**AVERAGES AND DIFFERENCES**

This is entirely different from the section on *power series*, but results turn out to be very similar. Here, we work with

\[ q_H(x,t) = q(x, H(x,t), t) \text{ and } q_B(x,t) = q(x, B(x,t), t) , \] (evaluations)

\[ q_D(x,t) = q_H - q_B \text{ with } z_D = F(x,t) = H - B , \] (differences)

\[ q_A(x,t) = \frac{1}{2}(q_H + q_B) \text{ with } z_A = Z(x,t) = \frac{1}{2}(H + B) . \] (averages)

The notations \( F \) and \( Z \) (*fatness* and mean height) will generally be used in preference to \( z_D \) and \( z_A \).

The kinematic conditions are

\[ H_t + u_H H_x = w_H \text{ and } B_t + u_B B_x = w_B , \]

and the average and difference of them gives the exact results,

\[ Z_t + u_A Z_x + \frac{1}{4} u_D F_x = w_A \text{ and } F_t + u_A F_x + u_D Z_x = w_D . \]

The *approximation* here is that \( u = u_A + (z - Z)u_D/F \), and the integral of the continuity equation then implies

\[ w_D = Z_x u_D - F u_A x , \quad F_t + u_A F_x + F u_A x = 0 \]

and \( w_A = B_t + u_A B_x - \frac{1}{2} F u_A x + \frac{1}{4} F_x u_D . \)

From the chain rule and the kinematic conditions it follows that

\[ (q_t + u q_x + w q_z)_H = q_H t + u_H q_{Hx} \text{ and } (q_t + u q_x + w q_z)_B = q_B t + u_B q_{Bx} . \]
Next, for $\& = H$ or $B$, comes

$$u_{kt} + u_\& u_{kx} + p_{kx} - \&xp_{z\&} = 0 \text{ and } w_{kt} + u_\& w_{kx} + p_{z\&} + g = 0,$$

and, finally,

\[
\begin{align*}
    u_{At} + u_A u_{Ax} + \frac{1}{4} u_D u_{Dx} + p_{Ax} - Z_x p_{zA} - \frac{1}{4} F_x p_{zD} &= 0, \\
    u_{Dt} + u_A u_{Dx} + u_D u_{Ax} + p_{Dx} - Z_x p_{zD} - F_x p_{zA} &= 0, \\
    w_{At} + u_A w_{Ax} + \frac{1}{4} u_D w_{Dx} + p_{zA} + g &= 0, \\
    \text{and } w_{Dt} + u_A w_{Dx} + \frac{1}{4} u_D w_{Ax} + p_{zD} &= 0.
\end{align*}
\]

Similarities between these equations and those derived from truncated Taylor series can be seen by the use of

\[
\begin{align*}
    u_0 &= u_A, \quad u_1 = \frac{u_D}{F}, \quad w_0 = w_A - \frac{1}{8} F^2 w_2, \quad w_1 = \frac{w_D}{F}, \\
    p_0 &= p_A - \frac{1}{8} F^2 p_2 \text{ and } p_1 = \frac{p_D}{F} - \frac{1}{24} F^2 p_3.
\end{align*}
\]

When the substitutions are used to transform the various equations from one form to the other, results are found to be very similar, but not identical! Other small differences in nonlinear terms follow from quite a few different ways to look for equations for $p_2$ and $p_3$. Some of the possibilities are:

- Take $p_2$ and $p_3$ from the Poisson equation (section on pressure equations). This will be taken to be the method of choice, to be used until some compelling argument to the contrary is found.
- Take the equation for $w_{2t}$ from the section on power series.
- Use $u_0 = u_A$, $w_0 = w_A - F^2 w_2/8$ and the equations that contain $w_{0t}$ and $w_{At}$ to derive an equation for $(F^2 w_2)_t$. The result is another, slightly different, equation for $w_{2t}$.
• Use the z-derivative of the vertical momentum equation to derive equations for \(w_zDt\) and \(w_zAt\). It’s not particularly difficult, and the result is yet another equation for \(w_2t\) and another equation for \(w_1t\).

• Introduce results from the section on means and moments, where the mean value of \(q\) is \(Q = \int q \, dz / F\), the deviation is \(\tilde{q} = q - Q\), and the second approximation is based upon

\[
\begin{align*}
    u & \to U + \tilde{z} \tilde{u}_1 \quad \text{and} \quad w \to W + \tilde{z} \tilde{w}_1 + \frac{1}{2} \left( \tilde{z}^2 - \frac{F^2}{12} \right) \tilde{w}_2 .
\end{align*}
\]

After that, counting the number of different equations for \(p_2\) and \(p_3\) becomes more of a challenge.

*Oh treacherous shallow waters - treacherous!*  

Equations for pressure coefficients, including \(p_2\) and \(p_3\), will be found in the section on pressure equations, and only the gravity equation will be discussed here. The representation of pressure is

\[
p = p_H + G(H - z) + \dot{\rho}(x, y, \tilde{z}, t) , \quad p_B = p_H + GF \quad \text{and} \quad \dot{p}_H = \dot{p}_B = 0 ,
\]

and the vector and matrix notations are

\[
\begin{align*}
    u &= (u, v)^\top , \quad \nabla = (\partial_x, \partial_y)^\top , \quad \Delta = \nabla^\top \nabla \quad \text{and} \quad \frac{D}{Dt} = \partial_t + u_A^\top \nabla .
\end{align*}
\]

Again, there are five degrees of freedom in

\[
\begin{align*}
    \frac{DH}{Dt} + F \nabla^\top u_A &= B_t + u_A^\top \nabla B , \\
    \frac{Du_A}{Dt} + \frac{1}{4} u_D^\top \nabla u_D + \nabla p_H + G \nabla H + \frac{1}{2} F \nabla G &= \dot{p}_zA \nabla Z + \frac{1}{4} \dot{p}_zD \nabla F , \\
    \frac{Du_D}{Dt} + u_D^\top \nabla u_A &= F \nabla G + \dot{p}_zD \nabla Z + \dot{p}_zA \nabla F .
\end{align*}
\]
The gravity equation comes from

\[ w_A = \frac{DB}{Dt} - \frac{1}{2} F \nabla^\top u_A + \frac{1}{4} u_D^\top \nabla F \quad \text{and} \]

\[ G = g + \frac{Dw_A}{Dt} + \frac{1}{4} u_D^\top \nabla (u_D \nabla Z - F \nabla^\top u_A) + \hat{p}_z A. \]

The process is the same as in the section on series: The evaluation of \( Dw_A/Dt \) contains quite a few parts that have no explicit reference to \( G, p_H \) or \( \hat{p} \), and three that do, namely \( (\nabla B)^\top D u_A/Dt \), \( F \nabla^\top D u_A/Dt \) and \( (\nabla F)^\top D u_D/Dt \). The result is

\[ \left(1 + \nabla H^\top \nabla B - \frac{1}{2} F \Delta H\right) G - \frac{1}{4F^2} \nabla^\top \left(F^4 \nabla G\right) = g + \Gamma + T, \]

where

\[ \Gamma = \left(\frac{1}{2} F \nabla - (\nabla B)\right)^\top \left(\nabla p_H - \hat{p}_z A \nabla Z - \frac{1}{4} \hat{p}_z D \nabla F\right) \]

\[ + \frac{1}{4} \hat{p}_z D (\nabla F)^\top \nabla Z + \hat{p}_z A \left(1 + \frac{|\nabla F|^2}{4}\right) \]

and \( T \) contains all the virtual algebra. Slightly different treatments of this can give different coefficients (not 1) for \( F (\nabla F)^\top \nabla G \), but the coefficient of \( F^2 \Delta G \) is always 1/4.