$\mathcal{M}EANS\ \ AND\ \ MOMENTS$

Another method for defining shallow water approximations employs the formulation of the Euler equations and a few others as the conservation laws

$$1_t + (1u)_x + (1w)_z = 0,$$

$$z_t + (uz)_x + (wz)_z = w,$$

$$u_t + (uu)_x + (wu)_z + p_x = 0,$$

$$w_t + (uw)_x + (ww)_z + p_z + g = 0.$$

The artificial conservation law for the constant 1 is familiar, and the one for z, less so. Both appear somewhat more naturally in work on atmospheres, where the density $\rho(x, z, t)$ is not a constant.

Most of the notation to be used for 2-D flow in $B(x,t) < z < H(x,t)$ is

$$q_H(x,t) = q(x, H(x,t), t) \text{ and } q_B(x,t) = q(x, B(x,t), t), \quad \text{(evaluations)}$$

$$\langle q \rangle = \int_B^H q dz \text{ with } \langle 1 \rangle = F = H - B \text{ and } \langle z \rangle = \frac{1}{2} F(H+B), \quad \text{(integrals)}$$

$$\text{and } Q(x,t) = \frac{\langle q \rangle}{F} \text{ with } Z(x,t) = \frac{1}{2}(H+B), \qquad \text{(mean values)}$$

and the kinematic conditions are

$$H_t + u_H H_x = w_H \text{ and } B_t + u_B B_x = w_B\,.$$

An attractive feature of this approach is the *inherited* conservation laws that follow from the kinematic conditions. We define *deviations*, $\tilde{q} = q - Q$, and observe that

$$\langle q_t + (uq)_x + (wq)_z \rangle = \langle q \rangle_t + \langle uq \rangle_x = (FQ)_t + (FUQ)_x + \langle \tilde{u}\tilde{q} \rangle_x \,.$$

The fatness $F(x,t)$ plays the the rôle of density and the integrals $\langle \tilde{u}\tilde{q} \rangle$ resemble Reynolds stresses in the inherited conservation laws,

$$F_t + (FU)_x = F_t + UF_x + FU_x = 0\,,$$

$$F(Z_t + UZ_x) + \langle \tilde{u}\tilde{z} \rangle_x = FW\,,$$

$$F(U_t + UU_x) + \langle \tilde{u}\tilde{u} \rangle_x + \langle p_x \rangle = 0\,,$$

$$F(W_t + UW_x) + \langle \tilde{u}\tilde{w} \rangle_x + Fg = p_B - p_H\,.$$

(The first of these has been used to rewrite the others in the alternate form.)

To find another equation to govern a second order theory where

$$u - U = \tilde{u} \to \tilde{z}\tilde{u}_1 \text{ and } w - W = \tilde{w} \to \tilde{z}\tilde{w}_1 + \frac{1}{2}\left(\tilde{z}^2 - \frac{F^2}{12}\right)\tilde{w}_2\,,$$

introduce the *first moments*,

$$\langle \tilde{z}q \rangle = \langle \tilde{z}\tilde{q} \rangle = \frac{1}{12}F^3\tilde{q}_1\,.$$

To find the time-derivatives of the first moments, observe

$$\begin{aligned}
\langle \tilde{z}\,(q_t + uq_x + wq_z) \rangle &= \langle (\tilde{z}q)_t + (u\tilde{z}q)_x + (w\tilde{z}q)_z - (\tilde{z}_t + u\tilde{z}_x + w\tilde{z}_z)q \rangle \\
&= \langle \tilde{z}\tilde{q} \rangle_t + \langle \tilde{z}uq \rangle_x + \langle (Z_t + uZ_x - w)q \rangle \\
&= \langle \tilde{z}\tilde{q} \rangle_t + \langle \tilde{z}uq \rangle_x - Q\langle \tilde{z}\tilde{u} \rangle_x + \langle (Z_x\tilde{u} - \tilde{w})\tilde{q} \rangle\,.
\end{aligned}$$

From the continuity equation,

$$Z_x \tilde{u} - \tilde{w} = \tilde{z} U_x + \frac{1}{2}\left(\tilde{z}^2 - \frac{F^2}{12}\right)\tilde{u}_{1x}\,,$$

and the result for $q = u$ is

$$\boxed{\langle \tilde{z}\tilde{u}\rangle_t + U\langle \tilde{z}\tilde{u}\rangle_x + 3U_x\langle \tilde{z}\tilde{u}\rangle + \langle \tilde{z}p_x\rangle = 0\,.}$$

(The result for $q = w$, which will not be used here, is slightly more complicated.)

As usual (now), before restoring v and y, rewrite the pressure as

$$p \to p_H + G(H - z) + \grave{p}\,,\ p_B = p_H + GF \text{ and } \grave{p} = \frac{1}{2}\left(\tilde{z}^2 - \frac{F^2}{4}\right)\left(p_2 + \frac{\tilde{z}}{3}p_3\right)\,,$$

and introduce the vector and matrix notations,

$$\mathbf{u} = (u, v)^\top\,,\quad \nabla = (\partial_x, \partial_y)^\top\,,\quad \Delta = \nabla^\top\nabla \text{ and } \frac{D}{Dt} = \partial_t + \mathbf{U}^\top\nabla$$

Then the five degrees of freedom are in

$$\boxed{\begin{gathered}\frac{DH}{Dt} + F\nabla^\top\mathbf{U} = B_t + \mathbf{U}^\top\nabla B\,,\\ \frac{D\mathbf{U}}{Dt} + \frac{1}{12F}\left(\nabla^\top F^3\tilde{\mathbf{u}}_1\tilde{\mathbf{u}}_1^\top\right)^\top + \nabla p_H + G\nabla H + \frac{1}{2}F\nabla G + \frac{1}{F}\nabla\langle\grave{p}\rangle = 0\,,\\ \text{and } \frac{D}{Dt}(F^3\tilde{\mathbf{u}}_1) + F^3(2\tilde{\mathbf{u}}_1\nabla^\top + \tilde{\mathbf{u}}_1^\top\nabla)\mathbf{U} + 12\left(\nabla\langle\tilde{z}\grave{p}\rangle + \langle\grave{p}\rangle\nabla Z\right) = F^3\nabla G\,,\end{gathered}}$$

and the gravity equation comes from

$$W = \frac{DB}{Dt} - \frac{1}{2} F \nabla^\top \mathbf{U} + \frac{1}{12F} \nabla^\top F^3 \tilde{\mathbf{u}}_1$$

$$\text{and } G = g + \frac{DW}{Dt} + \frac{1}{12F} \nabla^\top F^3 \tilde{\mathbf{u}}_1 (\tilde{\mathbf{u}}_1^\top \nabla Z - \nabla^\top \mathbf{U}) \, .$$

This time the result is

$$\left(1 + \nabla H^\top \nabla B - \frac{1}{2} F \Delta H\right) G - \frac{1}{3F} \nabla^\top \left(F^3 \nabla G\right) = g + \Gamma + \mathcal{T} \, ,$$

where

$$\Gamma = \left(\frac{1}{2} F \nabla - (\nabla B)\right)^\top \left(\nabla p_H + \frac{1}{F} \nabla \langle \grave{p} \rangle\right) - \frac{1}{F} \nabla^\top \left(\nabla \langle \tilde{z} \grave{p} \rangle + \langle \grave{p} \rangle \nabla Z\right)$$

and $\mathcal{T}$ can be evaluated whenever it is needed. As usual, the coefficient of $F(\nabla F)^\top \nabla G$ can be different (from 1) in slightly different derivations of a gravity equation.

Except for the gravity equations, the three second order approximations are all pretty simple, so simplicity of their governing equations doesn't provide much of a basis for comparison. The simplification that comes from the use of means and moments appears in the formulation of approximations of higher orders. From the counting of degrees of freedom it follows that only one equation is needed for DF/Dt, and as Airy knew, the introduction of the mean values, U and V gives a *final determination* of that equation. Other equations for DF/Dt gain new terms in higher approximations. Keeping W among the properties of w simplifies the approximations too. When the gravity equation is derived from W, $\mathbf{U}$ and $\langle \tilde{z} \mathbf{u} \rangle$, it is in a final form, with references to other properties of $\mathbf{u}$ all virtually buried in $\mathcal{T}$. Further uses of orthogonal polynomials will be discussed in a separate section .