# *AVERAGES AND DIFFERENCES*

This is entirely different from the section on *power series*, but results turn out to be very similar. Here, we work with

$$q_H(x,t) = q(x, H(x,t), t) \text{ and } q_B(x,t) = q(x, B(x,t), t)\,, \quad \text{(evaluations)}$$

$$q_D(x,t) = q_H - q_B \text{ with } z_D = F(x,t) = H - B\,, \qquad \text{(differences)}$$

$$q_A(x,t) = \frac{1}{2}(q_H + q_B) \text{ with } z_A = Z(x,t) = \frac{1}{2}(H+B)\,. \qquad \text{(averages)}$$

The notations $F$ and $Z$ (*fatness* and mean height) will generally be used in preference to $z_D$ and $z_A$.

The kinematic conditions are

$$H_t + u_H H_x = w_H \text{ and } B_t + u_B B_x = w_B\,,$$

and the average and difference of them gives the *exact* results,

$$Z_t + u_A Z_x + \frac{1}{4} u_D F_x = w_A \text{ and } F_t + u_A F_x + u_D Z_x = w_D\,.$$

The *approximation* here is that $u = u_A + (z - Z)u_D/F$, and the integral of the continuity equation then implies

$$w_D = Z_x u_D - F u_{Ax}\,, \quad F_t + u_A F_x + F u_{Ax} = 0$$
$$\text{and } w_A = B_t + u_A B_x - \frac{1}{2} F u_{Ax} + \frac{1}{4} F_x u_D\,.$$

From the chain rule and the kinematic conditions it follows that

$$(q_t + u q_x + w q_z)_H = q_{Ht} + u_H q_{Hx} \text{ and } (q_t + u q_x + w q_z)_B = q_{Bt} + u_B q_{Bx}\,.$$

Next, for $\& = H \; or \; B$, comes

$$u_{\&t} + u_\& u_{\&x} + p_{\&x} - \&_x p_{z\&} = 0 \text{ and } w_{\&t} + u_\& w_{\&x} + p_{z\&} + g = 0\,,$$

and, finally,

$$\boxed{\begin{gathered} u_{At} + u_A u_{Ax} + \frac{1}{4} u_D u_{Dx} + p_{Ax} - Z_x p_{zA} - \frac{1}{4} F_x p_{zD} = 0\,, \\ u_{Dt} + u_A u_{Dx} + u_D u_{Ax} + p_{Dx} - Z_x p_{zD} - F_x p_{zA} = 0\,, \\ w_{At} + u_A w_{Ax} + \frac{1}{4} u_D w_{Dx} + p_{zA} + g = 0\,, \\ \text{and } w_{Dt} + u_A w_{Dx} + \frac{1}{4} u_D w_{Ax} + p_{zD} = 0\,. \end{gathered}}$$

Similarities between these equations and those derived from truncated Taylor series can be seen by the use of

$$u_0 = u_A\,,\ u_1 = \frac{u_D}{F}\,,\ w_0 = w_A - \frac{1}{8} F^2 w_2\,,\ w_1 = \frac{w_d}{F}\,,$$

$$p_0 = p_A - \frac{1}{8} F^2 p_2 \text{ and } p_1 = \frac{p_D}{F} - \frac{1}{24} F^2 p_3\,.$$

When the substitutions are used to transform the various equations from one form to the other, results are found to be very similar, *but not identical*! Other small differences in nonlinear terms follow from quite a few different ways to look for equations for $p_2$ and $p_3$. Some of the possibilities are:

- Take $p_2$ and $p_3$ from the Poisson equation (section on *pressure equations*). This will be taken to be the *method of choice*, to be used until some compelling argument to the contrary is found.
- Take the equation for $w_{2t}$ from the section on *power series*.
- Use $u_0 = u_A$, $w_0 = w_A - F^2 w_2/8$ and the equations that contain $w_{0t}$ and $w_{At}$ to derive an equation for $(F^2 w_2)_t$. The result is another, slightly different, equation for $w_{2t}$.

- Use the z-derivative of the vertical momentum equation to derive equations for $w_{zDt}$ and $w_{zAt}$. It's not particularly difficult, and the result is yet another equation for $w_{2t}$ and another equation for $w_{1t}$.
- Introduce results from the section on *means and moments*, where the mean value of $q$ is $Q = \int q\,dz/F$, the *deviation* is $\tilde{q} = q - Q$, and the second approximation is based upon

  $$u \to U + \tilde{z}\tilde{u}_1 \text{ and } w \to W + \tilde{z}\tilde{w}_1 + \frac{1}{2}\left(\tilde{z}^2 - \frac{F^2}{12}\right)\tilde{w}_2\,.$$

  After *that*, counting the number of different equations for $p_2$ and $p_3$ becomes more of a challenge.

*Oh treacherous shallow waters - treacherous*!

Equations for pressure coefficients, including $p_2$ and $p_3$, will be found in the section on *pressure equations*, and only the gravity equation will be discussed here. The representation of pressure is

$$p = p_H + G(H - z) + \grave{p}(x, y, \tilde{z}, t)\,,\ p_B = p_H + GF \text{ and } \grave{p}_H = \grave{p}_B = 0\,,$$

and the vector and matrix notations are

$$\mathbf{u} = (u, v)^\top\,,\ \ \nabla = (\partial_x, \partial_y)^\top\ ,\ \ \Delta = \nabla^\top\nabla \ \text{ and }\ \frac{D}{Dt} = \partial_t + \mathbf{u}_A^\top\nabla\,.$$

Again, there are five degrees of freedom in

$$\boxed{\begin{gathered}\frac{DH}{Dt} + F\nabla^\top\mathbf{u}_A = B_t + \mathbf{u}_A^\top\nabla B\,,\\ \frac{D\mathbf{u}_A}{Dt} + \frac{1}{4}\mathbf{u}_D^\top\nabla\mathbf{u}_D + \nabla p_H + G\nabla H + \frac{1}{2}F\nabla G = \grave{p}_{zA}\nabla Z + \frac{1}{4}\grave{p}_{zD}\nabla F\,,\\ \frac{D\mathbf{u}_D}{Dt} + \mathbf{u}_D^\top\nabla\mathbf{u}_A = F\nabla G + \grave{p}_{zD}\nabla Z + \grave{p}_{zA}\nabla F\,.\end{gathered}}$$

The gravity equation comes from

$$w_A = \frac{DB}{Dt} - \frac{1}{2}F\nabla^\top \mathbf{u}_A + \frac{1}{4}\mathbf{u}_D^\top \nabla F \text{ and}$$
$$G = g + \frac{Dw_A}{Dt} + \frac{1}{4}\mathbf{u}_D^\top \nabla(\mathbf{u}_D^\top \nabla Z - F\nabla^\top \mathbf{u}_A) + \grave{p}_{zA}\,.$$

The process is the same as in the section on series: The evaluation of $Dw_A/Dt$ contains quite a few parts that have no explicit reference to $G$, $p_H$ or $\grave{p}$, and three that do, namely $(\nabla B)^\top D\mathbf{u}_A/Dt$, $F\nabla^\top D\mathbf{u}_A/Dt$ and $(\nabla F)^\top D\mathbf{u}_D/Dt$. The result is

$$\left(1 + \nabla H^\top \nabla B - \frac{1}{2}F\Delta H\right) G - \frac{1}{4F^2}\nabla^\top \left(F^4 \nabla G\right) = g + \Gamma + \mathcal{T}\,,$$

where

$$\begin{aligned}\Gamma &= \left(\frac{1}{2}F\nabla - (\nabla B)\right)^\top \left(\nabla p_H - \grave{p}_{zA}\nabla Z - \frac{1}{4}\grave{p}_{zD}\nabla F\right) \\ &+ \frac{1}{4}\grave{p}_{zD}(\nabla F)^\top \nabla Z + \grave{p}_{zA}\left(1 + \frac{|\nabla F|^2}{4}\right)\end{aligned}$$

and $\mathcal{T}$ contains all the virtual algebra. Slightly different treatments of this can give different coefficients (not $1$) for $F(\nabla F)^\top \nabla G$, but the coefficient of $F^2 \Delta G$ is always $1/4$.