ORTHOGONAL POLYNOMIALS

The orthogonal polynomials on $-1 < \mu < 1$ with the weight function "1" are the Legendre polynomials, and in a fourth order approximation for shallow water waves and flows, we would need the first five:

$$P_0 = 1\,,\ P_1 = \mu\,,\ P_2 = \frac{1}{2}(3\mu^2-1)\,,\ P_3 = \frac{1}{2}\mu(5\mu^2-3)\,,\ P_4 = \frac{1}{8}(35\mu^4-30\mu^2+3)\,.$$

(The last one is for w, which has an extra power of $\tilde{z}$.)

The substitution, $\mu = 2\tilde{z}/F$, converts the domain to $-F/2 < \tilde{z} < F/2$, and the orthogonal polynomials on that interval are

$$\phi_0 = P_0 = 1\,,\quad \phi_1 = \frac{F}{2}P_1 = \tilde{z}\,,\quad \phi_2 = \frac{1}{3}\left(\frac{F}{2}\right)^2 P_2 = \frac{1}{2}\left(\tilde{z}^2 - \frac{1}{12}F^2\right)\,,$$

$$\phi_3 = \frac{1}{15}\left(\frac{F}{2}\right)^3 P_3 = \frac{1}{6}\tilde{z}\left(\tilde{z}^2 - \frac{3}{20}F^2\right) \text{ and}$$

$$\phi_4 = \frac{1}{105}\left(\frac{F}{2}\right)^4 P_4 = \frac{1}{24}\left(\tilde{z}^4 - \frac{3}{14}\tilde{z}^2F^2 + \frac{3}{560}F^4\right)\,.$$

The integral of P_n^2 is $2/(2n+1)$, and

$$\langle\phi_0^2\rangle = \frac{F}{2}\cdot 2 = F\,,\ \langle\phi_1^2\rangle = \left(\frac{F}{2}\right)^3\cdot\frac{2}{3} = \frac{1}{12}F^3\,,\ \langle\phi_2^2\rangle = \left(\frac{F}{2}\right)^5\cdot\frac{1}{9}\cdot\frac{2}{5} = \frac{1}{720}F^5\,,$$

$$\langle\phi_3^2\rangle = \left(\frac{F}{2}\right)^7\cdot\frac{1}{15^2}\cdot\frac{2}{7} = \frac{1}{100800}F^7 \text{ and } \langle\phi_4^2\rangle = \left(\frac{F}{2}\right)^9\cdot\frac{1}{105^2}\cdot\frac{2}{9} = \frac{1}{25401600}F^9\,.$$

The expansion scheme for the set $\{\phi_k(\tilde{z}, F)\}$ is

$$q = Q + \sum \tilde{q}_k\phi_k \text{ with } \langle q\rangle = QF \text{ and } \langle\phi_k\tilde{q}_k\rangle = \langle\phi_k^2\rangle\tilde{q}_k\,.$$

The differential equations for the moments follow more-or-less as in the section on *means and moments*, with

$$\phi_k(q_t + uq_x + vq_y + wq_z)\rangle = \langle\phi_k q\rangle_t + \langle u\phi_k q\rangle_x + \langle v\phi_k q\rangle_y - \langle q\frac{d\phi_k}{dt}\rangle\,,$$

where $d/dt = \partial_t + u\partial_x + v\partial_y + w\partial_z$.

In the usual notation, where

$$\mathbf{u} = (u,v)^\top\,,\ \ \nabla = (\partial_x,\partial_y)^\top\ ,\ \ \Delta = \nabla^\top\nabla \ \text{ and }\ \frac{D}{Dt} = \partial_t + \mathbf{U}^\top\nabla\,,$$

the equations for the moments of $\mathbf{u}$ are

$$\langle\phi_k\mathbf{u}\rangle_t + \left(\nabla^\top\langle\phi_k\mathbf{u}\mathbf{u}^\top\rangle\right)^\top - \langle\mathbf{u}\frac{d\phi_k}{dt}\rangle + \langle\phi_k\nabla p\rangle = 0\,.$$

After the pressure has been rearranged as,

$$p = p_H + G(H-z) + \grave{p}\ \text{ with }\ \grave{p}_H = \grave{p}_B = 0\,,$$

$$\text{and }\ \nabla p = \nabla p_H + G\nabla H + \frac{1}{2}F\nabla G - \tilde{z}\nabla G + \nabla\grave{p}\,,$$

the nine degrees of freedom of the fourth-order approximation are in

$$\frac{DH}{Dt} + F\nabla^\top\mathbf{U} = B_t + \mathbf{U}^\top\nabla B\,,$$

$$\frac{D\mathbf{U}}{Dt} + \frac{1}{F}\left(\nabla^\top\langle\tilde{\mathbf{u}}\tilde{\mathbf{u}}^\top\rangle\right)^\top + \nabla p_H + G\nabla H + \frac{1}{2}F\nabla G + \frac{1}{F}\nabla\langle\grave{p}\rangle = 0\,,$$

$$\langle\tilde{z}\mathbf{u}\rangle_t + \left(\nabla^\top\langle\tilde{z}\mathbf{u}\mathbf{u}^\top\rangle\right)^\top - \langle\mathbf{u}(w - \frac{dZ}{dt})\rangle = \frac{F^3}{12}\nabla G - \langle\tilde{z}\nabla\grave{p}\rangle\,,$$

$$\langle\phi_k\mathbf{u}\rangle_t + \left(\nabla^\top\langle\phi_k\mathbf{u}\mathbf{u}^\top\rangle\right)^\top - \langle\mathbf{u}\frac{d\phi_k}{dt}\rangle + \langle\phi_k\nabla\grave{p}\rangle = 0 \quad (k=2,3)\,.$$

Now

$$\frac{d\phi_k}{dt} = \phi_{k\tilde{z}}(w - \frac{dZ}{dt}) + \frac{\partial \phi_k}{\partial F}\frac{dF}{dt}$$

gives

$$\frac{d\phi_2}{dt} = \tilde{z}(w - \frac{dZ}{dt}) - \frac{F}{12}\frac{dF}{dt} \quad \text{and}$$

$$\frac{d\phi_3}{dt} = \frac{1}{2}\left(\tilde{z}^2 - \frac{F^2}{20}\right)(w - \frac{dZ}{dt}) - \frac{\tilde{z}F}{20}\frac{dF}{dt}\,.$$

and the gravity equation follows from

$$W = \frac{DB}{Dt} - \frac{1}{2}F\nabla^\top \mathbf{U} + \frac{1}{F}\nabla^\top\langle\tilde{z}\tilde{\mathbf{u}}\rangle$$

$$\text{and } G = g + \frac{DW}{Dt} + \frac{1}{F}\nabla^\top\langle\tilde{\mathbf{u}}\tilde{w}\rangle\,.$$

This time the result is

$$\left(1 + \nabla H^\top \nabla B - \frac{1}{2}F\Delta H\right)G - \frac{1}{3F}\nabla^\top\left(F^3\nabla G\right) = g + \Gamma + \mathcal{T}\,.$$

where

$$\Gamma = \left(\frac{1}{2}F\nabla - (\nabla B)\right)^\top\left(\nabla p_H + \frac{1}{F}\nabla\langle\grave{p}\rangle\right) - \frac{1}{F}\nabla^\top\langle\tilde{z}\nabla\grave{p}\rangle$$

There is a lot of virtual algebra to be done if we find a need for higher order approximations, but the last one of these that requires any *thought* to speak of is the third order approximation.