$\mathcal{GREEN} - \mathcal{NAGHDI}\ \ \mathcal{THEORY}$

What with all the virtual algebra in these pieces, it seemed appropriate that the details should be worked out in at least one example. In my opinion, this is the most interesting of the first order approximations, but I can't recommend *any* first approximation for a description of three dimensional flows. This will be limited to the two dimensional flows where

$$u \to U(x,t)\,,\ w \to W + \tilde{z}\tilde{w}_1 \ \text{ and }\ p \to p_H + G(H-z) + \frac{1}{2}\left(\tilde{z}^2 - \frac{F^2}{4}\right)p_2\,.$$

As in other sections, U and W are mean values, G is the effective gravity, F is $H-B$, Z is $B+F/2$, and $\tilde{z}$ is $z-Z$. The two degrees of freedom are in

$$\frac{DF}{Dt} + FU_x = 0 \ \text{ with }\ \frac{D}{Dt} = \partial_t + U\partial_x \ \text{ and}$$

$$\frac{DU}{Dt} + p_{Hx} + GH_x + \frac{1}{2}FG_x - \frac{1}{12F}\left(F^3 p_2\right)_x = 0\,.$$

The gravity equation follows from

$$W = \frac{DB}{Dt} - \frac{1}{2}FU_x \ \text{ and}$$

$$G = g + \frac{DW}{Dt} \quad = \quad \frac{p_B - p_H}{F}\,,$$

and it is

$$\left(1 + H_x B_x - \frac{1}{2}FH_{xx}\right)G - \frac{1}{4F}(F^3G_x)_x = g + \Gamma + \mathcal{T}\,,$$

where

$$\Gamma = \frac{1}{2}F\left(p_{Hx} - \frac{1}{12F}\left(F^3p_2\right)_x\right)_x - B_x\left(p_{Hx} - \frac{1}{12F}\left(F^3p_2\right)_x\right)$$

$$\text{and } \mathcal{T} = B_{tt} + 2UB_{xt} + U^2B_{xx} + FU_x^2\,.$$

(Note that the coefficients of F^2G_{xx} and FF_xG_x are different from second order results.)

First approximations of this kind can differ in the way in which the pressure coefficient p_2 is evaluated. In all cases

$$u_0 = u_A = U\,, \quad w_0 = w_A = W \;\text{ and }\; w_1 = \frac{w_D}{F} = \tilde{w}_1 = -U_x\,,$$

and some of the choices are:

- Use the Poisson equation, $p_{xx} + p_{zz} + 2U_x^2 = 0$. From the section on *pressure equations*,

$$(p_H + \frac{1}{2}GF - \frac{1}{8}F^2p_2)_{xx} + \frac{1}{G}(G^2Z_x)_x + (1 + Z_x^2)p_2 + 2U_x^2 = 0\,.$$

- Use the equation for w_1. From the section on *series*,

$$\frac{D}{Dt}U_x - U_x^2 - p_2 = 0\,.$$

- Use the equation for w_D. From the section on *averages and differences*,

$$\frac{D}{Dt}\left(FU_x\right) - Fp_2 = 0\,.$$

- Use the equation for $\langle\tilde{z}w\rangle$. From the section on *means and moments*,

$$\frac{D}{Dt}\left(\langle\tilde{z}^2\rangle U_x\right) + 2\langle\tilde{z}^2\rangle U_x^2 - \langle\tilde{z}^2\rangle p_2 = 0\,.$$

- Set $p_2 = 0$ (*not qualitatively different from the others*).

In what may be seen as a test of virtual algebra, the three expressions that are derived from a vertical momentum equation all give the same result,

$$\left(p_{Hx} + GH_x + \frac{1}{2}FG_x - \frac{1}{12F}\left(F^3 p_2\right)_x\right)_x + 2U_x^2 + p_2 = 0\,.$$

The result from the Poisson equation *differs* from the others, and that's how it is.

A familiar kind of comparison of shallow water approximations follows from the linearized (small amplitude) equations for the case where $B = 0$, p_H is constant, $H = h + \zeta(x,t)$ and $G = g + \gamma(x,t)$. In terms of the operators, $\omega q = i\, q_t$, $kq = -i\, q_x$ and $\kappa q = hkq$, the linearized equations are:

$\omega\zeta = \kappa U\,,\ \omega U = gk\zeta + \frac{1}{2}\kappa\gamma - \frac{1}{12}h\kappa p_2\,,\ \left(1 + \frac{1}{4}\kappa^2\right)\gamma + \frac{1}{2}gk\kappa\zeta = \frac{1}{24}h\kappa^2 p_2$ and

$p_2 = 0\,,\ \left(1 + \frac{1}{12}\kappa^2\right)p_2 = gk^2\zeta + \frac{1}{2}k\kappa\gamma$ or $\left(1 + \frac{1}{8}\kappa^2\right)p_2 = gk^2\zeta + \frac{1}{2}k\kappa\gamma\,.$

No more virtual algebra - this is real and it's fairly nasty, with a few treacherous, unexpected cancellations of terms. The elimination of hp_2 from the equation for U gives $\omega U + 2\kappa^{-1}\gamma = 0$, and hp_2 can also be eliminated from the pressure and gravity equations. When ω and κ are treated as numbers, rather than as operators, the three *dispersion relations* are,

$$\frac{\omega^2}{gk} = \frac{\kappa}{1 + \kappa^2/4}\,, = \frac{\kappa}{1 + \kappa^2/3} \text{ or } = \frac{\kappa(1 + \kappa^2/24)}{1 + 3\kappa^2/8 + \kappa^4/96}\,.$$

So which result, if any, is *better*? Well, Airy's exact result is $\omega^2/gk = \tanh\kappa$, and

- $p_2 = 0$ leads to the approximations, $\omega^2/gk \to \kappa(1 - \kappa^2/4)$ as $\kappa \to 0$ and $\to 4/\kappa$ as $\kappa \to \infty$. The first result is not quite the first two terms of the Taylor series for $\tanh\kappa$, and that has led some of my former colleagues to conclude that it cannot possibly be acceptable. The second result is not a particularly good approximation of $\tanh\kappa \to 1$ as $\kappa \to \infty$.

- When p_2 is included, and the vertical momentum equation is used to determine it, the corresponding results are $\omega^2/gk \to \kappa(1-\kappa^2/3)$ as $\kappa \to 0$ and $\to 3/\kappa$ as $\kappa \to \infty$. This time, the long wave limit is marginally better, and the short wave limit is marginally worse.

- When p_2 is included, and the pressure equation is used to determine it, the corresponding results are $\omega^2/gk \to \kappa(1-\kappa^2/3)$ as $\kappa \to 0$ and $\to 4/\kappa$ as $\kappa \to \infty$. Much as I would like to say *Aha!*, I think the appropriate comment is *So what?*.

The next stage of linear approximation within this modified Green-Naghdi approximation comes from a consideration of two more terms in

$$p \to p_H + G(H-z) + \grave{p} \text{ with } \grave{p} = \frac{1}{2}\left(\tilde{z}^2 - \frac{F^2}{4}\right)\left(p_2 + \frac{\tilde{z}}{3}p_3\right) + \frac{1}{24}\left(\tilde{z}^4 - \frac{F^4}{16}\right)p_4 \,.$$

Then the linearized pressure equations imply $p_4 = \kappa^2 p_2$ and, in turn, $\langle \grave{p} \rangle = -(F^3/12)(1+\kappa^2/40)p_2$. ($(1+\kappa^2)p_3 + \kappa^2\gamma = 0$ doesn't figure in these results.) The resulting approximation of the dispersion relation is

$$\frac{\omega^2}{gk} = \frac{\kappa\,(1+\kappa^2/24+\kappa^4/1920)}{1+3\kappa^2/8+5\kappa^4/384+\kappa^6/7680} \,.$$

As higher approximations of $\tanh\kappa$ go, this is a rather disappointing one: it misses the $\cdots + 2\kappa^5/15$ when $\kappa \to 0$, and its shortwave limit is still $4/\kappa$ when $\kappa \to \infty$. More accurate approximations will be given in a separate section on *dispersion relations*.