1 Differential equations

Differential equations are found in many areas of science. A differential equation is an equation relating an unknown function or functions to one or more of its derivatives. For example:

\[
\frac{dx}{dt}(t) = \sin(x(t)) \tag{1}
\]

\[
\frac{d^2x}{dt^2} - \frac{dx}{dt} - 2x = 0 \tag{2}
\]

\[
\frac{\partial^2u}{\partial x^2} + \frac{\partial^2u}{\partial y^2} = u^2 \tag{3}
\]

\[
\begin{cases}
\frac{du}{dt} = u + v \\
\frac{dv}{dt} = u - v \\
\frac{dx}{dt} + t^2x = \sin(t) \\
\frac{dx}{dy} = \frac{x}{y^2}
\end{cases} \tag{4}
\]

\[
\frac{dx}{dt} + \sin(t) \tag{5}
\]

\[
\frac{dx}{dy} = \frac{x}{y^2} \tag{6}
\]

1.1 Equation Characteristics

Categorizing an equation is important. It enables us to use information that has already been studied about an equation of interest. Identifying the dependent and independent variables of a differential equation is helpful for its characterization. The dependent variables of the equation are the solutions we are looking for. They are always defined in terms of the independent variables. For example, in equation (1), \(x\) is dependent and \(t\) is independent (a solution is given by the function \(x(t)\)).

- **ODE vs. PDE** When the equation involves derivatives that are taken with respect to a single (independent) variable, we say that this is an Ordinary Differential Equation (ODE). When the equation involves derivatives taken with respect to two or more (independent) variables, we say that this is a Partial Differential Equation (PDE). Equations (1), (2) are examples of ODEs whereas Equation (3) is an example of a PDE. In this course we will be studying only ODEs.

- **The order of a differential equation** is that of the highest derivative that it contains. For example, equation (1) is a first order equation, while equations (2), (3) are second order.

- **The dimension of a differential equation** is the number of its dependent variables. For example, equations (1), (2), (3) are one dimensional whereas equation (4) is two dimensional.

- **Linear and Non-linear**: A differential equation is linear if it is linear in its dependent variables and their derivatives.
1.2 Analysis

There are three general approaches for analyzing differential equations:

1. Qualitative
2. Numerical
3. Analytical

Using the qualitative approach we determine the behavior of the solutions without actually getting a formula for them. This is somewhat similar to the qualitative analysis of functions in calculus (e.g. finding its minima maxima and saddle points).

Using the numerical approach we compute numerical values of the dependent variables (solutions) for a specific range of values set for the independent variables.

Using the analytical approach provides an explicit analytic formula for the solution. In general, this is very hard and sometimes even impossible.

2 First order, 1-D differential equations

Examples:

\[
\frac{dv}{dt} = -\alpha v \\
\frac{dv}{dt} = -\alpha v + t \\
\frac{dv}{dt} = -\alpha v + w f(v)
\]

The most general form for first order one dimensional differential equation is:

\[
\frac{dv}{dt} = F(v, t) \quad (7)
\]

Where \( F(v, t) \) is some function of the dependent variable \( v \) and the independent variable \( t \). In the above examples we have:

\[
F(v, t) = \begin{cases} 
-\alpha v \\
-\alpha v + t \\
-\alpha v + w f(v)
\end{cases} \quad (8)
\]

2.1 Initial Value Problems

We have already noted in the introduction, that a differential equation on its own does not necessarily have a unique solution. It is usually necessary to specify some initial conditions. For example, if we are driving our car at a constant speed \( a \) then our motion can be described using the following differential equation:

\[
\frac{dx}{dt} = a \quad (9)
\]

Where \( x(t) \) represents our position at time \( t \). However, note that our position at time \( t \) depends on our initial position where we started to drive. If we started at NY, our position after driving one hour would probably (and hopefully) be different from our position after one hour starting at Boston. For the solution to be unique (e.g. our distance from NY at time \( t \)) we must also specify our position \( x(t = t_0) = x_0 \) at time \( t_0 = 0 \) (other times for \( t_0 \) are also OK). This specification gives a unique solution to the position at time \( t \):

\[
x(t - t_0) = a \ast (t - t_0) + x_0 \quad (10)
\]

These kind of problems are called initial value problems. Using our general form, we can state an initial value problem as: Find a function \( v(t) \) that solves equation (7) and also satisfies: \( v(t = t_0) = v_0 \).
3 Numerical integration

Let’s recall the definition of derivative of a function $v(t)$ with respect to $t$:

$$\frac{dv}{dt} = \lim_{\epsilon \to 0} \frac{v(t + \epsilon) - v(t)}{\epsilon} \quad (11)$$

For $\epsilon$ sufficiently small, the derivative can therefore be approximated by:

$$\frac{dv}{dt} \approx \frac{v(t + \epsilon) - v(t)}{\epsilon} \quad (12)$$

In other words, we can approximate:

$$\frac{dv}{dt} = F(v, t) \quad (13)$$

by:

$$v(t + \epsilon) = v(t) + \epsilon F(v, t) \quad (14)$$

The last equation gives us a numerical method for solving the differential equation in (7) with initial value $v(0) = v_0$:

1. Choose $\epsilon$ small.
2. Initialize $v(0) = v_0$ using initial value.
3. Compute:

$$v(\epsilon) = v(0) + \epsilon F(v(0), 0)$$
$$v(2\epsilon) = v(\epsilon) + \epsilon F(v(\epsilon), \epsilon)$$
$$v(3\epsilon) = v(2\epsilon) + \epsilon F(v(2\epsilon), 2\epsilon)$$
$$\vdots$$