MULTI – LAYER MODELS

The introduction of multiple layers leads to models that look more and more like numerical integrations of the full three-dimensional unsteady flow. The major purpose of this piece is to investigate how some features of shallow water approximations can be incorporated in the multilayer models. There will be untested speculation here, and these ideas will probably need modification if anyone ever gets around to implementing them.

The two-layer model is almost exactly like the one that has been called MAD, so a four-layer example will be based upon equations for time rates of change of horizontal velocity components,

$$\boxed{\mathbf{u}_k(x,y,t) = \mathbf{u}(x,y,Z_k(x,y,t),t)\,,\ k = 0\cdots 4\,,}$$

defined at equally spaced levels $z = Z_k$ that run from $Z_0 = H$ to $Z_4 = B$. This will have *eleven* degrees of freedom after the introduction of an equation for H, and to make it more like familiar numerical algorithms for 3-D hydrodynamics, there will also be equations for time rates of change of w_1 to w_3. Equations for $Z(= Z_2)$ and W will be introduced for use in ways that do not add to the number of degrees of freedom.

All this is to be accompanied by a Poisson equation for pressure, and that equation is to be integrated numerically by the use of central differences, with values of pressure, $p_H \cdots p_B$, defined at the levels Z_k. The optional use of extra levels at the midpoints of the layers can be introduced to bring the solution of the Poisson equation somewhat closer to an exact solution, thus to improve the description of short waves (we hope). As in most other sections, the representation of pressure is

$$\boxed{p = p_H(x,y,t) + G(x,y,t)(H - z) + \grave{p}(x,y,z,t) \ \text{ with } \ \grave{p}_H = \grave{p}_B = 0\,.}$$

One of the aims here is to make the implementation of the Poisson equation for $\grave{p}$ as easy as possible.

The *eleven* rate equations are

$$\frac{DH}{Dt} + F\nabla^\top \mathbf{U} = \dot{B} \quad \left(= \frac{DB}{Dt}\right),$$

$$\mathbf{u}_{kt} + \mathbf{u}_k^\top \nabla \mathbf{u}_k + (w_k - Z_{kt} - \mathbf{u}_k^\top \nabla Z_k)\mathbf{u}_{zk} + \nabla p_k - p_{zk}\nabla Z_k = 0\,.$$

In these $F = H - B$ and $Dq/Dt = q_t + \mathbf{U}^\top \nabla q$. The introduction of $\dot{B}$ serves to emphasize the fact that DB/Dt is not to be treated as part of a differential equation for B. As they stand the rate equations are *exact*, and the approximations that will be proposed for them are the replacements,

$$\mathbf{u}_{zk} \to 2\frac{\mathbf{u}_{k-1} - \mathbf{u}_{k+1}}{F} \quad \left(-\frac{1}{6}\left(\frac{F}{4}\right)^2 \mathbf{u}_{3z*}\right),$$

for $k = 1$ to 3, and

$$\mathbf{U} \to \frac{1}{12}\left(\mathbf{u}_0 + 4\mathbf{u}_1 + 2\mathbf{u}_2 + 4\mathbf{u}_3 + \mathbf{u}_4\right).$$

In the momentum equations for $\mathbf{u}_0$ and $\mathbf{u}_4$, the coefficients of $\mathbf{u}_{zk}$ are *zero*.

Equations for time rates of change of vertical components of velocity are

$$w_{kt} + \mathbf{u}_k^\top \nabla w_k + (w_k - Z_{kt} - \mathbf{u}_k^\top \nabla Z_k) w_{zk} + p_{zk} + g = 0\,,$$

for $k = 1$ to 3. Notably absent from these are equations for w_{0t} and w_{4t}, and that is because $B(x,y,t)$ is prescribed boundary data and there is an equation for the determination of $H(x,y,t)$. There are also equations for the determination of $\mathbf{u}_0$ and $\mathbf{u}_4$, so the kinematic conditions are *evaluations* of

$$w_0 = \frac{DH}{Dt} + (\mathbf{u}_0 - \mathbf{U})^\top \nabla H \quad \text{and} \quad w_4 = B_t + \mathbf{u}_4^\top \nabla B\,.$$

Divergence-free flow in shallow water approximations is insured if the equation of continuity is used explicitly to find the vertical velocity. The practice in hydrodynamics codes is often to use the vertical momentum equations, even though they do not represent honest degrees of freedom. The argument is that if $\nabla^\top \mathbf{u} + w_z = 0$ is used to derive the Poisson equation,

$$\Delta\, p + \phi = 0 \text{ with } \phi = u_x^2 + v_y^2 + w_z^2 + 2(u_y v_x + v_z w_y + w_x u_z)\,,$$

then that pressure equation and the momentum equations imply

$$(\partial_t + \mathbf{u}^\top \nabla + w\partial_z)(\nabla^\top \mathbf{u} + w_z) = 0\,.$$

Thus divergence-free initial conditions imply divergence-free flow *in the continuous limit*. (The related result for the Navier Stokes equations is better; it has ν times the Laplacian of $(\nabla^\top \mathbf{u} + w_z)$ on the right-hand-side.) Other than to observe that discrete models of the Euler equations should perhaps be checked for deviations from divergence-free flow now and again, no more will be said here about the use of the momentum equations for w_k. (There is an entirely different approach to all this in the section on direct methods.)

Next comes the pressure equation: Given algorithms to produce values of $\mathbf{u}$ and w, central differences (or other means) can be used to estimate the interior nonhomogeneous terms, $\phi_k(x,y,t)$ for $k = 1$ to 3. Of course the surfaces $z = Z_k$ are tilted in general, and the crudest way to account for that is to use interpolations to estimate

$$q(x \pm dx, y, Z_k(x,y,t), t) \text{ and } q(x, y \pm dy, Z_k(x,y,t), t)\,.$$

In terms of $\grave{p}$, the pressure equation at interior points (x,y,t) is

$$(\Delta \grave{p})_k + \grave{p}_{zzk} + \Delta p_H + 2(\nabla G)^\top \nabla H + (H - Z_k)\Delta G + \phi_k = 0\,,$$

where $\Delta = \nabla^\top \nabla = \partial_x^2 + \partial_y^2$. The boundary conditions are $\grave{p}_0 = \grave{p}_4 = 0$, and the crude interpolations suggested above can be used to design a truly simple algorithm for the pressure equation. The cost of that major simplification: find an equation for G.

In this particular scheme, $Z_2 = Z = (H + B)/2 = B + F/2$, and the equation,

$$\frac{DZ}{Dt} + \frac{1}{F}\nabla^\top \langle \tilde{z}\tilde{\mathbf{u}} \rangle = W \, ,$$

is *exact*, in shallow water or deep water. The ingredients, again, are $\langle q \rangle$ which is the integral from B to H of $q\,dz$, the *fatness* $F = \langle 1 \rangle = H - B$, the *means* $Q = \langle q \rangle / F$, and the *deviations* $\tilde{q} = q - Q$. The approximations in the present use of it are the estimate of $\mathbf{U}$ from $\mathbf{u}_k$ and Simpson's rule, and the similar estimate of $\langle \tilde{z}\tilde{\mathbf{u}} \rangle$. In conjunction with the equation for DH/Dt it gives the approximate evaluation of

$$W = \dot{B} - \frac{1}{2} F \nabla^\top \mathbf{U} + \frac{1}{F}\nabla^\top \langle \tilde{z}\tilde{\mathbf{u}} \rangle \, ,$$

and the effective gravity is defined by

$$G = g + \frac{DW}{Dt} + \frac{1}{F}\nabla^\top \langle \tilde{w}\tilde{\mathbf{u}} \rangle \, .$$

The further approximation in the present use of this exact result comes from the use of Simpson's rule to estimate $\langle \tilde{w}\tilde{\mathbf{u}} \rangle$.

The matter of how the last two equations are to be used to design an algorithm for G will be left open for now. The continuous derivation, using $(D/Dt)DB/Dt + \cdots$ and other results from the MM approximation, contains virtual algebra I am not willing to do just yet. *(And with a little bit of luck $\cdots$)*