DISPERSION RELATIONS

In discussions of linearized approximations for surface waves over a flat bottom at $z = B = 0$ the the direct use of the coordinate z is sometimes more convenient than use of $\tilde{z} = z - H/2$. In a two-dimensional flow where

$$p_{xx} + p_{zz} = 0 \text{ and } p_H(x,t) = 0 \text{ at } z = H(x,t) = h + \zeta \sin(kx - \omega t)\,,$$

the pressure is

$$p = \pi_0 + \pi_1 z + (\pi_2 \cosh kz + \pi_3 \sinh kz) \sin(kx - \omega t)\,.$$

Then

$$w_B = B_t = 0 \to p_{zB} + g = 0 \to \pi_1 = -g \text{ and } \pi_3 = 0\,,$$

$$p_H = 0 \;\to\; \pi_0 = gh \text{ and } g\zeta = \pi_2 \cosh kh \;\text{ and}$$

$$w_H = H_t \to H_{tt} + p_{zH} + g = 0 \to \omega^2 \zeta = k\pi_2 \sinh kh\,.$$

Thus Airy's dispersion relation, $\omega^2 = gk \tanh kh$, follows from *what appears to be not very much information about the flow*. This *kind* derivation of the result has been around since Poincaré's work or before, but it is seldom seen, except in treatments of rotating flows, where the ever popular assumption of potential flow is not an option.

The simple result serves here to introduce the subject of this piece,

rational function approximations of $\tanh \kappa$
that follow from polynomial approximations of the pressure.

In the first examples the pressure is

$$p = \pi_0 + \pi_1 z + (\pi_2 C_m(kz) + \pi_3 S_n(kz)) \sin(kx - \omega t)\,,$$

where C_m and S_n are are truncated Taylor series of degrees m and n of $cosh$ and $sinh$. The results, $\pi_3 = 0$ and $\pi_1 = -g$, follow as before, and the other conditions are

$$g\zeta = \pi_2 C_m(\kappa) \text{ and } \omega^2\zeta = k\pi_2 S_{m-1}(\kappa)\,,$$

where $\kappa = kh$. The C's are even functions, and the approximate results

$$\frac{\omega^2}{gk} = \frac{S_{2\ell-1}(\kappa)}{C_{2\ell}(\kappa)}\,,$$

imply one of several sequences of rational approximations of $\tanh\kappa$. From

$$C_{2\ell}(\kappa) \sim \cosh\kappa - \frac{\kappa^{2\ell+2}}{(2\ell+2)!} \;\text{ and }\; S_{2\ell-1}(\kappa) \sim \sinh\kappa - \frac{\kappa^{2\ell+1}}{(2\ell+1)!}\,,$$

estimates of errors in the long-wave limits ($\kappa \to 0$) can easily be found by anyone who thinks it worth the effort, and the short wave limit is

$$\frac{S_{2\ell-1}(\kappa)}{C_{2\ell}(\kappa)} \to \frac{2\ell}{\kappa} \;\text{ as }\; \kappa \to \infty\,.$$

An observation: *These are not very good approximations of* $\tanh\kappa$.

Somewhat better rational approximations of $\tanh\kappa$ follow from rewriting the expression for pressure as

$$p = \varpi_0 + \varpi_1 z + \Big(\varpi_2 \cosh k(z - \frac{1}{2}h) + \varpi_3 \sinh k(z - \frac{1}{2}h)\Big)\sin(kx - \omega t)\,.$$

It is the same solution of $p_{xx} + p_{zz} = 0$, rewritten in terms of the linearized version of $\tilde{z} = z - H/2$. The condition at $z = 0$ now implies

$$\varpi_1 = -g \;\text{ and }\; \varpi_2 \sinh\frac{1}{2}\kappa = \varpi_3 \cosh\frac{1}{2}\kappa\,,$$

and the conditions at $z = h + \zeta\sin(kx - \omega t)$, suitably linearized, are

$$g\zeta = \varpi_2 \cosh\frac{1}{2}\kappa + \varpi_3 \sinh\frac{1}{2}\kappa \text{ and } \omega^2\zeta = k\varpi_2 \sinh\frac{1}{2}\kappa + \varpi_3 \cosh\frac{1}{2}\kappa\,.$$

The *exact* dispersion relation is

$$\frac{\omega^2}{gk} = \frac{2\sinh(\kappa/2)\cosh(\kappa/2)}{\cosh^2(\kappa/2)+\sinh^2(\kappa/2)} = \tanh\kappa\,,$$

and the approximations that follow from

$$p = \varpi_0 + \varpi_1 z + \Big(\varpi_2 C_m(z-\frac{1}{2}h) + \varpi_3 S_n(z-\frac{1}{2}h)\Big)\sin(kx-\omega t)$$

are

$$\frac{\omega^2}{gk} = \frac{2\mathcal{S}_{m-1}(\kappa/2)\mathcal{C}_{n-1}(\kappa/2)}{\mathcal{C}_m(\kappa/2)\mathcal{C}_{n-1}(\kappa/2)+\mathcal{S}_n(\kappa/2)\mathcal{S}_{m-1}(\kappa/2)}\,.$$

where $\mathcal{C}_m = C_m(\kappa/2)$ and $\mathcal{S}_n = S_n(\kappa/2)$. The results for even and odd truncations of the series for $p(z-h/2)$ are

$$p = P_{2\ell}\ :\quad \omega^2 = gk\,\frac{2\mathcal{S}_{2\ell-1}\mathcal{C}_{2\ell-2}}{\mathcal{C}_{2\ell}\mathcal{C}_{2\ell-2}+\mathcal{S}^2_{2\ell-1}} \ \to\ 4\frac{4\ell-1}{\kappa} \text{ as } \kappa\to\infty\,,$$

$$p = P_{2\ell+1}\ :\quad \omega^2 = gk\,\frac{2\mathcal{S}_{2\ell-1}\mathcal{C}_{2\ell}}{\mathcal{C}^2_{2\ell}+\mathcal{S}_{2\ell+1}\mathcal{S}_{2\ell-1}} \ \to\ 4\frac{4\ell+1}{\kappa} \text{ as } \kappa\to\infty\,.$$

Long wave limits of these are improved because of the powers of *two* in

$$\mathcal{C}_{2\ell} \sim \cosh(\kappa/2) - \frac{\kappa^{2\ell+2}}{2^{2\ell+2}(2\ell+2)!} \text{ and } \mathcal{S}_{2\ell-1} \sim \sinh(\kappa/2) - \frac{\kappa^{2\ell+1}}{2^{2\ell+1}(2\ell+1)!}\,,$$

and the short wave limits are somewhat better than $\to 2\ell/\kappa$.

To find results from shallow water approximations, we shall need the linearized version of $p = p_H + G(H-z) + \grave{p}$ and $\grave{p}_B = \grave{p}_H = 0$ that follows when $B = 0$,

$H = h + \zeta \sin(kx - \omega t)$ and $G = g + \gamma \sin(kx - \omega t)$ The hard way (not recommended) is to sum the series that appear in results from the section on *pressure equations*. From general results of this section and the single condition, $p_H = 0$, it follows that

$$p = \frac{1}{2}gh - g\tilde{z} + \tilde{p}\sin(kx - \omega t)\,, \quad \text{where} \quad \tilde{z} = z - \frac{h}{2}$$

$$\text{and} \quad \tilde{p} = \left(g\zeta + \frac{1}{2}h\gamma\right)\frac{\cosh k\tilde{z}}{\cosh \kappa/2} - \frac{1}{2}h\gamma\,\frac{\sinh k\tilde{z}}{\sinh \kappa/2}\,.$$

From this follows

$$\grave{p} = \left(\left(g\zeta + \frac{1}{2}h\gamma\right)\left(\frac{\cosh k\tilde{z}}{\cosh \kappa/2} - 1\right) - \frac{1}{2}h\gamma\left(\frac{\sinh k\tilde{z}}{\sinh \kappa/2} - \frac{2}{h}\tilde{z}\right)\right)\sin(kx - \omega t).$$

Shallow water approximations that include the equation for the mean value, U, imply the linearized relations,

$$U_t + gH_x + \frac{1}{2}hG_x + \frac{1}{h}\langle\grave{p}\rangle_x = 0\,,\ \langle\grave{p}\rangle = \int_0^h \grave{p}\,dz = \int_{-h/2}^{h/2} \grave{p}\,d\tilde{z}\,,$$

$$H_t + hU_x = 0 \ \text{ and } \ H_{tt} = h\left(gH_{xx} + \frac{1}{2}hG_{xx}\right) + \langle\grave{p}\rangle_{xx}.$$

From these comes

$$\omega^2\zeta = (gk\kappa\zeta + \frac{1}{2}\kappa^2\gamma)\left(1 + \frac{2}{\kappa}\tanh\frac{\kappa}{2} - 1\right)\,.$$

The linearized gravity equation follows from

$$Z = \frac{1}{2}H\,,\ W = Z_t + \frac{1}{h}\langle\tilde{z}\tilde{u}\rangle_x \ \text{ and } \ G - g = W_t = \frac{1}{2}H_{tt} + \frac{1}{h}\langle\tilde{z}u\rangle_{xt}\,.$$

The final result depends upon the expression that is used for $\langle\tilde{z}u\rangle_t$, and the one that *works best* comes from the sections on *means and moments* and/or *orthogonal polynomials*. The linearized equation,

$$\langle \tilde{z} u \rangle_t = \frac{1}{12} h^3 G_x - \langle \tilde{z} \hat{p} \rangle_x \,,$$

implies

$$\gamma = -\frac{1}{2}\omega^2 \zeta - \frac{1}{12}\kappa^2 \gamma - \frac{1}{2} k^2 \gamma \left(\frac{2}{k^2} \left(\frac{\kappa}{2} \coth \kappa/2 - 1 \right) - \frac{1}{6} h^2 \right) .$$

After all the cancellations in this and the previous equation for ζ and γ have been used up, it follows easily that

$$\frac{\omega^2}{gk} = \frac{2 \tanh \kappa/2}{1 + \tanh^2 \kappa/2} = \tanh \kappa = \frac{2 \tanh \kappa/2 \coth \kappa/2}{\coth \kappa/2 + \tanh \kappa/2} .$$

The last identity was included to facilitate the writing of the rational approximations of $\tanh \kappa$ that follow when the power series expansion of the pressure is truncated. The terms that sum to $\tanh \kappa/2$ come from the integral $\langle \cosh k\tilde{z} \rangle$, and the ones that sum to $\coth \kappa/2$ come from $\langle \tilde{z} \sinh k\tilde{z} \rangle$. Their truncated versions are:

$$\tanh \frac{\kappa}{2} \;\rightarrow\; \frac{\mathcal{S}_{2\ell+1}}{\mathcal{C}_{2\ell}} \quad \text{and} \quad \coth \frac{\kappa}{2} \;\rightarrow\; \frac{\mathcal{C}'_{2m}}{\mathcal{S}_{2m-1}} \,,$$

$$\text{where} \;\; \mathcal{C}'_{2m} \;=\; \mathcal{C}_{2m} - \frac{2}{\kappa}(\mathcal{S}_{2m+1} - \mathcal{S}_{2m-1}) \;=\; \mathcal{C}_{2m-2} + \frac{2m}{(2m+1)!}\left(\frac{\kappa}{2}\right)^{2m} .$$

The results for even and odd truncations of the series for $p(\tilde{z})$ are

$$p = P_{2\ell} \;:\;\; \frac{\omega^2}{gk} = \frac{2\mathcal{S}_{2\ell+1}\mathcal{C}'_{2\ell}}{\mathcal{C}_{2\ell}\mathcal{C}'_{2\ell} + \mathcal{S}_{2\ell-1}\mathcal{S}_{2\ell+1}} \;\rightarrow\; \frac{\kappa}{4\ell+2} \;\; \text{as} \;\; \kappa \rightarrow \infty \,,$$

$$p = P_{2\ell-1} \;:\;\; \frac{\omega^2}{gk} = \frac{2\mathcal{S}_{2\ell-1}\mathcal{C}'_{2\ell}}{\mathcal{C}_{2\ell-2}\mathcal{C}'_{2\ell} + \mathcal{S}_{2\ell-1}\mathcal{S}_{2\ell-1}} \;\rightarrow\; \frac{\kappa}{4\ell} \;\; \text{as} \;\; \kappa \rightarrow \infty \,.$$

Recovery of ever more terms in the Taylor series expansion of $\tanh \kappa$ is guaranteed, and there is not much point to doing the algebra.

These shallow water results came from the second order approximation, MM (*mean and one moment*), and a notable feature is that the approximations have short wave limits that approach the result, $\tanh\kappa \to 1$ as $\kappa \to \infty$, from *above*. The second order approximation, AD (*average and difference*), has all the information about boundary conditions on pressure, with no approximations other than estimates of p_H, p_B, p_{zH} and p_{zB}. Thus it could be rigged to include Airy's result with approximations that have short wave limits that approach the result, $\tanh\kappa \to 1$ as $\kappa \to \infty$, from *below*. None of the first order approximations that have been discussed here contains Airy's result, and the one that appears to come closest to it is M, which has no equation for $\langle\tilde{z}u\rangle$. Setting $\langle\tilde{z}u\rangle = 0$ eliminates most of the content of the gravity equation, and the result is

$$\frac{\omega^2}{gk} \;=\; \frac{2\tanh\kappa/2}{1+(\kappa/2)\tanh\kappa/2}\,.$$

In the long wave limit, two terms of the Taylor series for $\tanh\kappa$ are recovered, and the short wave limit is $\omega^2/gk \to 4/\kappa$ as $\kappa \to \infty$.

Now we have two second order approximations that pass the dispersion relation test perfectly and at least one fourth order one, MMAD, that does the same in at least two entirely different ways. As for my opinion of the use of Airy's result in arguments over which of several nonlinear approximations might be *more correct* than another - well, it's just an opinion, and not vulnerable to anything more treacherous than ridicule, so here it is:

this toy is broken.