POWER SERIES

For 2-D flow between $z = B(x,t)$ and $z = H(x,t)$ we first introduce the *fatness* and the mean (or average) height,

$$F(x,t) = H - B \text{ and } Z(x,t) = \frac{1}{2}(H+B) = B + \frac{1}{2}F\,.$$

This second approximation is based upon the substitutions,

$$u \rightarrow u_0 + \tilde{z}u_1\,,\ w \rightarrow w_0 + \tilde{z}w_1 + \frac{1}{2}\tilde{z}^2 w_2\,,\ \text{and } p \rightarrow p_0 + \tilde{z}p_1 + \frac{1}{2}\tilde{z}^2 p_2 + \frac{1}{6}\tilde{z}^3 p_3\,,$$

where $\tilde{z} = z - Z$ and $-F/2 \leq \tilde{z} \leq F/2$. Note the use of $\rightarrow$ to signify an *association*, rather than $q = q_0 + \cdots$. The second notation is too suggestive of Taylor series expansions and final determination of coefficients.

The expressions above are to be substituted in the Euler equations, and for that purpose it is convenient to make the change of variable in

$$u_x - Z_x u_{\tilde{z}} + w_{\tilde{z}} = 0,$$

$$u_t + uu_x + (w - Z_t - uZ_x)u_{\tilde{z}} + p_x - Z_x p_{\tilde{z}} = 0,$$

$$w_t + uw_x + (w - Z_t - uZ_x)w_{\tilde{z}} + p_{\tilde{z}} + g = 0.$$

Then

$$w_1 = Z_x u_1 - u_{0x}\,,\quad w_2 = -u_{1x}\,,$$

and from the kinematic conditions follows

$$w - Z_t - uZ_x = \frac{1}{8}\left(F^2 u_1\right)_x - \tilde{z}u_{0x} - \frac{1}{2}\tilde{z}^2 u_{1x}\,.$$

The three *essential* time-derivatives are in

$$F_t + u_0 F_x + F u_{0x} = 0\,,$$

$$u_{0t} + u_0 u_{0x} + \frac{1}{8} u_1 \left(F^2 u_1\right)_x + p_{0x} - Z_x p_1 = 0\,,$$

$$u_{1t} + u_0 u_{1x} + u_1 u_{0x} - u_1 u_{0x} + p_{1x} - Z_x p_2 = 0\,.$$

(The cancellation of terms shown above changes when y and v are restored.) From $Z = B + \frac{1}{2}F$ follows

$$w_0 = B_t + u_0 B_x - \frac{1}{2} F u_{0x} + \frac{1}{8}\left(F^2 u_1\right)_x\,,$$

and three conditions on the pressure coefficients follow from

$$w_{0t} + u_0 w_{0x} + \frac{1}{8}\left(F^2 u_1\right)_x (Z_x u_1 - u_{0x}) + p_1 + g = 0\,,$$

$$w_{1t} + u_0 w_{1x} + u_1 w_{0x} - u_{0x}(Z_x u_1 - u_{0x}) - \frac{1}{8}\left(F^2 u_1\right)_x u_{1x} + p_2 = 0\,,$$

$$w_{2t} + u_0 w_{2x} + 2u_1 (Z_x u_1 - u_{0x})_x + 2u_{0x} u_{1x} + p_3 = 0\,.$$

Conceptually, it looks like that's it: w's have been evaluated in terms of u's and their x-derivatives. The w_t's provide relations between p's, u's and first and second x-derivatives of both. But these are treacherous waters, and there are several entirely different ways to treat the pressure. A different way to find pressure coefficients will appear in the section on *pressure equations*.

The coordinate y and the velocity component v will now be reintroduced. Before writing the equations, the representation of pressure will first be rearranged as

$$p_H + G(H-z) + \grave{p} \text{ and } \grave{p} = \frac{1}{2}\left(\tilde{z}^2 - \frac{F^2}{4}\right)\left(p_2 + \tilde{z}\frac{p_3}{3}\right) .$$

That incorporates p_H explicitly and makes $p_B = p_H + GF$. Also, the vector and matrix notations,

$$\mathbf{u} = (u,v)^\top , \quad \nabla = (\partial_x, \partial_y)^\top , \quad \Delta = \nabla^\top \nabla \text{ and } \frac{D}{Dt} = \partial_t + \mathbf{u}_0^\top \nabla ,$$

will be used. The five degrees of freedom appear in

$$\frac{DH}{Dt} + F\nabla^\top \mathbf{u}_0 = B_t + \mathbf{u}_0^\top \nabla B ,$$

$$\frac{D\mathbf{u}_0}{Dt} + \frac{1}{8}\mathbf{u}_1 \nabla^\top F^2 \mathbf{u}_1 + \nabla p_H + G\nabla H + \frac{1}{2}F\nabla G + (\nabla \grave{p})_0 = 0 ,$$

$$\frac{D\mathbf{u}_1}{Dt} + \mathbf{u}_1^\top \nabla \mathbf{u}_0 - \mathbf{u}_1 \nabla^\top \mathbf{u}_0 + (\nabla \grave{p})_1 = \nabla G ,$$

The *formulas* for the vertical velocity components are

$$w_0 = \frac{DB}{Dt} - \frac{1}{2}F\nabla^\top \mathbf{u}_0 + \frac{1}{8}\left(\nabla^\top F^2 \mathbf{u}_1\right) ,$$

$$w_1 = \mathbf{u}_1^\top \nabla Z - \nabla^\top \mathbf{u}_0 \text{ and } w_2 = -\nabla^\top \mathbf{u}_1 ,$$

and the vertical momentum equations are,

$$\frac{Dw_1}{Dt} + \cdots + p_2 = 0 , \quad \frac{Dw_2}{Dt} + \cdots + p_3 = 0 ,$$

$$\text{and } \frac{Dw_0}{Dt} + \frac{1}{8}w_1 \nabla^\top F^2 \mathbf{u}_1 + \grave{p}_1 + g = G .$$

As suggested by the last equation, G will be called *effective gravity*, and the equation that follows from it, *the gravity equation*. The use of all three vertical momentum equations gives a direct derivation of three conditions on the pressure coefficients, but that willl not be done here. The section on *pressure equations* contains results that depend only the gravity equation and the Poisson equation that follows directly from the Euler equations and the continuity equation. The gravity equation has references to $\grave{p}$ in it, but there will be no equation for determination of $\grave{p}$ in this section.

Typical steps in the derivation of the gravity equation are

$$\frac{D\nabla^\top \mathbf{u}_k}{Dt} = \nabla^\top \frac{D\mathbf{u}_k}{Dt} - \Big((\nabla \mathbf{u}_0^\top)\nabla\Big)^\top \tilde{\mathbf{u}}_k$$

and

$$\frac{D}{Dt}\frac{DB}{Dt} = B_{tt} + 2\mathbf{u}_0^\top \nabla B_t + \mathbf{u}_0^\top(\nabla\nabla^\top)\mathbf{u}_0 + (\nabla B)^\top \frac{D\mathbf{u}_0}{Dt}.$$

Substitutions for $D\mathbf{u}_0/Dt$ and $D\mathbf{u}_1/Dt$ give the equation for G:

$$\boxed{\Big(1 + (\nabla H)^\top \nabla B - \frac{1}{2}F\Delta H\Big)G - \frac{3}{8}\nabla^\top F^2 \nabla G = g + \Gamma + \mathcal{T}\,,}$$

where

$$\Gamma = \Big(\frac{1}{2}F\nabla - (\nabla B)\Big)^\top (\nabla p_H - (\nabla\grave{p})_0) + (\grave{p})_1 - \frac{1}{8}F^2\nabla^\top(\nabla\grave{p})_1\,,$$

and $\mathcal{T}$ contains all the terms that have no explicit reference to G, p_H or $\grave{p}$ in them. Evaluation of $\mathcal{T}$ is *virtual algebra* that needn't be performed until someone decides to write a numerical implementation of this approximation.

Caution: (*and this is a really treacherous point!)* - the coefficient of $F(\nabla F)^\top \nabla G$ in the gravity equation can be $3/4$, $7/8$ or 1, depending on whether $D\mathbf{u}_1/Dt$, $(D/Dt)F\mathbf{u}_1$ or $(D/Dt)F^2\mathbf{u}_1$ is used in the derivation. The choice is largely a matter of convenience. Further comments on this and other, similar *gravity equations* are in the section named that.