INTRODUCTION AND OVERVIEW

This is about shallow water theory, and an appropriate subtitle for it might be:

shallow waters can be treacherous

The reason for that is not because shallow water approximations are hard to find, but rather because it is pretty easy to find a great many different ones. Various results differ from one another, and the choice of one approximation in preference to another may depend upon the problems that are to be treated. Some different *styles* of approximation that will be discussed in sections to follow are:

- Power series • Averages and differences • Means and moments
- Mixed approximations • Direct methods • Multiple layers

The physical situation in all cases is that of 3-D unsteady flow of a liquid with a constant density. The flow takes place between a *specified* lower surface at $z = B(x, y, t)$ and a *free* upper surface at $z = H(x, y, t)$. Generally speaking, a shallow water approximation will be taken to be any description of the 3-D flow that is defined by equations of motion for any number of 2-D quantities, Q(x,y,t). Capital letters will sometimes be used for 2-D quantities that are very closely related to certain 3-D quantities, q(x,y,z,t), but there are many cases where that can't be done. In other cases the differential equations that govern the q(x,y,z,t) and the corresponding Q(x,y,t) look alike, but they have to be interpreted in entirely different ways.

The underlying (Euler) equations that govern the three-dimensional, unsteady flow are

$$\boxed{\begin{gathered} u_x + v_y + w_z = 0, \\ u_t + uu_x + vu_y + wu_z + p_x = 0\,, \\ v_t + uv_x + vv_y + wv_z + p_y = 0\,, \\ w_t + uw_x + vw_y + ww_z + p_z + g = 0\,. \end{gathered}}$$

In these $p(x, y, z, t)$ is the physical pressure divided by the (constant) density.

These equations are accompanied by the kinematic boundary conditions at the upper and lower surfaces,

$$\boxed{\begin{aligned} H_t + u_H H_x + v_H H_y &= w_H \, , \\ B_t + u_B B_x + v_B B_y &= w_B \, . \end{aligned}}$$

The notation is that the subscripts t, x, y, and z denote partial derivatives, and the subscripts H and B denote the *evaluations*,

$$q_H(x,y,t) = q(x,y,H(x,y,t)) \text{ and } q_B(x,y,t) = q(x,y,B(x,y,t)) \, .$$

Here and in sections to follow much use is made of the chain rules (e.g.)

$$q_{Ht} = q_{tH} + H_t q_{zH}, \; q_{Bt} = q_{tB} + B_t q_{zB}, \; \cdots \; .$$

The third boundary condition is that the pressure is specified at the upper surface, i.e. that values of $p_H(x,y,t)$ are externally specified data. The pressure at the bottom $p_B(x,y,t)$ is somehow to be predicted by the equations of motion. The last condition, also a boundary condition, is that the position of the lower surface $B(x,y,t)$ is an externally specified function. Time dependence of B is not usually included in theories of water waves, but it might as well be included since it doesn't introduce any significant difficulties.

To begin a discussion of equations that have exactly the same *form* but have entirely different *functions*, let it be supposed that at some specific time t we have whatever information we might need about $u(x,y,z;t)$, $v(x,y,z;t)$ and $H(x,y;t)$. The aim then is to see if *that (with semicolons)* plus the external data $B(x,y,t)$ and $p_H(x,y,t)$ *(without semicolons)* is sufficient information to evaluate $u_t(x,y,z;t)$, $v_t(x,y,z;t)$ and $H_t(x,y;t)$. If so, there is sufficient information to solve initial value problems, and the semicolons can be promoted to commas.

Consider first the lower kinematic condition: since the left-hand-side contains information that is *known* at time t, it is an evaluation of

$$w_B(x,y;t) = B_t + u_B B_x + v_B B_y \, .$$

The equation of continuity, can now be integrated to define

$$w(x,y,z;t) = w_B - \int_B^z (u_x + v_y)\,dz\,,$$

and given that, the upper kinematic condition is an evaluation of

$$H_t(x,y;t) = w_B - \int_B^H (u_x + v_y)\,dz - u_H H_x - v_H H_y\,.$$

One down - two to go, and note the entirely different uses of the kinematic conditions.

A fairly direct approach to the evaluation of $u_t(x,y,z;t)$ and $v_t(x,y,z;t)$ follows from the pressure equation,

$$\Delta\, p + \phi = 0 \text{ with } \phi = u_x^2 + v_y^2 + w_z^2 + 2(u_y v_x + v_z w_y + w_x u_z)\,.$$

The boundary values, $p_H(x,y,t)$ are specified, and if p_B were known, the evaluations of p, p_x, p_y, u_t and v_t would follow directly. The bottom pressure is *not* specified, however, and the lower boundary condition for p will have to be derived from what is already here. To find it, introduce the the time derivative of the lower kinematic condition,

$$w_{Bt} = B_{tt} + u_B B_{xt} + v_B B_{yt} + B_x u_{Bt} + B_y v_{Bt}\,.$$

Now it follows from the same kinematic condition that

$$(q_t + u q_x + v q_y + w q_z)_B = q_{Bt} + u_B q_{Bx} + v_B q_{By}\,.$$

(Similar results that follow from the upper kinematic condition are used in the sections on *averages and differences*, *mixed approximations* and *dispersion relations*.) From the result above,

$$w_{Bt} + u_B w_{Bx} + v_B w_{By} + p_{zB} + g = 0\,,$$

$$u_{Bt} + u_B u_{Bx} + v_B u_{By} + p_{Bx} = B_x p_{zB}\,,$$

$$v_{Bt} + u_B v_{Bx} + v_B v_{By} + p_{By} = B_y p_{zB}\,.$$

The lower boundary condition for pressure that follows from the four equations that contain u_{Bt}, v_{Bt} and w_{Bt} is

$$\left(1 + B_x^2 + B_y^2\right) p_{zB} - B_x p_{Bx} - B_y p_{By} + \Phi = 0\,,$$

where

$$\Phi = g + B_{tt} + 2\left(u_B B_{xt} + v_B B_{yt}\right) + u_B^2 B_{xx} + 2 u_B v_B B_{xy} + v_B^2 B_{yy}\,.$$

The only time derivatives in the boundary condition are derivatives of the prescribed function $B(x,y,t)$, and that establishes the result. Well posed initial value problems can be based upon the process,

given $p_H(x,y,t)$, $B(x,y,t)$ *and the kinematic conditions*,

$$(u,v,H) \to w(x,y,z;t) \to p(x,y,z;t) \to (u_t, v_t, H_t)\,.$$

The excuse for deriving a result that was probably known by Poincaré, and perhaps before him, is that it is the *paradigm* that reappears in all the shallow water approximations. Their shared structure can be described by an *order*, which is the number of properties that are included in the description of the horizontal velocity components, and a number of *degrees of freedom*, which is the number of time derivatives that appear in the formulation of initial value problems. In all cases the number of degrees of freedom is twice the order plus one.

A simple example is the approximate result of *Green & Naghdi*, which is formulated in terms of mean values of velocity components, $U(x,y,t)$ and $V(x,y,t)$, and the *fatness*, $F(x,y,t) = H - B$. From results in the sections on *means and moments* and *Green-Naghdi theory*, the time derivatives that cannot be eliminated by the use of the continuity equation and the kinematic conditions are in the equations

$$F_t + U F_x + V F_y + F(U_x + V_y) = 0\,,$$

$$U_t + UU_x + VU_y + \frac{1}{F}\int\limits_B^H p_x\,dz = 0\,,$$

$$V_t + UV_x + VV_y + \frac{1}{F}\int\limits_B^H p_y\,dz = 0\,.$$

The number of degrees of freedom is *three*, and equations to determine suitable approximation(s) of the pressure can be found. The well posed initial value problem is: *given* $p_H(x,y,t)$*,* $B(x,y,t)$ *and the kinematic conditions*,

$$(U,V,F) \to w(x,y,z;t) \to p(x,y,z;t) \to (U_t,V_t,F_t)\,.$$

To indicate just how treacherous shallow waters can be, let it be noted that there are countless numbers of ways one or two properties of the horizontal velocity components can be chosen to define first or second order shallow water approximations. None of them is inherently more *correct* than another, but some may be preferred for no other reason than that they are simpler than others. Three fairly simple second order approximations are developed in the sections named *power series* , *averages and differences* and *means and moments*. The three approximations really are different from one another, and in the section named *mixed approximations* it is shown how simple second order results can be used in tandem to find relatively simple third, fourth, fifth and sixth order results. The section named *direct methods* treats Galerkin methods, and a different direct method is described in the section named *orthogonal polynomials*. In most of these sections, derivations are carried out for the 2-D flows defined by variables, $q(x,z,t)$, and 3-D results are cited at the end. The section named *multiple layers* comes near to describing a full-blown numerical simulation of three dimensional, unsteady flow. There are also sections named *pressure equations* and *gravity equations* that contain results that apply to all the others, and sections named *Green-Naghdi theory* and *dispersion relations* where some details are worked out.