AM261 – Recent Applications of Probability and Statistics

Project 2: Weak coupling in large systems ("local chaos")

(Don't forget to show your work and explain your reasoning.)

Problem 1.

Background

Consider the uniform probability distribution on the surface of the sphere in R^N, with radius $\sqrt{N}$. Each of the coordinate values, $X_1, X_2, \ldots, X_N$, is a random variable, and collectively X_1^N satisfies

$$\sum_{k=1}^{N} X_k^2 = N \tag{1}$$

In class we conjectured that X_1 is approximately $N(0,1)$, motivated through a "local chaos hypothesis" as follows:

(i) The $X_k's$ are exchangeable and only weakly coupled by equation (1). Hence, for any fixed M, $X_1, X_2, \ldots, X_M$ should be nearly *iid* when N is large ("local chaos").

(ii) X_1 is the projection of X_1^N onto the unit vector $(1,0,0,\ldots,0)$ (i.e. $X_1 = X_1^N \cdot (1,0,0,\ldots,0)$).

(iii) Since X_1^N is uniform on the sphere, the distribution of the projection of X_1^N onto a unit vector is the same for all unit vectors (symmetry). Hence

$$\begin{aligned}
\mathcal{L}\{X_1\} &= \mathcal{L}\{X_1^N \cdot (1,0,0,\ldots,0)\} \\
&= \mathcal{L}\{X_1^N \cdot (\frac{1}{\sqrt{M}}, \frac{1}{\sqrt{M}}, \ldots, \frac{1}{\sqrt{M}}, 0, 0, \ldots, 0)\} \\
&\quad (\tfrac{1}{\sqrt{M}} \text{ repeated } M \text{ times, followed by } N-M \text{ zeros}) \\
&= \mathcal{L}\{\frac{1}{\sqrt{M}} \sum_{k=1}^{M} X_k\}
\end{aligned}$$

(iv) If $N >> M >> 1$ then $X_1, \ldots, X_M$ is approximately *iid* and $\frac{1}{\sqrt{M}} \sum_{k=1}^{M} X_k$ is approximately normal (Central Limit Theorem). Thus X_1 is approximately normal, $N(\mu, \sigma^2)$.

(v) Clearly $\mu = 0$. As for σ^2: $\sigma^2 = EX_1^2 = EX_k^2 \; \forall k$ (symmetry), and hence

$$\sigma^2 = \frac{1}{N} \sum_{k=1}^{N} EX_k^2 = E[\frac{1}{N} \sum_{k=1}^{N} X_k^2] = E[\frac{1}{N} \cdot N] = 1$$

In summary, the conjecture is that $X_1 \to N(0,1)$ as $N \to \infty$, i.e.

$$Pr\{a \leq X_1 \leq b\} \stackrel{N \to \infty}{\longrightarrow} \int_a^b \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx$$

Assignment

(a) Test the conjecture by Monte Carlo simulation. For each of $N = 5, 10, 15, 30$ and 100, generate at least $100,000$ independent samples from the uniform distribution on the N-sphere. For each N, use the samples to generate an estimator of the density of X_1. (If, for a given N, $x_1(k)$, $1 \leq k \leq 100,000$ is the set of samples of X_1, then using a bin size of .01,

$$\begin{aligned} f_{X_1}(x) &\approx 100 Pr\{X_1 \in (x - .005, x + .005)\} \\ &\approx 100 \frac{\#\{k : x_1(k) \in (x - .005, x + .005)\}}{100,000} \end{aligned}$$

Which can then be plotted for each x from -6 to $+6$ in step sizes of 0.01.) Plot the estimated density on the same graph as the density of the unit normal distribution:

$$\frac{1}{\sqrt{2\pi}} e^{-x^2/2} \tag{2}$$

One easy way to generate samples from the uniform distribution on the sphere is to choose $Y_1, Y_2, \ldots, Y_N$ *iid* $N(0,1)$ and then normalize to the N-sphere of radius $\sqrt{N}$:

$$X_1^N = \frac{\sqrt{N} Y_1^N}{\sqrt{\sum_{k=1}^{N} Y_k^2}} \tag{3}$$

Then $\sum_1^N X_k^2 = N$, and X_1^N is uniform on the sphere since $f_{Y_1,\ldots,Y_N}(y_1,\ldots,y_N)$ is spherically symmetric: $f_{Y_1,\ldots,Y_N}(y_1,\ldots,y_N) = \frac{1}{\sqrt{2\pi}}^N e^{-|\vec{y}|^2/2}$.

(In fact, equation (3) leads to an easy proof of the conjecture. The outline is this: divide the numerator and denominator by $\sqrt{N}$ and apply the law of large numbers to $\frac{1}{N}\sum_1^N Y_k^2$: $\frac{1}{N}\sum_1^N Y_k^2 \to 1$ as $N \to \infty$, and hence X_1^N behaves like Y_1^N, which is $N(0,1)$.)

(b) For each of $N = 2, 3, 5, 10, 15, 30$, and 200, plot the actual density of X_1 and the unit normal density (2) on the same graph, over the range $-\sqrt{N} \leq x \leq \sqrt{N}$. To get the density of X_1, reason as follows: The area of the sphere of radius R in N dimensions is

$$\frac{2\pi^{\frac{N}{2}} R^{N-1}}{\Gamma(\frac{N}{2})} \tag{4}$$

The surface area of the portion of the sphere with first coordinate $x \in [-\sqrt{N}, \xi]$ is the integral with respect to x, from $-\sqrt{N}$ to ξ, of the surface area of the sphere in $N-1$ dimensions with radius $\sqrt{N-x^2}$, but including (i.e. with the integrand multiplied by) the area element

$$\sqrt{1 + (\frac{d}{dx} Radius(x))^2} = \sqrt{1 + (\frac{d}{dx}\sqrt{N-x^2})^2}$$

The probability distribution function of X_1 ($Pr\{X_1 \leq \xi\}$) is then this integral normalized by (4), and the density is $\frac{d}{d\xi} Pr\{X_1 \leq \xi\}$.

(c) Prove the conjecture, using the explicit formula for the density of X_1 (see problem 2). Do not use the proof, via the law of large numbers, outlined in problem 1.

Here are two helpful facts:

(i)

$$(1 + \frac{x}{n})^n \overset{n\to\infty}{\longrightarrow} e^x$$

(So, for example, $(1 + \frac{x}{n})^{n/2} = \sqrt{(1 + \frac{x}{n})^n} \to e^{x/2}$.)

(ii) When z is an integer, $\Gamma(z+1) = z!$, and in fact Stirling's formula applies to $\Gamma(z)$ for z real and $z \to \infty$:

$$\frac{\Gamma(z+1)}{z^z e^{-z}\sqrt{2\pi z}} \overset{z\to\infty}{\longrightarrow} 1$$

One consequence is that

$$\frac{\frac{\Gamma(z)}{\Gamma(z-1/2)}}{\sqrt{z}} \overset{z\to\infty}{\longrightarrow} 1$$

Problem 2. Consider the system of N linear equations in N unknowns:

$$X_i = \frac{1}{N}\sum_{j=1}^{N} w_{ij}X_j + 1 \qquad 1 \le i \le N \tag{5}$$

where $\{w_{ij}\}_{1\le i,j\le N}$ are N^2 *iid* $N(\mu, 1)$ random variables. We will study the behavior of the solution, X_1^N, as a function of μ.

Since $W_N \doteq \{w_{ij}\}_{1\le i,j\le N}$ is random, and since $X_1^N = X_1^N(W_N)$ (the solution is a function of the coefficients in the linear system in Equation 5), X_1^N is a random vector. By symmetry, each component $X_1, X_2, \ldots, X_N$ has the same distribution.

(a) Make a conjecture about the asymptotic (large N) distribution of $X_1, X_2, \ldots, X_N$, and make note of its dependency on μ. Explain your reasoning. (If you get stuck, look to your computer for inspiration: skip to (b) and come back to (a) later. But give (a) a good try first – you'll learn more.)

There is an "exceptional" value of μ at which you might expect a departure from the conjectured behavior. What is it?

(b) Explore the solution

$$X_1^N = (I - W_N/N)^{-1}1_N$$

(where 1_N is the $N \times 1$ vector of $1's$), at each of several values of μ, using several independent draws of W_N. (Leave the "exceptional" value of μ for later – see below.) The larger N the better, but choose N so that the solution does not take more than a few seconds on your computer. Certainly $N = 1,000$ is adequate.

Compare the results to your conjecture.

(c) Explore X_1^N at and around the "exceptional" value of μ, using several draws of W_N. What, if anything, can you say about the random vector X_1^N?