PRESSURE EQUATIONS

The pressure has been written in several different ways, e.g.

$$p \to p_H + G(H-z) + \frac{1}{2}\left(\tilde{z}^2 - \frac{F^2}{4}\right)\hat{p}_2 + \frac{1}{6}\tilde{z}\left(\tilde{z}^2 - \frac{F^2}{4}\right)\hat{p}_3 + \grave{p}\,,$$

$$\text{and } p \to p_0 + \tilde{z}p_1 + \frac{1}{2}\tilde{z}^2 p_2 + \frac{1}{6}\tilde{z}^3 p_3 + \grave{p}\,,$$

where $F = H - B$, $Z = (H+B)/2$ and $\tilde{z} = z - Z$. In general, the coefficients $\{\hat{p}_k\}$ and $\{p_k\}$ are different from one another, but either set can be expressed in terms of the other when the degree of the polynomial approximations has been specified. (It is *three* in the examples.) The functions $\grave{p}(x, y, \tilde{z}, t)$ are different too, depending on what is intended for further approximations: functions with $\grave{p} = \grave{p}_{\tilde{z}} = 0$ at $\tilde{z} = \pm F/2$ is a possible choice for the first example, while further terms of the Taylor series is implied in the second. The first example, or any other that consistently employs $\grave{p}_H = \grave{p}_B = 0$, is useful for the writing of a gravity equation that defines p_B, albeit implicitly, and the second is far more suitable for use with the Poisson equation,

$$\boxed{p_{xx} + p_{yy} + p_{zz} + \phi = 0 \ \text{ with } \ \phi = u_x^2 + v_y^2 + w_z^2 + 2(u_y v_x + v_z w_y + w_x u_z)\,.}$$

The change of variables is

$$q(x,y,z,t) \to q(x,y,\tilde{z},t)\,, \quad q_t \to q_t - Z_t q_{\tilde{z}} \ \text{ and } \ q_{tt} \to q_{tt} - Z_{tt}q_{\tilde{z}} - 2Z_t q_{\tilde{z}t} + Z_t^2 q_{\tilde{z}\tilde{z}}\,,$$

with similar expressions for $x-$ and $y-$derivatives. The advantage of the representation of $p(\tilde{z})$ as a Taylor series is that the recurrence relation for pressure coefficients is fixed. After ϕ has been written in the form $\phi = \sum \phi_k \tilde{z}^k/k!$,

$$\boxed{\Delta p_k - \frac{1}{p_{k+1}}\nabla^\top p_{k+1}^2 \nabla Z + (1 + |\nabla Z|^2)p_{k+2} + \phi_k = 0,}$$

where $\nabla = (\partial_x, \partial_y)^\top$ and $\Delta = \nabla^\top\nabla$. The result looks like a sequence of evaluations of p_{k+2}, but that is not quite true because of the way coefficients have been rearranged to formulate the gravity equation.

For the treatment of relatively high orders of approximation, it may be preferable to write the entire series as

$$p = p_H + G(H - z) + \sum_{\ell=1}^{\infty} \frac{1}{(2\ell)!} \left(\tilde{z}^{2\ell} - (F/2)^{2\ell}\right) \left(p_{2\ell} + \frac{p_{2\ell+1}}{2\ell+1}\right) .$$

Then

$$\boxed{p_0 = p_H + \frac{1}{2}GF - \sum_{\ell=1}^{\infty} \frac{p_{2\ell}}{(2\ell)!}(F/2)^{2\ell} \text{ and } p_1 = -G - \frac{2}{F}\sum_{\ell=1}^{\infty} \frac{p_{2\ell+1}}{(2\ell+1)!}(F/2)^{2\ell+1}}$$

Take note: There are quite a few ways to derive *different* equations for $\{p_k\}$. One example of another way will be discussed in the section on *Green-Naghdi theory*.

The related problem, to find the Taylor series for ϕ, is most easily done by first changing the description of $\mathbf{u}$ from coefficients $\hat{\mathbf{u}}_k$ or $\tilde{\mathbf{u}}_k$ (or others) to the coefficients $\mathbf{u}_k$, For example, to the fourth order in orthogonal polynomials,

$$\begin{aligned} \mathbf{u} \quad \rightarrow \quad & \mathbf{U} + \tilde{z}\tilde{\mathbf{u}}_1 + \frac{1}{2}\left(\tilde{z}^2 - \frac{F^2}{12}\right)\tilde{\mathbf{u}}_2 + \frac{1}{6}\tilde{z}\left(\tilde{z}^2 - \frac{3F^2}{20}\right)\tilde{\mathbf{u}}_3 \\ = \quad & \left(\mathbf{U} - \frac{F^2}{24}\tilde{\mathbf{u}}_2\right) + \tilde{z}\left(\tilde{\mathbf{u}}_1 - \frac{F^2}{40}\tilde{\mathbf{u}}_3\right) + \frac{1}{2}\tilde{z}^2\tilde{\mathbf{u}}_2 + \frac{1}{6}\tilde{z}^3\tilde{\mathbf{u}}_3 \,. \end{aligned}$$

Given the $\mathbf{u}_k$'s, the w's after w_0 are defined by the fixed recurrence relation

$$w_{k+1} = \mathbf{u}_k^\top \nabla Z - \nabla^\top \mathbf{u}_k \,,$$

and the rest is just more virtual algebra.