According to the definition of order of the approximations that was adopted in the introduction, each independent property of the horizontal velocities counts as one in specifications of order and two in the number of degrees of freedom. There are six independent properties in the three second order theories. To see it explicitly, consider the rearrangements of $q$, $q(x, y, \bar{z}, t) \rightarrow q_0 + \bar{z} q_1 + \frac{1}{2} \bar{z}^2 q_2 + \frac{1}{6} \bar{z}^3 q_3 + \frac{1}{24} \bar{z}^4 q_4 + \frac{1}{120} \bar{z}^5 q_5$

$$= q_A + \frac{\bar{z} q_D}{F} + \frac{1}{2} \left( \bar{z}^2 - \frac{F^2}{4} \right) \left( \bar{q}_2 + \frac{\bar{z}}{3} \bar{q}_3 \right) + \cdots$$

$$= Q + \bar{z} \bar{q}_1 + \frac{1}{2} \left( \bar{z}^2 - \frac{F^2}{12} \right) \bar{q}_2 + \frac{1}{6} \bar{z} \left( \bar{z}^2 - \frac{3F^2}{20} \right) \bar{q}_3 + \cdots$$

where $\bar{z} = z - Z$, $Z = B + F/2$ and $F = H - B$. From second order approximations, we have equations for $q_0$, $q_1$, $q_A$, $q_D$, $Q$ and $\bar{q}_1 = 12 \langle \bar{z} q \rangle / F^3$. By their definitions,

$$q_A = q_0 + \frac{1}{8} F^2 q_2 + \frac{1}{384} q_4, \quad \frac{q_D}{F} = q_1 + \frac{1}{24} F^2 q_3 + \frac{1}{1920} F^4 q_5,$$

where $q_A = q_B + q_D/2$ and $q_D = q_H - q_B$, and

$$Q = q_0 + \frac{1}{24} F^2 q_2 + \frac{1}{1920} F^4 q_4$$

and

$$\bar{q}_1 = q_1 + \frac{1}{40} F^2 q_3 + \frac{1}{4480} F^4 q_5,$$

where $\langle q \rangle$ is the integral, $\int q \, dz$, from $B$ to $H$ and $Q = \langle q \rangle / F$. These can be solved for the coefficients, $u_0$ to $u_5$ of the power series, the continuity equation gives $w_0$ to $w_6$. and after the forms of the rearranged expressions have been specified, the $\hat{u}$'s, $\hat{w}$'s, $\tilde{u}$'s and $\tilde{w}$'s can all be evaluated.

By comparison of the numerical coefficients, the three kinds of approximation can be ranked, according to how well they do at making higher order terms seem less important, with series $< \text{averages} < \text{moments}$. Rather than to dismiss the
supposed lesser approximations, it can be more to our advantage to use them to
define lesser approximations of higher orders. Because of the simplifications they
bring, the equations for mean values, $U$, $V$ and $W$ lead to the cleanest equations
for $DF/Dt$ and $G$. Otherwise, there are lots of choices, but only three will be
discussed. They are MMAD (mean, moment, average, difference), merely MAD
(mean, average, difference), and truly MMMAD. The MMAD and MMMAD
approximations will be described in a section of their own.

Very little needs to be changed to derive the third order approximation, MAD. In it, the horizontal velocity is

$$u \rightarrow u_A + \frac{z u_D}{F} + \frac{1}{2} \left( \frac{z^2 - F^2}{4} \right) \hat{u}_2 = U + \frac{z}{2} \hat{u}_1 + \frac{1}{2} \left( \frac{z^2 - F^2}{12} \right) \hat{u}_2,$$

the vertical velocity is

$$w \rightarrow W + \frac{z}{2} w_1 + \frac{1}{2} \left( \frac{z^2 - F^2}{12} \right) \hat{w}_2 + \frac{1}{6} \left( \frac{z^2 - 3F^2}{20} \right) \hat{w}_3,$$

and the pressure is

$$p \rightarrow p_H + G(H - z) + \frac{1}{2} \left( \frac{z^2 - F^2}{4} \right) \left( \hat{p}_2 + \frac{z}{3} \hat{p}_3 + \frac{1}{12} \left( \frac{z^2 - F^2}{4} \right) \hat{p}_4 \right).$$

These rearrangements of polynomials of various degrees may seem arbitrary, but there are reasons (maybe bad, maybe good) for every one of them. Hence, these comments:

- There are at least three expressions for $u$ in any approximation that includes equations for $u_{Ht}$, $u_{Bt}$ and $U_t$. In this case they give the evaluations,

$$2u_0 = 3U - u_A, \quad u_1 = \frac{u_D}{F} = \hat{u}_1 = u_D/F \quad \text{and} \quad u_2 = \hat{u}_2 = \hat{u}_2 = 12(u_A - U)/F^3.$$

- The use of orthogonal polynomials to represent $\hat{u}$ puts the equation for $Z_t$ in its final form where

$$\langle \hat{z} \hat{u} \rangle = \frac{1}{12} F^3 \hat{u}_1 = \frac{1}{12} F^2 u_D.$$
It also decreases the number of terms that would otherwise be present in approximations of $\langle \tilde{u} \tilde{u}^\top \rangle$ that appear in the equation for $U_t$. In this case, the approximation is,

$$\langle \tilde{u} \tilde{u}^\top \rangle = \frac{1}{12} F u_D u_D^\top + \frac{1}{720} F^5 \tilde{u}_2 \tilde{u}_2^\top .$$

- The use of orthogonal polynomials to represent $w$ further decreases the number of terms that would otherwise appear in approximations of $\langle \tilde{u} \tilde{w} \rangle$ that appear in the equation for $W_t$. In this case, $\tilde{w}_3$ does not appear in the approximation,

$$\langle \tilde{u} \tilde{w} \rangle = \frac{1}{12} F^2 u_D \tilde{w}_1 + \frac{1}{720} F^5 \tilde{u}_2 \tilde{w}_2 .$$

- The writing of $p = p_H + G(H - z) + \hat{p}$ with $\hat{p}_H = \hat{p}_B = 0$ is the device by which the pressure equation (for $\hat{p}$) is converted to a nonhomogeneous Poisson equation with homogeneous boundary conditions at $z = H$ and $z = B$. It’s the usual device, and the cost of it in shallow water approximations is the introduction of the gravity equation. (A way to simplify the treatment of the gravity equation in numerical algorithms will be proposed presently.)

- The writing of $\hat{p} \rightarrow \cdots + \hat{p}$ with $\hat{p}_z H = \hat{p}_z B = 0$ (as above) puts the equations for $u_H$ and $u_B$ in their final form where no pressure coefficients beyond $\hat{p}_3$ ever appear.

In the MAD approximation, $D/Dt = \partial_t + U^\top \nabla$. The gravity equation follows from

$$W = \frac{DB}{Dt} - \frac{1}{2} F \nabla^\top U + \frac{1}{F} \nabla^\top \langle \tilde{z} \tilde{u} \rangle$$

and $G = g + \frac{DW}{Dt} + \frac{1}{F} \nabla \langle \tilde{u} \tilde{w} \rangle$. 

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and it determines \( p_B = p_H + GF \). The seven degrees of freedom are in

\[
\frac{DF}{Dt} + F\nabla^\top U = 0 ,
\]

\[
\frac{Du_H}{Dt} + (u_H - U)^\top \nabla u_H + \nabla p_H + (G - \dot{p}_z H) \nabla H = 0 ,
\]

\[
\frac{DU}{Dt} + \frac{1}{F} (\nabla^\top \langle \tilde{u} \tilde{u}^\top \rangle)^\top + \nabla p_H + G \nabla H + \frac{1}{2} F \nabla G + \frac{1}{F} \nabla \langle \dot{p} \rangle = 0 ,
\]

\[
\frac{Du_B}{Dt} + (u_B - U)^\top \nabla u_B + \nabla p_B + (G - \dot{p}_z B) \nabla B = 0 .
\]

The pressure equations come from results in the section on pressure equations, and there are layers of approximation in which ever more pressure coefficients are included. A reasonable place to stop is the quartic approximation, where \( \hat{p}_2 \) and \( \hat{p}_3 \) appear in the evaluations of \( \hat{p}_z H \) and \( \hat{p}_z B \), while \( \hat{p}_2 \) and \( \hat{p}_4 \) contribute to \( \langle \dot{p} \rangle \).

In the form that is most like the result in the section on means and moments, a gravity equation is

\[
\left(1 + \nabla H^\top \nabla B - \frac{1}{2} F \Delta H\right) G - \frac{1}{3F} \nabla^\top \left(F^3 \nabla G\right) = g + \Gamma + T .
\]

where

\[
\Gamma = \left(\frac{1}{2} F \nabla - (\nabla B)\right)^\top \left(\nabla p_H + \frac{1}{F} \nabla \langle \dot{p} \rangle\right) + \frac{1}{12F} \nabla^\top F^2 \left(\hat{p}_z D \nabla Z + \hat{p}_z A \nabla F\right),
\]

and the trash (pardon that) is all in \( T \). (An equation for \( DF^2 u_B / Dt \) follows easily from the ones of this section, and not all of the algebra has to be done.)