsf{P}\left(T_b \le t, B_t \ge b\right) + \mathsf{P}\left(T_b \le t, B_t \le b\right) = \mathsf{P}\left(B_t \ge b\right) + \mathsf{P}\left(T_b \le t, B_t \le b\right)$$

However, by refliction principle,

$$\mathsf{P}\left(T_b \le t, B_t \le b\right) = \mathsf{P}(B_t \le b \,\big|\, T_b \le t) \cdot \mathsf{P}(T_b \le t) = \frac{1}{2}\mathsf{P}(T_b \le t).$$

It follows that

$$\mathsf{P}(T_b \le t) = 2\mathsf{P}(B_t \ge b) = 2\Phi\left(-\frac{b}{\sqrt{t}}\right).$$

The density of T_b is then

$$\mathsf{P}(T_b \in dt) = \frac{b}{\sqrt{2\pi t^3}} e^{-\frac{b^2}{2t}}\, dt.$$

In particular, $\mathsf{P}(T_b < \infty) = 1$. □

Exercise: Show that the joint distribution between Brownian motion B and its *running maximum* $M_t \doteq \max_{0\le s\le t} B_s$ is given by the following:

$$\mathsf{P}(B_t \le a, M_t \ge b) = \Phi\left(\frac{a-2b}{\sqrt{t}}\right)$$

for all $b \ge 0$ and $a \le b$.

2.3 Martingales associated with Brownian motion

There are several very useful martingales associated with the Brownian motion.

Lemma: Suppose $W = (W_t, \mathcal{F}_t)$ is a standard Brownian motion.

1. For every $\theta \in \mathsf{R}$, the process $(X_t, \mathcal{F}_t)$ where

   $$X_t \doteq e^{\theta W_t - \frac{1}{2}\theta^2 t}$$

   is a martingale.
2. The Brownian motion it self is a martingale.
3. The process $(W_t^2 - t, \mathcal{F}_t)$ is a martingale.

The proof is left as an exercise.

Example: Another way to determine the distribution of the first passage time T_b is to find its Laplace transform $\mathsf{E}\left[e^{-\alpha T_b}\right]$. Now suppose $\theta > 0$ and consider the exponential martingale

$$X_t \doteq e^{\theta W_t - \frac{1}{2}\theta^2 t}$$

It follows from optional sampling theorem that

$$\mathsf{E}[X_{t\wedge T_b}] = \mathsf{E}[X_0] = 1, \quad \forall\, t \ge 0.$$

However, $0 \le X_{t\wedge T_b} \le e^{\theta b}$ is bounded. It follows from DCT that

$$\mathsf{E}[X_{T_b}] = \lim_{t\to\infty} \mathsf{E}[X_{t\wedge T_b}] = 1 \quad \Rightarrow \quad e^{\theta b - \frac{1}{2}\theta^2 T_b} = 1.$$

Letting $\theta = \sqrt{2\alpha}$, we have

$$\mathsf{E}\left[e^{-\alpha T_b}\right] = e^{-\sqrt{2\alpha} b}.$$

We can inverse the Laplace transform to get the density of T_b. □

Example: Let $a < 0 < b$ and $T \doteq \inf\{t \ge 0;\ B_t \notin (a,b)\}$. Show that

$$\mathsf{P}(B_T = a) = \frac{b}{b-a}, \quad \mathsf{P}(B_T = b) = \frac{-a}{b-a}, \quad \mathsf{E}T = -ab$$

Proof: By optional sampling theorem $\mathsf{E}[B_{e\wedge T}] = 0$ for all $t \ge 0$. However, $a \le B_{t\wedge T} \le b$ for all t. It follows from DCT that

$$0 = \lim_{t\to\infty} \mathsf{E}[X_{t\wedge T}] = \mathsf{E}[B_T] = a\mathsf{P}(B_T = a) + b\mathsf{P}(B_T = b).$$

We can solve it with $\mathsf{P}(B_T = a) + \mathsf{P}(B_T = b) = 1$. In order to calculate $\mathsf{E}T$, consider the following martingale (exercise!)

$$X_t \doteq (B_t - a)(b - B_t) + t, \quad t \ge 0.$$

It follows from optional sampling theorem again that

$$-ab = \mathsf{E}[X_0] = \mathsf{E}\left[X_{t\wedge T}\right] = \mathsf{E}[t\wedge T] + \mathsf{E}\left[(B_{t\wedge T} - a)(b - B_{t\wedge T})\right].$$

Letting $t \to \infty$, by MCT and DCT, we have

$$-ab = \mathsf{E}T + \mathsf{E}\left[(B_T - a)(b - B_T)\right] = \mathsf{E}T.$$

This completes the proof. □

3 Some basic properties of Brownian motion

Here we list some elementary properties of Brownian motion without proof. Throughout the section, we assume $B = (B_t, \mathcal{F}_t)$ be a standard Brownian motion.

- (*SLLN:*) With probability one,
$$\lim_{t\to\infty} \frac{B_t}{t} = 0.$$
- (*Scaling:*) The process $X = (X_t, \mathcal{F}_{ct})$ is a standard Brownian motion, where
$$X_t = \frac{1}{\sqrt{c}} B_{ct}; \qquad t \ge 0$$

- (Time-inversion:) The process $Y = (Y_t, \mathcal{F}_t^Y)$ is a standard Brownian motion, where

$$Y_t \doteq \begin{cases} tB_{\frac{1}{t}} & ; \quad t > 0 \\ 0 & ; \quad t = 0 \end{cases}$$

- (*Time-reversal:*) The process $Z = (Z_t, \mathcal{F}_t^Z;\ 0 \le t \le T)$ is a standard Brownian motion, where

$$Z_t = B_T - B_{T-t}; \quad 0 \le t \le T$$

- (*Law of Iterated Algorithm:*) With probability one,

$$\limsup_{t\to 0} \frac{B_t}{\sqrt{2t\log\log(\frac{1}{t})}} = 1, \qquad \liminf_{t\to 0} \frac{B_t}{\sqrt{2t\log\log(\frac{1}{t})}} = -1;$$
$$\limsup_{t\to \infty} \frac{B_t}{\sqrt{2t\log\log(t)}} = 1, \qquad \liminf_{t\to \infty} \frac{B_t}{\sqrt{2t\log\log(t)}} = -1.$$

The Brownian sample paths exhibit some interesting features.

- (*Brownian sample path:*) With probability one, the Brownian motion
  - is monotone in NO interval.
  - the set of local maxima is *countable, dense* in $[0, \infty)$, and all local maxima are *strict.*
  - is NO-where differentable.
- (*Zero crossings:*) For every sample path ω, define the *zero set* $\mathcal{Z}_\omega \doteq \{t \ge 0;\ \omega(t) = 0\}$. With probability one, the zero set $\mathcal{Z}_\omega$
  - has lebesgue measure zero.
  - is closed and unbounded.
  - has NO isolated points.

Proof. "Monotone in no interval": Denote by F the set of $\omega \in \Omega$ such that $W.(\omega)$ is monotone on some interval. It is not difficult to see that

$$F = \cup_{s,t\in\mathsf{Q}^+} \{\omega \in \Omega;\ W.(\omega) \text{ is monotone on interval } [s,t]\}.$$

It is sufficient to show that

$$p \doteq \mathsf{P}\{\omega \in \Omega;\ W.(\omega) \text{ is monotone on interval } [s,t]\} = 0, \qquad \forall\ s,t \in \mathsf{Q}^+,\ s < t.$$

Without loss generality, assume $[s,t] = [0,1]$. It follows from symmetry that, $\forall\ n \ge 1$,

$$p \le 2\mathsf{P}\left(\cap_{j=0}^{n-1} \{\omega \in \Omega;\ W_{(i+1)/n}(\omega) - W_{i/n}(\omega) \ge 0\}\right) = 2 \cdot 2^{-n} \quad \to \quad 0.$$

This completes the proof. $\square$

4 Connection with PDEs

The Brownian motion (or more precisely, general SDE driven by Brownian motion) can be used to represent the solution of general elliptical PDEs with Dirichlet or Neumman boundary conditions. A comprehensive treatment requires the theory of Itô integral and SDE. Below we only give some elementary examples.

Example: Suppose $f \in C_0^2(\mathsf{R})$ (twice continuously differentiable with compact support). Show that

$$\lim_{t\to 0} \frac{\mathsf{E}^x f(B_t) - f(x)}{t} = \frac{1}{2} f''(x), \qquad \forall\, x \in \mathsf{R}.$$

Argue from here that the process $(M_t, \mathcal{F}_t)$

$$M_t \doteq f(B_t) - \int_0^t \frac{1}{2} f''(B_s)\, ds, \qquad t \geq 0$$

is a martingale. Here $B = (B_t, \mathcal{F}_t)$ is a Brownian motion.

Proof: It is easy to see that

$$\frac{\mathsf{E}^x f(B_t) - f(x)}{t} = \int_{\mathsf{R}} \frac{f(x+y) - f(x)}{t} \cdot \frac{1}{\sqrt{2\pi t}} e^{-\frac{y^2}{2t}}\, dy = \int_{\mathsf{R}} \frac{f(x + y\sqrt{t}) - f(x)}{t} \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\, dy$$

However, Taylor expansion yields

$$f(x + y\sqrt{t}) - f(x) = f'(x) y \sqrt{t} + \frac{1}{2} f''(x + \theta y \sqrt{t}) y^2 t, \qquad \text{for some } \theta \in [0,1].$$

It follows from DCT that

$$\lim_{t\to 0} \frac{\mathsf{E}^x f(B_t) - f(x)}{t} = \lim_{t\to 0} \frac{1}{2} \int_{\mathsf{R}} f''(x + \theta y \sqrt{t}) y^2 \cdot \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\, dy = \frac{1}{2} f''(x) \int_{\mathsf{R}} y^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{y^2}{2}}\, dy = \frac{1}{2} f''(x).$$

It remains to show that M is a martingale, or

$$\mathsf{E}\left[f(B_t) - f(B_s) - \int_s^t \frac{1}{2} f''(B_u)\, du \,\middle|\, \mathcal{F}_s \right] = 0, \quad \forall\, t \geq s.$$

which is equivalent to

$$\mathsf{E}^x \left[f(B_t) - f(x) - \int_0^t \frac{1}{2} f''(B_s)\, ds \right] = 0, \qquad \forall\, x \in \mathsf{R},\ t \geq 0.$$

Fix an arbitrary $x \in \mathsf{R}$, define $g(t) \doteq \mathsf{E}^x f(B_t)$. We have

$$D^+ g(t) \doteq \lim_{h\downarrow 0} \frac{\mathsf{E}^x f(B_{t+h}) - \mathsf{E}^x f(B_t)}{h} = \lim_{h\downarrow 0} \mathsf{E}^x \left[\mathsf{E}^x \left(\frac{f(B_{t+h}) - f(B_t)}{h} \,\middle|\, B_t \right) \right].$$

It is not difficult to see that the random variable

$$\left| \mathsf{E}^x \left(\frac{f(B_{t+h}) - f(B_t)}{h} \,\middle|\, B_s \right) \right| \leq \frac{1}{2} \| f'' \|_\infty.$$

We can then apply the DCT to obtain that

$$D^+ g(t) = \frac{1}{2}\mathsf{E} f''(B_t).$$

Similarly, we can approve $D^- g(t) = D^+ g(t) = \frac{1}{2}\mathsf{E} f''(B_t) = g'(t)$. Therefore,

$$\mathsf{E}^x \left[f(B_t) - f(x) - \int_0^t \frac{1}{2} f''(B_s)\, ds \right] = g(t) - g(0) - \int_0^t g'(t)\, dt = 0.$$

We complete the proof. □

Representation to the solution of ODE: Consider the following 1-dim Dirichlet problem

$$u''(x) = g(x), \qquad \forall\, x \in [a,b], \qquad u(a) = A, \quad u(b) = B, \qquad a < b;$$

here $g(x)$ is some continuous function. The solution of the equation can be represented via a Brownian motion. Suppose u is the solution. Clearly, we can assume $u \in \mathcal{C}_0^2(\mathsf{R})$ by extension. Let $W = (W_t, \mathcal{F}_t)$ be a standard Brownian motion with $W_0 = x \in [a,b]$. Also define the exit time

$$\tau \doteq \inf\{t \geq 0; \quad W_t \notin [a,b]\},$$

which has finite expectation. Note that the process $M = (M_t, \mathcal{F}_t)$ is a martingale, where

$$M_t \doteq u(W_t) - \int_0^t \frac{1}{2} u''(W_s)\, ds = u(W_t) - \int_0^t \frac{1}{2} g(W_s)\, ds.$$

It follows from optional sampling theorem that

$$M_{t\wedge\tau} = \mathsf{E} M_0 = u(x) \qquad \Rightarrow \qquad \mathsf{E} M_\tau = u(x) \qquad \text{by DCT.}$$

Hence we have

$$u(x) = \mathsf{E}^x \left[\int_0^\tau \frac{1}{2} g''(W_s)\, ds + A 1_{\{W_\tau = a\}} + B 1_{\{W_\tau = b\}} \right].$$

This representation implies the uniqueness of the solution. □

Example: (*harmonic function*) Suppose $u \in \mathcal{C}_0^2(\mathsf{R}^2)$. Show that

$$M_t \doteq u\left(W_t^{(1)}, W_t^{(2)}\right) - \int_0^t \frac{1}{2} \triangle u \left(W_s^{(1)}, W_s^{(2)}\right)\, ds$$

is a martingale wrt filtration $(\mathcal{F}_t)$; here $W = (W^{(1)}, W^{(2)})$ are two independent F-standard Brownian motions, and

$$\triangle u \doteq \frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} := u_{11} + u_{22}.$$

Proof. The proof is almost the same as the preceding example. We only need to observe the expansion

$$\begin{aligned} u(x_1 + z_1\sqrt{t}, x_2 + z_2\sqrt{t}) &= u(x_1, x_2) + (u_1 z_1 + u_2 z_2)(x_1, x_2)\sqrt{t} \\ &\quad + \frac{1}{2}\left(u_{11} z_1^2 + u_{22} z_2^2 + 2u_{12} z_1 z_2\right)(x_1 + \theta z_1\sqrt{t}, x_2 + \theta z_2\sqrt{t}) t, \qquad \text{for some } 0 \leq \theta \leq 1; \end{aligned}$$

and that

$$\mathsf{E}(Z_1^2) = \mathsf{E}(Z_2^2) = 1, \qquad \mathsf{E}(Z_1 Z_2) = 0,$$

where Z_1, Z_2 are independent $N(0,1)$ random variables. Fill in the details (exercise!) □

Remark: Suppose $u \in \mathcal{C}_0^2(\mathsf{R}^d)$, and $W = (W_t, \mathcal{F}_t)$ is a d-dim standard Brownian motion (i.e., W consists of d independent 1-dim standard Brownian motion). Then the process

$$M_t \doteq u(W_t) - \int_0^t \frac{1}{2} \triangle u(W_s)\, ds; \qquad \forall\, t \geq 0$$

is a martingale wrt $(\mathcal{F}_t)$.

Representation of solution to PDE: Suppose D is a bounded open region in space R^d, and $u \in^{\in} (\bar{\mathcal{D}})$ is a solution to the Dirichlet problem

$$\triangle u = -g; \quad \text{in } D, \qquad u = f; \quad \text{on } \partial D.$$

Here $g : D \to \mathsf{R}$, $f : \partial D \to \mathsf{R}$ are bounded, continuous functions. Then the solution u has the representation

$$u(x) = \mathsf{E}^x \left[f(W_\tau) + \int_0^\tau \frac{1}{2} g(W_t)\, dt \right], \qquad \forall\, x \in \bar{D};$$

here τ is the exit time

$$\tau \doteq \inf\{t \geq 0; \quad W_t \notin D\}.$$

The proof of this result is exactly like the 1-dim case. We only need to observe that

$$\mathsf{E}^x(\tau) < \infty. \qquad \text{(why?)}$$

Remark: In case when $D = B_r = \{x;\ \|x\| < r\}$ and $f = 0$, $g = 2$, we have

$$u(x) = \mathsf{E}^x \tau.$$

It is not hard to verify that

$$u(x) \doteq \frac{r^2 - \|x\|^2}{d}, \qquad \forall\, x \in \bar{B}_r,$$

is a solution to the corresponding PDE. Hence

$$\mathsf{E}^x \tau = \frac{r^2 - \|x\|^2}{d}, \quad \text{where } \tau \doteq \inf\{t \geq 0; \quad W_t \notin B_r\}.$$

5 A collection of Exercises

Exercise: $X_n \xrightarrow{\mathcal{D}} X$ if and only if the induced probability measure on (S, ρ) by X_n converges weakly to the probability measure induced by X.

Exercise: Suppose $X_n \to X$ in probability. Show that $X_n \xrightarrow{\mathcal{D}} X$.

Exercise: Suppose $X_n \xrightarrow{\mathcal{D}} X$ and $\phi : (S, \rho) \to (S_1, \rho_1)$ is a continuous mapping. Show that

$$\phi(X_n) \xrightarrow{\mathcal{D}} \phi(X).$$

Exercise: Suppose (X_n, Y_n) and X are random variables taking values in (S, ρ). If $X_n \xrightarrow{\mathcal{D}} X$ and $\rho(X_n, Y_n) \to 0$ in probability, show that $Y_n \xrightarrow{\mathcal{D}} X$.

Exercise (continuous mapping theorem): Suppose $\mathsf{P}_n \Rightarrow \mathsf{P}$ on space (S, ρ). If $\pi : (S, \rho) \to (S_1, \rho_1)$ is a continuous mapping, then

$$\mathsf{P}_n \pi^{-1} \Rightarrow \mathsf{P} \pi^{-1}.$$

Exercise: Suppose $W = (W_t, \mathcal{F}_t)$ is a standard 1-dim Brownian motion with $W_0 = 0$. Let

$$M_t \doteq \max_{0 \le s \le t} W_s, \quad \forall\, t \ge 0$$

be the *running maxima*. Show that $\left\{ Y_t \doteq M_t - W_t;\ t \ge 0 \right\}$ is a Markov process, and has the same law as the process $\{|W_t|;\ t \ge 0\}$.

Exercise: Suppose $W = (W_t, \mathcal{F}_t)$ is a standard 1-dim Brownian motion with $W_0 = 0$. Let

$$Y \doteq \int_0^T W_t \, dt;$$

here $T > 0$ is a fixed terminal time. Compute the joint distribution of (Y, W_T) and the conditional probability of Y given W_T.