ORTHOGONAL POLYNOMIALS

The orthogonal polynomials on \(-1 < \mu < 1\) with the weight function ”1” are the Legendre polynomials, and in a fourth order approximation for shallow water waves and flows, we would need the first five:

\[ P_0 = 1, \quad P_1 = \mu, \quad P_2 = \frac{1}{2}(3\mu^2-1), \quad P_3 = \frac{1}{2}\mu(5\mu^2-3), \quad P_4 = \frac{1}{8}(35\mu^4-30\mu^2+3). \]

(The last one is for \(w\), which has an extra power of \(\tilde{z}\).)

The substitution, \(\mu = 2\tilde{z}/F\), converts the domain to \(-F/2 < \tilde{z} < F/2\), and the orthogonal polynomials on that interval are

\[ \phi_0 = P_0 = 1, \quad \phi_1 = \frac{F}{2} P_1 = \tilde{z}, \quad \phi_2 = \frac{1}{3} \left( \frac{F}{2} \right)^2 P_2 = \frac{1}{2} \left( \tilde{z}^2 - \frac{1}{12} F^2 \right), \]

\[ \phi_3 = \frac{1}{15} \left( \frac{F}{2} \right)^3 P_3 = \frac{1}{6} \tilde{z} \left( \tilde{z}^2 - \frac{3}{20} F^2 \right) \quad \text{and} \quad \phi_4 = \frac{1}{105} \left( \frac{F}{2} \right)^4 P_4 = \frac{1}{24} \left( \tilde{z}^4 - \frac{3}{14} \tilde{z}^2 F^2 + \frac{3}{560} F^4 \right). \]

The integral of \(P_n^2\) is \(2/(2n + 1)\), and

\[ \langle \phi_0^2 \rangle = \frac{F}{2} \cdot 2 = F, \quad \langle \phi_1^2 \rangle = \left( \frac{F}{2} \right)^3 \cdot \frac{2}{3} = \frac{1}{12} F^3, \quad \langle \phi_2^2 \rangle = \left( \frac{F}{2} \right)^5 \cdot \frac{1}{9} \cdot \frac{2}{5} = \frac{1}{720} F^5, \]

\[ \langle \phi_3^2 \rangle = \left( \frac{F}{2} \right)^7 \cdot \frac{1}{15^2} \cdot \frac{2}{7} = \frac{1}{100800} F^7 \quad \text{and} \quad \langle \phi_4^2 \rangle = \left( \frac{F}{2} \right)^9 \cdot \frac{1}{105^2} \cdot \frac{2}{9} = \frac{1}{25401600} F^9. \]

The expansion scheme for the set \(\{\phi_k(\tilde{z}, F)\}\) is

\[ q = Q + \sum \hat{q}_k \phi_k \quad \text{with} \quad \langle q \rangle = Q F \quad \text{and} \quad \langle \phi_k \hat{q}_k \rangle = \langle \phi_k^2 \rangle \hat{q}_k. \]
The differential equations for the moments follow more-or-less as in the section on means and moments, with

$$\phi_k (q_t + u q_x + v q_y + w q_z) = \langle \phi_k q \rangle_t + \langle u \phi_k q \rangle_x + \langle v \phi_k q \rangle_y - \langle q \frac{d\phi_k}{dt} \rangle,$$

where \( d/dt = \partial_t + u \partial_x + v \partial_y + w \partial_z \).

In the usual notation, where \( u = (u, v) \) and \( \nabla = (\partial_x, \partial_y) \) and \( \Delta = \nabla^\top \nabla \) and \( \frac{D}{Dt} = \partial_t + U^\top \nabla \), the equations for the moments of \( u \) are

$$\langle \phi_k u \rangle_t + \left( \nabla^\top \langle \phi_k uu \rangle \right)^\top - \langle u \frac{d\phi_k}{dt} \rangle + \langle \phi_k \nabla p \rangle = 0.$$

After the pressure has been rearranged as,

$$ p = p_H + G(H - z) + \hat{p} \text{ with } \hat{p}_H = \hat{p}_B = 0, $$

and \( \nabla p = \nabla p_H + G \nabla H + \frac{1}{2} F \nabla G - \tilde{z} \nabla G + \nabla \tilde{p} \),

the nine degrees of freedom of the fourth-order approximation are in

$$\frac{DH}{Dt} + F \nabla^\top U = B_t + U^\top \nabla B,$$

$$\frac{DU}{Dt} + \frac{1}{F} \left( \nabla^\top \langle \tilde{z} u u \rangle \right)^\top + \nabla p_H + G \nabla H + \frac{1}{2} F \nabla G + \frac{1}{F} \nabla \langle \tilde{p} \rangle = 0,$$

$$\langle \tilde{z} u \rangle_t + \left( \nabla^\top \langle \tilde{z} uu \rangle \right)^\top - \langle u (w - \frac{dZ}{dt}) \rangle = \frac{F^3}{12} \nabla G - \langle \nabla \tilde{p} \rangle,$$

$$\langle \phi_k u \rangle_t + \left( \nabla^\top \langle \phi_k uu \rangle \right)^\top - \langle u \frac{d\phi_k}{dt} \rangle + \langle \phi_k \nabla \tilde{p} \rangle = 0 \ (k = 2, 3).$$
Now
\[
\frac{d\phi_k}{dt} = \phi_k \hat{z}(w - \frac{dZ}{dt}) + \frac{\partial \phi_k}{\partial F} \frac{dF}{dt}
\]
gives
\[
\frac{d\phi_2}{dt} = \hat{z}(w - \frac{dZ}{dt}) - \frac{F}{12} \frac{dF}{dt}
\]
and
\[
\frac{d\phi_3}{dt} = \frac{1}{2} \left( \hat{z}^2 - \frac{F^2}{20} \right) (w - \frac{dZ}{dt}) - \frac{\hat{z} F}{20} \frac{dF}{dt}.
\]
and the gravity equation follows from
\[
W = \frac{DB}{Dt} - \frac{1}{2} F \nabla^\top U + \frac{1}{F} \nabla^\top (\hat{z} \hat{u})
\]
and
\[
G = g + \frac{DW}{Dt} + \frac{1}{F} \nabla^\top (\hat{u} \hat{w}).
\]
This time the result is
\[
\left( 1 + \nabla H^\top \nabla B - \frac{1}{2} F \Delta H \right) G - \frac{1}{3F} \nabla^\top \left( F^3 \nabla G \right) = g + \Gamma + T.
\]
where
\[
\Gamma = \left( \frac{1}{2} F \nabla - (\nabla B) \right)^\top \left( \nabla p_H + \frac{1}{F} \nabla \langle \hat{p} \rangle \right) - \frac{1}{F} \nabla^\top \langle \hat{z} \nabla \hat{p} \rangle
\]
There is a lot of virtual algebra to be done if we find a need for higher order approximations, but the last one of these that requires any thought to speak of is the third order approximation.