MIXED APPROXIMATIONS

According to the definition of *order* of the approximations that was adopted in the introduction, each independent property of the horizontal velocities counts as *one* in specifications of order and *two* in the number of degrees of freedom, There are *six* independent properties in the three second order theories. To see it explicitly, consider the rearrangements of q,

$$\begin{aligned} q(x,y,\tilde{z},t) \quad &\rightarrow \quad q_0 + \tilde{z}q_1 + \frac{1}{2}\tilde{z}^2 q_2 + \frac{1}{6}\tilde{z}^3 q_3 + \frac{1}{24}\tilde{z}^4 q_4 + \frac{1}{120}\tilde{z}^5 q_5 \\ &= \quad q_A + \tilde{z}\frac{q_D}{F} + \frac{1}{2}\left(\tilde{z}^2 - \frac{F^2}{4}\right)\left(\hat{q}_2 + \frac{\tilde{z}}{3}\hat{q}_3\right) + \cdots \\ &= \quad Q + \tilde{z}\tilde{q}_1 + \frac{1}{2}\left(\tilde{z}^2 - \frac{F^2}{12}\right)\tilde{q}_2 + \frac{1}{6}\tilde{z}\left(\tilde{z}^2 - \frac{3F^2}{20}\right)\tilde{q}_3 + \cdots \end{aligned}$$

where $\tilde{z} = z - Z$, $Z = B + F/2$ and $F = H - B$. From second order approximations, we have equations for q_0, q_1, q_A, q_D, Q and $\tilde{q}_1 = 12\langle \tilde{z}q\rangle/F^3$. By their definitions,

$$q_A = q_0 + \frac{1}{8}F^2 q_2 + \frac{1}{384}q_4\,, \quad \frac{q_D}{F} = q_1 + \frac{1}{24}F^2 q_3 + \frac{1}{1920}F^4 q_5\,,$$

where $q_A = q_B + q_D/2$ and $q_D = q_H - q_B$, and

$$Q = q_0 + \frac{1}{24}F^2 q_2 + \frac{1}{1920}F^4 q_4 \ \text{ and } \ \tilde{q}_1 = q_1 + \frac{1}{40}F^2 q_3 + \frac{1}{4480}F^4 q_5\,,$$

where $\langle q\rangle$ is the integral, $\int q\,dz$, from B to H and $Q = \langle q\rangle/F$. These can be solved for the coefficients, $\mathbf{u}_0$ to $\mathbf{u}_5$ of the power series, the continuity equation gives w_0 to w_6. and after the forms of the rearranged expressions have been specified, the $\hat{\mathbf{u}}$'s, $\hat{w}$'s, $\tilde{\mathbf{u}}$'s and $\tilde{w}$'s can all be evaluated.

By comparison of the numerical coefficients, the three kinds of approximation can be ranked, according to how well they do at making higher order terms seem *less important*, with *series*$<$*averages*$<$*moments*. Rather than to dismiss the supposed *lesser* approximations, it can be more to our advantage to use them to define lesser approximations of higher orders. Because of the simplifications they bring, the equations for mean values, U, V and W lead to the cleanest equations for DF/Dt and G, Otherwise, there are lots of choices, but only three will be discussed. They are MMAD (mean, moment, average, difference), merely MAD (mean, average, difference), and truly MMMAD. The MMAD and MMMAD approximations will be described in a section of their own.

Very little needs to be changed to derive the third order approximation, MAD. In it, the horizontal velocity is

$$\mathbf{u} \to \mathbf{u}_A + \tilde{z}\frac{\mathbf{u}_D}{F} + \frac{1}{2}\left(\tilde{z}^2 - \frac{F^2}{4}\right)\hat{\mathbf{u}}_2 = \mathbf{U} + \tilde{z}\tilde{\mathbf{u}}_1 + \frac{1}{2}\left(\tilde{z}^2 - \frac{F^2}{12}\right)\tilde{\mathbf{u}}_2\,,$$

the vertical velocity is

$$w \to W + \tilde{z}\tilde{w}_1 + \frac{1}{2}\left(\tilde{z}^2 - \frac{F^2}{12}\right)\tilde{w}_2 + \frac{1}{6}\tilde{z}\left(\tilde{z}^2 - \frac{3F^2}{20}\right)\tilde{w}_3\,,$$

and the pressure is

$$p \to p_H + G(H-z) + \frac{1}{2}\left(\tilde{z}^2 - \frac{F^2}{4}\right)\left(\hat{p}_2 + \frac{\tilde{z}}{3}\hat{p}_3 + \frac{1}{12}\left(\tilde{z}^2 - \frac{F^2}{4}\right)\hat{p}_4\right)\,.$$

These rearrangements of polynomials of various degrees may seem arbitrary, but there are reasons *(maybe bad, maybe good)* for every one of them. Hence, these comments:

- There are at least three expressions for $\mathbf{u}$ in any approximation that includes equations for $\mathbf{u}_{Ht}$, $\mathbf{u}_{Bt}$ and $\mathbf{U}_t$. In this case they give the evaluations,

  $2\mathbf{u}_0 = 3\mathbf{U} - \mathbf{u}_A\,,\ \mathbf{u}_1 = \tilde{\mathbf{u}}_1 = \mathbf{u}_D/F$ and $\mathbf{u}_2 = \hat{\mathbf{u}}_2 = \tilde{\mathbf{u}}_2 = 12(\mathbf{u}_A - \mathbf{U})/F^3\,.$

- The use of orthogonal polynomials to represent $\tilde{\mathbf{u}}$ puts the equation for Z_t in its final form where

$$\langle \tilde{z}\tilde{\mathbf{u}}\rangle = \frac{1}{12}F^3\tilde{\mathbf{u}}_1 = \frac{1}{12}F^2\mathbf{u}_D\,.$$

It also decreases the number of terms that would otherwise be present in approximations of $\langle \tilde{\mathbf{u}}\tilde{\mathbf{u}}^\top \rangle$ that appear in the equation for $\mathbf{U}_t$. In this case, the approximation is,

$$\langle \tilde{\mathbf{u}}\tilde{\mathbf{u}}^\top \rangle = \frac{1}{12} F \mathbf{u}_D \mathbf{u}_D^\top + \frac{1}{720} F^5 \tilde{\mathbf{u}}_2 \tilde{\mathbf{u}}_2^\top \, .$$

- The use of orthogonal polynomials to represent w further decreases the number of terms that would otherwise be present in approximations of $\langle \tilde{\mathbf{u}}\tilde{w} \rangle$ that appear in the equation for W_t. In this case, $\tilde{w}_3$ does not appear in the approximation,

$$\langle \tilde{\mathbf{u}}\tilde{w} \rangle = \frac{1}{12} F^2 \mathbf{u}_D \tilde{w}_1 + \frac{1}{720} F^5 \tilde{\mathbf{u}}_2 \tilde{w}_2 \, .$$

- The writing of $p = p_H + G(H - z) + \grave{p}$ with $\grave{p}_H = \grave{p}_B = 0$ is the *device* by which the pressure equation (for $\grave{p}$) is converted to a nonhomogeneous Poisson equation with homogeneous boundary conditions at $z = H$ and $z = B$. It's the usual device, and the cost of it in shallow water approximations is the introduction of the gravity equation. (A way to simplify the treatment of the gravity equation *in numerical algorithms* will be proposed presently.)

- The writing of $\grave{p} \to \cdots + \grave{p}$ with $\grave{p}_{zH} = \grave{p}_{zB} = 0$ (as above) puts the equations for $\mathbf{u}_{Ht}$ and $\mathbf{u}_{Bt}$ in their final form where no pressure coefficients beyond $\hat{p}_3$ ever appear.

In the MAD approximation, $D/Dt = \partial_t + \mathbf{U}^\top \nabla$. The gravity equation follows from

$$\boxed{\begin{gathered} W = \frac{DB}{Dt} - \frac{1}{2} F \nabla^\top \mathbf{U} + \frac{1}{F} \nabla^\top \langle \tilde{z} \tilde{\mathbf{u}} \rangle \\ \text{and } G = g + \frac{DW}{Dt} + \frac{1}{F} \nabla \langle \tilde{\mathbf{u}} \tilde{w} \rangle \, , \end{gathered}}$$

and it determines $p_B = p_H + GF$. The seven degrees of freedom are in

$$\frac{DF}{Dt} + F\nabla^\top \mathbf{U} = 0\,,$$

$$\frac{D\mathbf{u}_H}{Dt} + (\mathbf{u}_H - \mathbf{U})^\top \nabla \mathbf{u}_H + \nabla p_H + (G - \grave{p}_{zH})\nabla H = 0\,,$$

$$\frac{D\mathbf{U}}{Dt} + \frac{1}{F}(\nabla^\top \langle \tilde{\mathbf{u}}\tilde{\mathbf{u}}^\top \rangle)^\top + \nabla p_H + G\nabla H + \frac{1}{2}F\nabla G + \frac{1}{F}\nabla \langle \grave{p} \rangle = 0\,,$$

$$\frac{D\mathbf{u}_B}{Dt} + (\mathbf{u}_B - \mathbf{U})^\top \nabla \mathbf{u}_B + \nabla p_B + (G - \grave{p}_{zB})\nabla B = 0\,.$$

The pressure equations come from results in the section on *pressure equations*, and there are layers of approximation in which ever more pressure coefficients are included. A reasonable place to stop is the quartic approximation, where $\hat{p}_2$ and $\hat{p}_3$ appear in the evaluations of $\grave{p}_{zH}$ and $\grave{p}_{zB}$, while $\hat{p}_2$ and $\hat{p}_4$ contribute to $\langle \grave{p} \rangle$.

In the form that is most like the result in the section on means and moments, a gravity equation is

$$\left(1 + \nabla H^\top \nabla B - \frac{1}{2}F\Delta H\right) G - \frac{1}{3F}\nabla^\top \left(F^3 \nabla G\right) = g + \Gamma + \mathcal{T}\,.$$

where

$$\Gamma = \left(\frac{1}{2}F\nabla - (\nabla B)\right)^\top \left(\nabla p_H + \frac{1}{F}\nabla \langle \grave{p} \rangle\right) + \frac{1}{12F}\nabla^\top F^2 \left(\grave{p}_{zD}\nabla Z + \grave{p}_{zA}\nabla F\right)\,,$$

and the trash (pardon that) is all in $\mathcal{T}$. (An equation for $DF^2\mathbf{u}_D/Dt$ follows easily from the ones of this section, and not all of the algebra has to be done.)