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THE SPECTRAL RADIUS OF LARGE RANDOM MATRICES[1]

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Let $\{m_{ij}\}$, $i = 1,2,\dots, j = 1,2,\dots$, be iid random variables with $Em_{11} = 0$ and $Em_{11}^2 = \sigma^2$. For each n define $M_n = \{m_{ij}\}_{1 \le i,\, j \le n}$, the $n \times n$ matrix whose (i, j) component is m_{ij}. We show that $\limsup_{n\to\infty} \rho_n \le \sigma$ a.s., where ρ_n is the spectral radius of $M_n/\sqrt{n}$. Evidence from computer experiments indicates that in fact $\rho_n \to \sigma$ a.s.

1. Introduction. Let $\{m_{ij}\}$, $i = 1,2,\dots, j = 1,2,\dots$, be iid (real-valued) random variables with $Em_{11} = 0$ and $Em_{11}^2 = \sigma^2$. For each n define $M_n = \{m_{ij}\}_{1 \le i,\, j \le n}$, the $n \times n$ matrix whose (i, j) component is m_{ij}. The limiting $(n \to \infty)$ behavior of the spectrum of $(1/n)M_n M_n^T$, and of various generalizations, has been thoroughly studied and is well understood; see Wigner ([11], [12]) for the earliest contributions, or Jonsson [7] and Wachter [10] for more recent advances. In contrast, almost nothing is known about the large n behavior of the spectrum of $M_n/\sqrt{n}$. If the entries of M_n are *complex* and *Gaussian*, then the spectrum is asymptotically uniform on the complex circle of radius σ, as shown by Ginibre [4]. Unfortunately, as observed by Ginibre, and later by Mehta [9], the methods do not extend to either the non-Gaussian or the real case. Of particular interest for certain applications to mathematical biology is the large n behavior of the *spectral radius*,

$$\rho_n = \max\{|\lambda|: \lambda \text{ eigenvalue of } M_n/\sqrt{n}\}.$$

In this paper we will show that $\limsup_{n\to\infty} \rho_n \le \sigma$ a.s., under a suitable moment condition on m_{11}, and in a slight generalization of the above setup.

The connection to mathematical biology is made by Hastings [6], who studies the stability of systems of difference equations

$$x_{t+1} = Mx_t, \qquad t = 0,1,\dots, \tag{1}$$

where M is a large $n \times n$ random matrix, and x_t, $t = 0,1,\dots$, is an n-component vector. These equations model temporal growth of a perturbed ecological system comprising n interacting species. It is well known that the stability of (1) hinges on whether or not the spectral radius of M is less than 1. Hastings' model is based upon an earlier and closely related model by May [8]. Unfortunately, the analytic treatments by May and Hastings of the stability of (1), and of related systems, contain errors. These were discovered by Cohen and Newman [2], who give counterexamples, and who study generalizations of (1) in which $M = M_t$ is

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an iid sequence of random matrices. In Section 4 of their paper, Cohen and Newman raise the specific question addressed here: Under what conditions is the limiting spectral radius of $M_n/\sqrt{n}$ equal to σ?

It is not hard to see that σ is the natural conjecture for the limiting spectral radius. Let $m_i(n) = 1/\sqrt{n}(m_{i1}, m_{i2}, \dots, m_{in})$ be the ith row of $M_n/\sqrt{n}$. Then $m_i(n) \cdot m_j(n) \to \delta_{ij}\sigma^2$ a.s., so $M_n/\sqrt{n\sigma^2}$ resembles a unitary matrix when n is large, and its spectral radius should be near to 1. Equivalently, $M_n/\sqrt{n}$ should have spectral radius near to σ. All that we have been able to show here is that $\limsup_{n\to\infty}\rho_n \le \sigma$ a.s.; the other inequality, $\liminf_{n\to\infty}\rho_n \ge \sigma$ a.s., appears to be more difficult. Nevertheless, at least in the Gaussian case ($w_{11} \sim N(0, \sigma^2)$), computer simulations overwhelmingly support the conjecture $\rho_n \to \sigma$ a.s. Incidently, these same computer experiments suggest that any limiting spectral distribution for $M_n/\sqrt{n\sigma^2}$ will have support on the entire unit disk, and thereby demonstrate the hazard of pushing too hard the analogy to a unitary matrix.

2. Main result.

THEOREM. *Let* $M_n = \{m_{ij}(n)\}_{1\le i,j\le n}$ *be a sequence of* $n \times n$ *random matrices with* $m_{ij}(n)$, $1 \le i, j \le n$, *iid for each* n. *Assume that for each* n

(a) $Em_{11}(n) = 0$,
(b) $Em_{11}^2(n) = \sigma^2$,
(c) $E|m_{11}(n)|^p \le p^{\alpha p}$, *for all* $p \ge 2$, *some* α.

Let

$$\rho_n = \text{spectral radius of } M_n/\sqrt{n} = \max\{|\lambda| : \lambda \text{ eigenvalue of } M_n/\sqrt{n}\}.$$

Then $\limsup_{n\to\infty}\rho_n \le \sigma$ *a.s.*

REMARKS. 1. The result has recently been improved upon: Bai and Yin [1] demonstrate the same asymptotic bound for ρ_n, but replace (c) by $E|m_{11}(n)|^4 \le \alpha$, some α. 2. Some of the above-cited applications in biology involve moment conditions that depend on n. A few small changes in our proof permit a weakening of condition (c): α can depend on n ($\alpha = \alpha(n)$), where $\alpha(n)$ can grow at least as fast as $(\log n)^{1-\varepsilon}$, $\varepsilon > 0$ fixed.

3. Proof of the theorem. The assertion is equivalent to $\limsup \rho_n \le 1$ if M_n is replaced by M_n/σ. Since $m_{ij}(n)/\sigma$ has variance 1, we shall assume w.l.o.g. that $\sigma = 1$. Also, as a further notational convenience we shall write m_{ij} instead of $m_{ij}(n)$.

Let $\|V\|$ denote the *Euclidean* norm of an $n \times n$ matrix $V = \{v_{ij}\}_{1\le i,j\le n}$:

$$\|V\|^2 = \sum_{i,j} v_{ij}^2.$$

If λ is an eigenvalue of V with eigenvector f, $\|f\| = 1$, then for any positive integer p

$$\lambda^p f = V^p f \Rightarrow |\lambda|^p = \|V^p f\| \le \|V^p\|$$
$$\Rightarrow |\lambda| \le \|V^p\|^{1/p}.$$

Hence the spectral radius of V is bounded by $\|V^p\|^{1/p}$ for every $p = 1, 2, \dots$. (It is in fact well known that $\lim_{p\to\infty} \|V^p\|^{1/p}$ *equals* the spectral radius of V, but we will not make use of this relation.)

Fix $\beta > 1$, and suppose that for some sequence of positive integers $\{p_n\}_{n=1,2,\dots}$

$$E\sum_{n=1}^{\infty} \left\|\left(M_n/\sqrt{n}\right)^{p_n}\right\|^2/\beta^{2p_n} < \infty. \tag{2}$$

Then

$$\limsup_{n\to\infty} \left\|\left(M_n/\sqrt{n}\right)^{p_n}\right\|^{1/p_n} \le \beta \quad \text{a.s.},$$

and by the reasoning in the previous paragraph $\limsup_{n\to\infty}\rho_n \le \beta$ a.s. as well. Thus for the theorem, it is enough to demonstrate (2) for arbitrary but fixed $\beta > 1$ (p_n will depend upon β).

For the time being, we shall denote p_n simply by p. Concerning (2) we have:

$$\begin{aligned}
E&\sum_{n=1}^{\infty} \left\|\left(M_n/\sqrt{n}\right)^{p}\right\|^2/\beta^{2p} \\
&= \sum_{n=1}^{\infty} \frac{1}{\beta^{2p}n^p} \sum_{\substack{i,j\\ k_1,\dots,k_{p-1}\\ l_1,\dots,l_{p-1}}} Em_{ik_1}m_{k_1k_2}\cdots{}_{k_{p-1}j}\, m_{il_1}m_{l_1l_2}\cdots m_{l_{p-1}j} \qquad (3)\\
&\le \sum_{n=1}^{\infty} \frac{1}{\beta^{2p}n^p} \sum_{\substack{k_1,\dots,k_{p+1}\\ l_1,\dots,l_{p+1}}} \left|Em_{k_1k_2}m_{k_2k_3}\cdots m_{k_pk_{p+1}}m_{l_1l_2}m_{l_2l_3}\cdots m_{l_pl_{p+1}}\right| \\
&\le \sum_{n=1}^{\infty} \frac{1}{\beta^{2p}n^p} \sum{}^{*} E|m_{k_1k_2}\cdots m_{k_pk_{p+1}}m_{l_1l_2}\cdots m_{l_pl_{p+1}}|.
\end{aligned}$$

In this last expression, Σ^* denotes summation of all $E|m_{k_1k_2}\cdots m_{k_pk_{p+1}}m_{l_1l_2}\cdots m_{l_pl_{p+1}}|$ such that every matrix element appearing in the sequence $m_{k_1k_2}\cdots m_{k_pk_{p+1}}m_{l_1l_2}\cdots m_{l_pl_{p+1}}$ appears *at least twice*, and such that $1 \le k_i, l_i \le n$, $i = 1, 2, \dots, p+1$.

We will show that the expression in (3) is finite for a suitably growing sequence $p = p_n$. For this purpose we introduce a "taxonomy" for the terms appearing in Σ^*, chosen to conveniently bound contributions from collections of alike terms. This taxonomy is based upon one developed previously by Geman and Hwang [3] for a similar purpose. We begin with some preliminary definitions and conventions:

1. A particular sequence $m_{k_1k_2}\cdots m_{k_pk_{p+1}}m_{l_1l_2}\cdots m_{l_pl_{p+1}}$ appearing in Σ^* will be called a *chain*.
2. The first p elements of a chain, $m_{k_1k_2}\cdots m_{k_pk_{p+1}}$, and the last p elements of a chain, $m_{l_1l_2}\cdots m_{l_pl_{p+1}}$, will be referred to as *subchains*.

3. A *chain element* is a particular matrix element at a particular location in a chain.
4. The chain elements will be considered to be ordered by their left to right appearance, the left-most being the first.

Recall that the only chains in Σ^* are those for which each matrix element appears at least twice.

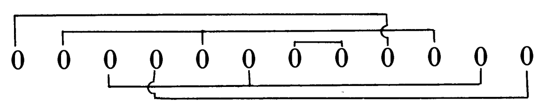
5. Call a chain element a *First* if it is not repeated to its left. All other chain elements are *Seconds*.

Every chain appearing in Σ^* can be uniquely classified according to its *pairing diagram*: For every Second, draw an arc which connects the location of that Second to the location of the (unique) First consisting of the same matrix element.

As an example, take $p = 6$, $n \geq 4$, and consider the following chain:

$$\begin{matrix} m_{12} & m_{23} & m_{34} & m_{42} & m_{23} & m_{34} & m_{11} & m_{11} & m_{12} & m_{23} & m_{34} & m_{42} \\ F & F & F & F & S & S & F & S & S & S & S & S, \end{matrix} \tag{4}$$

for which the chain elements have been labelled F or S to indicate First or Second. The pairing diagram for the chain in (4) is

(5) 

Pairing class will refer to the set of all chains with a given pairing diagram. Observe that for fixed p, and for n sufficiently large, the number of pairing classes depends only upon p. Furthermore, the elements of a given pairing class make identical contributions to the expression in (3).

The following attributes, n_s, n_r, and n_j, of a given pairing class will be of particular importance:

1. Let n_s denote the number of Seconds. Notice that $p \leq n_s \leq 2p - 1$.
2. A maximal consecutive sequence of Seconds contained in a subchain will be called a *run* (or *run of Seconds*). For the chain in (4), the fifth and sixth elements constitute a run, as do the eighth through twelfth elements. The chain $m_{11}m_{11}m_{11}m_{11}$ (with $p = 2$) has two runs: the second element and the third and fourth elements. Let n_r denote the number of runs of Seconds. Notice that $1 \leq n_r \leq n_s$.
3. We will say that two consecutive Seconds form a *junction* if (i) they are contained in the same subchain, and (ii) their corresponding Firsts are either in different subchains, or are not consecutive in the same order as the respective Seconds. For the illustration used in (4) the eighth and ninth chain elements constitute a junction. The third and fourth elements of $m_{12}m_{21}m_{21}m_{12}$ ($p = 2$) also constitute a junction. Let n_j denote the number of junctions. Notice that $0 \leq n_j \leq n_s - 1$.

The proof of the Theorem is based upon the following three lemmas. These place bounds on (i) the contribution to the expression in (3) by an element of a pairing class; (ii) the number of pairing classes; and (iii) the number of elements within a pairing class.

LEMMA i. *For any chain* $m_{k_1k_2} \cdots m_{k_pk_{p+1}} m_{l_1l_2} \cdots m_{l_pl_{p+1}}$ *in a pairing class with* n_s *Seconds,*

$$E|m_{k_1k_2} \cdots m_{k_pk_{p+1}} m_{l_1l_2} \cdots m_{l_pl_{p+1}}| \le (2p)^{2\alpha}(2p)^{2\alpha(n_s-p)}.$$

LEMMA ii. *Independent of n, the number of pairing classes with* n_s *Seconds,* n_r *runs, and* n_j *junctions is no larger than*

$$(2p)^{3n_r+2n_j}.$$

LEMMA iii. *For any n and any p, the number of chains in a pairing class with* n_s *Seconds,* n_r *runs, and* n_j *junctions is no larger than*

$$n^{2p+2-n_s-n_r-\frac{1}{2}[n_j-8n_s+8p-2n_r-4]^+}$$

where $[x]^+$ *denotes x when x is positive, and zero otherwise.*

From these lemmas, and from the expression in (3), we obtain:

$$
\begin{aligned}
&E\sum_{n=1}^{\infty}\left\|\left(M_n/\sqrt{n}\right)^p\right\|^2/\beta^{2p} \\
&\le \sum_{n=1}^{\infty}\frac{1}{\beta^{2p}n^p}\sum_{n_s=p}^{2p-1}\sum_{n_r=1}^{n_s}\sum_{n_j=0}^{n_s-1}(2p)^{3n_r+2n_j} \\
&\quad \times n^{2p+2-n_s-n_r-\frac{1}{2}[n_j-8n_s+8p-2n_r-4]^+}(2p)^{2\alpha}(2p)^{2\alpha(n_s-p)} \\
&= \left(\text{letting } \Delta_s = n_s - p \text{ and } \Delta_j = n_j - 8\Delta_s - 2n_r - 4\right) \\
&\quad \sum_{n=1}^{\infty}\frac{1}{\beta^{2p}n^p}\sum_{\Delta_s=0}^{p-1}\sum_{n_r=1}^{\Delta_s+p}\sum_{\Delta_j=-8\Delta_s-2n_r-4}^{p-7\Delta_s-2n_r-5}(2p)^{(16+2\alpha)\Delta_s+7n_r+2\Delta_j+8+2\alpha} \\
&\quad \times n^{p+2-\Delta_s-n_r-\frac{1}{2}[\Delta_j]^+} \\
&\le \sum_{n=1}^{\infty}\frac{n^2}{\beta^{2p}}\sum_{\Delta_s=0}^{p-1}\sum_{n_r=1}^{\Delta_s+p}\sum_{\Delta_j=-c_1p}^{c_1p}\left[\frac{(2p)^{c_1}}{n}\right]^{\Delta_s+n_r} \\
&\quad \times (2p)^{2\Delta_j}n^{-\frac{1}{2}[\Delta_j]^+},
\end{aligned}
\tag{6}
$$

for some sufficiently large c_1, provided that $p = p_n \ge 1$ for all n. Recall that p_n is an arbitrary sequence of positive integers; we now choose p_n such that $p_n \ge 1$, $n = 1, 2, \ldots$, $p_n \sim k \log n$, and $\sum_{n=1}^{\infty} n^2 p_n/\beta_n^{2p_n} < \infty$. In this case, for some sufficiently large c_2,

$$\sup_n \sum_{\Delta_s=0}^{p-1}\sum_{n_r=1}^{\Delta_s+p}\left[\frac{(2p)^{c_1}}{n}\right]^{\Delta_s+n_r} \le c_2.$$

Finally, we use this in (6):

$$E\sum_{n=1}^{\infty}\left\|\left(M_n/\sqrt{n}\right)^p\right\|^2/\beta^{2p} \le c_2\sum_{n=1}^{\infty}\frac{n^2}{\beta^{2p}}\sum_{\Delta_j=-c_1p}^{c_1p}(2p)^{2\Delta_{j_n}-\frac{1}{2}[\Delta_j]^+}$$

$$= c_2\sum_{n=1}^{\infty}\frac{n^2}{\beta^{2p}}\left[\sum_{\Delta_j=-c_1p}^{-1}(2p)^{2\Delta_j}+\sum_{\Delta_j=0}^{c_1p}\left[\frac{(2p)^2}{n^{1/2}}\right]^{\Delta_j}\right]$$

$$\le c_3\sum_{n=1}^{\infty}\frac{n^2p}{\beta^{2p}} < \infty,$$

for sufficiently large c_3.

We now turn to the proofs of the lemmas.

PROOF OF LEMMA i. Fix a chain $m_{k_1k_2}\cdots m_{k_pk_{p+1}}m_{l_1l_2}\cdots m_{l_pl_{p+1}}$ in a pairing class with n_s Seconds. Evidently, this chain has $2p-n_s$ Firsts. For each $1\le k\le 2p-n_s$ let n_k-1 be the number of Seconds to which the kth First is paired. Then $n_k\ge 2$,

$$\sum_{k=1}^{2p-n_s} n_k = 2p$$

and

$$E|m_{k_1k_2}\cdots m_{k_pk_{p+1}}m_{l_1l_2}\cdots m_{l_pl_{p+1}}| = \prod_{k=1}^{2p-n_s}E|m_{11}|^{n_k}.$$

As a consequence of Muirhead's Theorem (Hardy et al. [5], page 44), $\prod E|m_{11}|^{n_k}$ is maximized (under the constraints $n_k\ge 2$ and $\sum n_k = 2p$) by choosing

$$n_1 = n_2 = \cdots = n_{2p-n_s-1} = 2$$

and

$$n_{2p-n_s} = 2p - 2(2p-n_s-1) = 2n_s - 2p + 2.$$

Recall that $Em_{11}^2 = 1$:

$$E|m_{k_1k_2}\cdots m_{k_pk_{p+1}}m_{l_1l_2}\cdots m_{l_pl_{p+1}}| \le E|m_{11}|^{2n_s-2p+2}$$

$$\le (2n_s-2p+2)^{\alpha(2n_s-2p+2)}.$$

And, finally, since $n_s\le 2p-1$:

$$E|m_{k_1k_2}\cdots m_{k_pk_{p+1}}m_{l_1l_2}\cdots m_{l_pl_{p+1}}| \le (2p)^{2\alpha}(2p)^{2\alpha(n_s-p)}. \qquad \square$$

PROOF OF LEMMA ii. A run of Seconds may start at any of at most $2p$ positions, corresponding to one of the $2p$ elements of the chain. Since there are n_r runs, there are no more than $(2p)^{n_r}$ ways to configure the starting locations of these runs. Given the placement of the starts of the runs, each run could contain no more than n_s Seconds. Hence there are no more than $(n_s)^{n_r}\le(2p)^{n_r}$

ways to distribute the Seconds among the n_r runs. Now, given the locations and sizes of runs, each of the n_j junctions can be placed at fewer than $2p$ locations, and hence there are no more than $(2p)^{n_j}$ placements for the n_j junctions. Finally, given the locations and sizes of runs, and given the placements of junctions, the pairing diagram is fully determined by specifying (i) which First is to be paired to each of the n_r Seconds that begins a run, and (ii) which First is to be paired to each of the n_j Seconds that is the right-hand member of a junction. These specifications can be done in no more than $(2p)^{n_r+n_j}$ ways. Hence, the number of pairing classes with n_s Seconds, n_r runs, and n_j junctions is no larger than

$$(2p)^{n_r}(2p)^{n_r}(2p)^{n_j}(2p)^{n_r+n_j} = (2p)^{3n_r+2n_j}. \qquad \square$$

PROOF OF LEMMA iii. (In the course of proving Lemma iii, the term "index" will be used somewhat ambiguously. At times it will refer to a particular subscript of a particular chain element, whereas at other times it will refer to the numerical value of such a subscript. In each instance, the context should clarify our meaning.) Fix a pairing class with n_s Seconds, n_r runs, and n_j junctions. The number of chains in this pairing class is determined by the number of indices $k_1, k_2, \dots k_{p+1}, l_1, l_2, \dots l_{p+1}$ left free after taking into account the matches dictated by the pairing diagram. In order to count the number of free indices associated with a pairing class, we shall introduce a procedure for resolving matches through a relabelling of indices.

Before describing the general procedure, it will be helpful to work through a specific example. For this purpose, we consider again the pairing class defined by the pairing diagram in (5). With no pairings taken into account, all indices are free:

$$m_{k_1k_2}m_{k_2k_3}m_{k_3k_4}m_{k_4k_5}m_{k_5k_6}m_{k_6k_7}m_{l_1l_2}m_{l_2l_3}m_{l_3l_4}m_{l_4l_5}m_{l_5l_6}m_{l_6l_7}.$$

Observe from the pairing diagram (5) that the first chain element is paired to the ninth chain element, and hence the indices belonging to these elements must be equal:

$$m_{k_1k_2}m_{k_2k_3}m_{k_3k_4}m_{k_4k_5}m_{k_5k_6}m_{k_6k_7}m_{l_1l_2}m_{l_2k_1}m_{k_1k_2}m_{k_2l_5}m_{l_5l_6}m_{l_6l_7}.$$

Notice that indices on the eighth and tenth chain elements which are shared with the ninth chain element have been changed as a result of the pairing. We proceed now to the second First (in this case, the second chain element), and modify indices to reflect its pairings to the fifth and tenth chain elements:

$$m_{k_1k_2}m_{k_2k_3}m_{k_3k_4}m_{k_4k_2}m_{k_2k_3}m_{k_3k_7}m_{l_1l_2}m_{l_2k_1}m_{k_1k_2}m_{k_2k_3}m_{k_3l_6}m_{l_6l_7}.$$

The third and fourth Firsts are the third and fourth chain elements. The third First is paired to the sixth and the eleventh chain elements, whereas the fourth First is paired only to the last chain element:

$$m_{k_1k_2}m_{k_2k_3}m_{k_3k_4}m_{k_4k_2}m_{k_2k_3}m_{k_3k_4}m_{l_1l_2}m_{l_2k_1}m_{k_1k_2}m_{k_2k_3}m_{k_3k_4}m_{k_4k_2}.$$

Finally, we take into account the pairing between the First at the seventh position and the Second at the eighth position:

$$m_{k_1k_2}m_{k_2k_3}m_{k_3k_4}m_{k_4k_2}m_{k_2k_3}m_{k_3k_4}m_{k_1k_1}m_{k_1k_1}m_{k_1k_2}m_{k_2k_3}m_{k_3k_4}m_{k_4k_2}.$$

In this way, we arrive at a *generic* description of chain elements in the pairing class. Notice that there are four free indices in this description. Hence there are no more than n^4 chain elements in this pairing class. (Actually, there are fewer than n^4: these four "free" indices must be chosen so as to avoid further matches which would place the chain element into a different pairing class.)

Let us now formalize the procedure for resolving matches within a pairing class. The indices $k_1, \dots, k_{p+1}, l_1, \dots, l_{p+1}$ will be considered ordered in the following way: $k_1 < k_2 < \cdots < k_{p+1} < l_1 < l_2 < \cdots < l_{p+1}$. Thus we say, for example, that "l_7 is of higher order than l_2" or that "l_1 is of higher order than k_3". The relabeling procedure is this:

1. Begin with the left-most First and proceed to the right through all Firsts.
2. For each First, begin with its left-most Second and proceed to the right through all Seconds paired to that First.
3. Relabel indices to reflect the matching of a Second to its First (as defined by the pairing diagram).
   a. Begin by resolving the match of the first index of the First to the first index of the Second. Then resolve the matchings of the second indices.
   b. Always relabel the index of higher order.
   c. Any time an index is relabeled, relabel all occurrences of that index in the chain.

We begin with $2p + 2$ free indices. How many of these indices are lost in the derivation of a generic description? We observe first that all free indices originally belonging to Seconds no longer appear after the relabeling procedure. To see this, observe that the order of the index at a given location is never increased. If a free index of a Second element is unchanged at the time at which that element is matched with its First, then the corresponding index of the First, being to the left, must be of lower order (the index of the First may have been changed, but not to an index of higher order). Hence, the free index of the Second will be lost upon relabeling. On the other hand, if a free index of a Second were changed before matching, then, since all occurrences of that index were changed, it is already lost from the chain.

The number of free indices belonging to Seconds before the pairing procedure is exactly $n_s + n_r$. Hence, there are no more than $2p + 2 - n_s - n_r$ free indices in the generic chain. In fact, if there were no junctions then there would be *exactly* $2p + 2 - n_s - n_r$ free indices. Unfortunately, the number of pairing classes grows rapidly with the number of junctions (see Lemma ii), and we must be careful to identify a compensating decrease in the number of chains (equivalently, the number of free indices) within pairing classes with large numbers of junctions. Observe that the existence of a junction implies that two nonconsecutive Firsts must share at least one index. Typically, this matching will imply the loss of an index not already counted among the $n_s + n_r$ Second indices. (But not always: for example, this match may be between indices shared by Seconds and thus already accounted for.) We will estimate the number of additional indices lost as a result of n_j junctions in a pairing class.

The first step is to introduce some additional terminology:

1. A *multiple First* is a First that is paired to two or more Seconds.
2. A *multiple Second* is a Second paired with a multiple First.
3. A *neighbor* of chain element "e" is a chain element that is within the same subchain as e and either immediately precedes or immediately follows e.
4. An *end element* is the first or last chain element in a subchain (there are four end elements in each chain).
5. A *pure junction* is a junction having the properties that
   a. neither of the two junction Seconds is a multiple Second,
   b. neither of the two Firsts paired to the junction Seconds has a neighboring Second, and
   c. neither of the two Firsts paired to the junction Seconds is an end element.
6. An *F-pair* is a pair of neighboring Firsts. The index of an F-pair is the shared index: the second index on the left member of the pair and the first index on the right member.

The point of the last definition is that indices of F-pairs are not yet taken into consideration in the previous accounting of $n_s + n_r$ lost indices. Any constraints among these indices translate into *additional* losses of indices in the derivation of a generic chain. We will show that pure junctions typically imply such constraints.

LEMMA iv. *If a pairing class has n_{pj} pure junctions, then there are at least $n_{pj}/2$ indices lost in the pairing procedure, in addition to the already considered $n_s + n_r$ indices belonging to Seconds.*

LEMMA v. *The number of pure junction indices is at least*

$$\left[n_j - 8n_s + 8p - 2n_r - 4\right]^+.$$

If we put together Lemmas iv and v then we get Lemma iii: We start with $2p + 2$ free indices, of which the $n_s + n_r$ that belong to Seconds are lost. The pure junctions "cost" an additional $\frac{1}{2}[n_j - 8n_s + 8p - 2n_r - 4]^+$ indices, leaving $2p + 2 - n_s - n_r - \frac{1}{2}[n_j - 8n_s + 8p - 2n_r - 4]^+$ indices free. The free indices can range from 1 to n (barring the creation of pairings that are not dictated by the pairing diagram), leaving no more than

$$n^{2p+2-n_s-n_r-\frac{1}{2}[n_j-8n_s+8p-2n_r-4]^+}$$

chains in the pairing class. □

PROOF OF LEMMA iv. Fix a pure junction and consider the pair of Firsts that match the two junction Seconds. Let F_a designate that First matched to the left member of the junction, and let F_b designate that First matched to the right member. If F_a were a left neighbor of F_b then the two matched Seconds would not form a junction. Hence F_a is the left member of an F-pair and F_b is the right member of a distinct F-pair. Now observe that since the original junction Seconds are neighbors, the indices of these two F-pairs must be identical. In this way, each pure junction implies a matching between the indices of two distinct F-pairs. Notice that any such matching leads to one fewer free index in the generic chain element.

An F-pair can be involved in a matching through either of its two neighboring Firsts. Thus n_{pj} pure junctions imply that at least n_{pj} (rather than $2n_{pj}$) F-pairs are involved in matches. That is, each of at least n_{pj} F-pairs is matched to some other F-pair. It follows that each of at least n_{pj} F-pair indices are matched to some other F-pair index. These matchings, among n_{pj} distinct F-pair indices, must lead to a loss of at least $n_{pj}/2$ indices in the derivation of the generic chain. □

PROOF OF LEMMA v. Let us first bound the number of possible multiple Seconds. Since there are n_s Seconds and $2p - n_s$ Firsts, there are $n_s - (2p - n_s) = 2n_s - 2p$ Seconds "left over" after pairing each First to one Second. Each of these left-over Seconds could join (match) with one already paired Second to create two multiple Seconds. If a left-over Second joins a Second that is already a multiple Second, then it adds only one (instead of two) multiple Seconds. Thus there are at most $2(2n_s - 2p) = 4n_s - 4p$ multiple Seconds, each of which could be involved in two junctions. Hence, of the n_j junctions, at most $8n_s - 8p$ can contain multiple Seconds.

Among those junctions that do not contain multiple Seconds, at most four can contain Seconds that are paired to end Firsts (there are only four end elements in a chain). Concerning Firsts that have neighboring Seconds, there are at most $2n_r$ of these. Hence at most $2n_r$ of those junctions having no multiple Seconds could have Seconds paired to such Firsts. Of the n_j junctions, we are left with at least $[n_j - (8n_s - 8p) - 4 - 2n_r]^+$ pure junctions. □

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