Geometric Measure Theory at Brown in the 1960s

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Part I - Mathematical Results

1 Introduction.

The field of geometric measure theory (GMT) is at an interface of problems in mathematical analysis and geometry. This article is intended as a historical retrospective, with emphasis on the decade 1960-1969. This was a time of rapid development of GMT, and Brown University was at the forefront. It is a revision and extension of material presented at a miniconference on April 16, 2011 in memory of Herbert Federer, who died in April 2010.

Federer joined the Mathematics Department in 1945, and remained at Brown throughout his career. He is remembered for his many deep and original contributions to GMT, beginning with his paper [Fe45] on the Gauss-Green theorem. It is difficult to imagine that the rapid growth of GMT beginning in the 1960s, as well as its subsequent influence on other areas of mathematics, could have happened without Federer's ground breaking efforts. His book *Geometric Measure Theory* [Fe69] is a classic reference. Reference [P12] is a scientific obituary article about him.

At Federer's initiative I came to Brown in 1958, and had the good fortune to participate in GMT research during the exciting years immediately afterwards. In particular, I had the good fortune during my first year at Brown to collaborate with Federer on our joint paper *Normal and Integral Currents* [FF60].

My purpose is to give some remembrances of developments in GMT through the 1960s, with selected references to more recent work. By 1970, I had left GMT to work on other research topics. As mentioned above [Fe69] is a definitive treatment of results in GMT up to its date of publication. Reference [Fe78] is based on Federer's 1977 American Mathematical Society Colloquium lectures. It can be a useful complement to the more detailed development in [Fe69]. Another thorough introduction to GMT is L. Simon's book [SL83]. It is often

recommended to students and others wishing to learn this subject. F. Morgan's book [MF00] provides a readable introduction to concepts and results in GMT, with many references. F.J. Almgren's survey paper [AF93] provides a good, concise overview of concepts and results in GMT, with emphasis on area minimizing surfaces.

These notes are organized as follows. Part I discusses mathematical results. We begin in Section 2 with a brief review of pre-1960 background. Among the developments during the 1950s which directly influenced developments in GMT afterward were deRham's theory of currents, L.C. Young's generalized surfaces and Whitney's geometric integration theory (Section 2, (v)-(vii)). Sections 3 and 4 focus on my paper with Federer [FF60] and subsequent results by Federer. This paper is in a deRham current setting, which involves orientations of k-dimensional "surfaces." Section 5 describes an alternative setting, in terms of Whitney's flat chains, which does not involve orientations of surfaces.

The next Section 6 introduces the Plateau (minimum area) problem, in both the Reifenberg [Re60] and integral current formulations. This is followed in Section 7 by a brief summary of results about the notoriously difficult questions about regularity of solutions to the higher dimensional Plateau problem. Section 8 discusses solutions to geometric problems in the calculus of variations, in the setting of Young's generalized surfaces. Such solutions exist without assumptions of convexity or ellipticity, needed for existence and regularity of "ordinary" solutions as integral currents or flat chains. Some open questions about representations of generalized surface solutions are discussed in Appendix C. Almgren's varifolds, which are defined in a way formally similar to generalized surfaces, are also mentioned in Sections 7 and 8.

The remainder of these notes (Part II) present a shift from mathematical discussions to some remembrances of the milieu at Brown University in the 1960s, and of our graduate students and visitors in GMT during that period (Sections 9 and 10). I then give in Section

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11 some brief personal remembrances about F.J. Almgren, E. De Giorgi, H. Federer, E.R. Reifenberg and L.C. Young, who are no longer with us. Except for Reifenberg, who died in a tragic mountaineering accident in 1964, I cite other scientific obituary articles. There are volumes of selected works of Almgren [AF99] and De Giorgi [DG06].

I wish to thank William Allard and William Ziemer for their valuable comments on an earlier draft of this article.

Part I - Mathematical Results

2 Pre-1960 background.

We begin with a brief overview of work before 1960 which impacted subsequent developments in GMT afterward. Throughout the discussion, $\mathbb{R}^n$ denotes n-dimensional Euclidean space, and k is an integer with $0 \leq k \leq n$. Other notations are summarized in Appendix A.

i Structure of sets of finite Hausdorff measure.

During the early 20th century several definitions of k-dimensional measure of a set $K \subset \mathbb{R}^n$ were given. Among them the Hausdorff definition is now most widely used. The Hausdorff measure of K is denoted by $H^k(K)$. A set K is called k-rectifiable if it differs in arbitrarily small H^k measure from a finite union of closed sets $K_1, \ldots, K_m$, such that each K_i is the image of a set $D_i \subset \mathbb{R}^k$ under some Lipschitz function f_i. The k-rectifiable sets have an important role in the theory of rectifiable and integral currents (Section 3). A set K with $H^k(K)$ finite is called purely k-unrectifiable if K has the following property. Let ρ denote the orthogonal projection of $\mathbb{R}^n$ onto a k-dimensional plane π containing 0. Then $\rho(K)$ has k dimensional Lebesgue measure 0 for "almost all" such projections ρ.

Besicovitch showed for $k = 1$, $n = 2$, that any set $K \subset \mathbb{R}^2$ with $0 < H^1(K) < \infty$ is the union of 1-rectifiable and purely 1-unrectifiable subsets K_1, K_2. This result was extended

by Federer to arbitrary dimensions n and $k < n$, in his fundamental paper [Fe47]. He also showed that, if K is k-rectifiable, then all reasonable definitions of k-dimensional measure of K agree with $H^k(K)$. See also [Fe52,69].

ii   Gauss-Green (divergence) theorem.

The classical Gauss-Green theorem states that if $E \subset \mathbb{R}^n$ is a bounded set with smooth boundary B, then

$$(2.1) \qquad \int_E div\zeta(x)dx = \int_B \zeta(y) \cdot \nu(y)dH^{n-1}(y)$$

for any smooth $\mathbb{R}^n$-valued function ζ, where $\nu(y)$ is the exterior unit normal at y. An early achievement of GMT, by Federer [Fe45] and De Giorgi [DG55], was to obtain a version of (2.1) for a much larger class of sets E, without any smoothness assumptions about the boundary B. For any bounded Borel set $E \subset \mathbb{R}^n$, De Giorgi defined a quantity $P(E)$ called the perimeter of E. If E happens to have a smooth boundary B, then $P(E) = H^{n-1}(B)$. It turns out that a set E has finite perimeter $P(E)$ if and only if the first order partial derivatives of the indicator function 1_E (in the Schwartz distribution sense) are measures.

In [DG55] De Giorgi defined the reduced boundary B_r of a set E with $P(E) < \infty$. He also defined an approximate exterior unit normal $\nu(y)$ at each $y \in B_r$. See [DG55, Thm. III]. Then B_r is a $(n-1)$-rectifiable set and formula (2.1) holds, with B replaced by B_r. De Giorgi's paper was written in Italian. An English translation of it is included in the selected papers by De Giorgi book [DG06].

iii   The Plateau (least area) problem.

The classical Plateau problem for two dimensional surfaces in $\mathbb{R}^3$ is as follows. Find a surface S_0 of least area among all surfaces S with given boundary C. This is a geometric problem in the calculus of variations, which has been studied extensively. During the 1930s, J. Douglas

[Do31] and T. Rado [Ra30] independently gave solutions to a version of the Plateau problem. Their results were widely acclaimed. Douglas received a Fields Medal in 1936 for his work.

In the Douglas-Rado formulation, admissible surfaces were defined in terms of "parametric representations" of surfaces. Any such parametric representation f maps a simply connected region $D \subset \mathbb{R}^2$ into $\mathbb{R}^3$. The boundary condition is that the restriction of f to the boundary of D is a parametric representation of the boundary curve C, which is assumed to have no multiple points (a "simple closed curve"). The area $A(f)$ is given by its classical formula,

$$(2.2) \qquad A(f) = \int_D \int \left| \frac{\partial f}{\partial u_1} \wedge \frac{\partial f}{\partial u_2} \right| du_1 du_2.$$

(See Appendix A for notations). Special (conformal) parametric representations are chosen such that D is a circular disk and

$$(2.3) \qquad A(f) = \frac{1}{2} \int_D \int |\nabla f|^2 du_1 du_2,$$

which is the Dirichlet integral. It was then shown by Douglas and Rado that there is a conformal parametric representation which minimizes $A(f)$, subject to the boundary conditions. The components of this $\mathbb{R}^3$-valued function f are harmonic functions.

If the parameter domain D is simply connected, then the surface parameterized by f was said to be of the "topological type of a circular disk." The Douglas-Rado result was afterward generalized by Courant [C50] and Douglas [Do39] to give a solution to the Plateau problem for surfaces bounded by a finite number of disjoint curves and of prescribed finite Euler characteristic. However, in [Fl56] an example was given which shows that the problem of least area with unrestricted topological types may have no solution of finite topological type, even in case the boundary C consists of a single closed curve of finite length.

The Douglas-Rado methods depend on conformal parameterizations, and hence are intrinsically 2 dimensional. For these reasons, it was clear by the late 1950s that entirely new formulations were needed to study the Plateau problem for surfaces of dimension $k > 2$.

During the 1960s and 1970s, remarkable progress was made in that regard. See Sections 6 and 7.

iv Surface area theory.

The issue of giving a suitable definition of area for surfaces, without traditional smoothness assumptions, goes back to Lebesgue's thesis at the beginning of the 20th century. As in (iii) consider surfaces defined parametrically. Let f be a continuous function from a region $D \subset \mathbb{R}^2$ into $\mathbb{R}^3$. The Lebesgue area $A_\ell(f)$ is defined as the lower limit of the elementary areas of approximating polyhedra. Lebesgue area theory flourished from the 1930s through the 1950s. T. Rado and L. Cesari were leaders in the field, and their books [Ra48] [Ce56] are important sources. In the years after WW2, they were joined by Federer who contributed many of the most significant advances during this period.

One of the objections to the Lebesgue definition is that there are examples in which $A_\ell(f)$ is finite but the set $f(D)$ has positive 3-dimensional Lebesgue measure. Besicovitch [B45] gave an alternative definition of area which is not subject to this objection. A basic question in area theory is to find an integer valued multiplicity function $\Theta(x)$ which yields $A_\ell(f)$, when integrated over $f(D)$ with respect to Hausdorff measure H^2. If f is Lipschitz, then $A_\ell(f) = A(f)$ as in (2.2) and one can take $\Theta(x) = N(x)$, where $N(x)$ is the number of points (u_1, u_2) in D such that $f(u_1, u_2) = x$. However, the task of defining a suitable multiplicity $\Theta(x)$ for every f with $A_\ell(f)$ finite presented a major challenge. For this purpose, corresponding multiplicity functions were first defined for mappings form $\mathbb{R}^2$ into $\mathbb{R}^2$ in terms of topological indices. Another important result was Cesari's inequality, which states that $A_\ell(f)$ is dominated by the sum of the Lebesgue areas of the projections of f onto the three coordinate planes.

A brief overview of Federer's important contributions to area theory is given in W. Ziemer's part of the scientific obituary article [P12], and also in [FZ14]. Federer extended

many results about Lebesgue area to k-dimensional surfaces, defined by mappings f from $D \subset \mathbb{R}^k$ into $\mathbb{R}^n$, with $2 < k \leq n$. In doing so, he used recent developments in algebraic topology. Topological indices for the case $k = 2$ were replaced by topological degrees, defined in terms of Čech cohomology groups.

v   deRham's currents.

The L. Schwartz theory of distributions appeared just at the end of WW2. Since then it has had a very profound influence on mathematical analysis. A Schwartz distribution T is defined as a linear functional on a space of smooth test functions on $\mathbb{R}^n$. Any such T has (by definition) partial derivatives of every order, which are also Schwartz distributions. Soon afterward, deRham's theory of currents appeared [Rh55]. Another approach, motivated by geometric problems in the calculus of variations, was L.C. Young's theory of generalized surfaces [Y51]. See part (vi), Section 8 and Appendix C. Both [Rh55] and [Y51] are formulated in functional analysis settings.

In [Rh55], deRham introduced the concept of currents on a smooth manifold V. He was motivated primarily by questions in algebraic topology and differential geometry. However, deRham's currents turned out to provide a very convenient framework for studying questions in geometric measure theory. This connection was first made in [FF60].

For simplicity, we consider only $V = \mathbb{R}^n$. A current of dimension k is defined as a linear functional on a space of $\mathcal{D}_k$ of smooth differential forms ω of degree k, which have compact support. (deRham calls T a current of degree $n - k$.)

Example 1. Let $k = n$ and g an integrable function on $\mathbb{R}^n$ with compact support. The corresponding current T_g of dimension n (degree 0) satisfies for every smooth test function ϕ on $\mathbb{R}^n$

(2.4)
$$T_g(\omega) = \int_{\mathbb{R}^n} g(x)\phi(x)dx$$

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where $\omega = \phi(x)dx_1 \wedge \cdots \wedge dx_n$ is the corresponding differential form of degree n.

Example 2. Let $S \subset \mathcal{M}$, where $\mathcal{M}$ is a smooth k-dimensional sub-manifold of $\mathbb{R}^n$ and S has an orientation specified by a continuously varying unit tangent k-vector $\tau(x)$, $x \in S$. The associated current T_S is defined by

$$(2.5) \qquad T_S(\omega) = \int_S \omega = \int_S \omega(x) \cdot \tau(x)dH^k(x),$$

for all $\omega \in \mathcal{D}_k$. In (2.5), $\tau(x) = |\alpha(x)|^{-1}\alpha(x)$ where $\alpha(x) = v_1(x) \wedge \cdots \wedge v_k(x)$ and $v_1(x), \ldots, v_k(x)$ are linearly independent tangent vectors to $\mathcal{M}$ at x. See Appendix A for notations. The unit k-vector $\tau(x)$ is determined, up to $a \pm$ sign, by the order of these basis vectors $v_1(x), \ldots, v_k(x)$ for the tangent space at x. Note that $-T_S$ has the opposite orientation from T_S.

For any current T of dimension $1 \leq k \leq n$, the boundary ∂T is defined as the current ∂T of dimension $k - 1$ such that

$$(2.6) \qquad \partial T(\omega) = T(d\omega)$$

for all $\omega \in \mathcal{D}_{k-1}$ where the k-form $d\omega$ is the exterior differential of ω. Formula (2.6) includes as a special case the classical theorem of Gauss-Green in (2.1). In this case $k = n - 1$ and $S = B$ is the (smooth) boundary of E. The $(k - 1)$-vector $\tau(x)$ in (2.5) is the adjoint of the unit exterior normal to E at the point $x \in B$. The $(n - 1)$-form ω in (2.5) is adjoint to the 1-form determined by ζ in (2.1). See [Fl77, Section 8.7] and Appendix A. The classical Stokes formula for surfaces in $\mathbb{R}^3$ can also be rewritten in the form (2.6). Let S be a smooth surface in $\mathbb{R}^3$ with smooth boundary C, and with consistent orientations chosen for S and C. Let T_C and T_S denote the corresponding currents of dimensions 1,2 respectively. The classical Stokes' formula is equivalent to $T_C(\omega) = T_S(d\omega)$ for every 1 form $\omega \in \mathcal{D}_1$. In the classical statement of Stokes' formula, the adjoint of the 2-form $d\omega$ corresponds to the curl of ω. See [Fl77, Section 8.8].

vi Young's generalized surfaces.

Young is well known for his work during the 1930s on generalized curves. This early work provided solutions to calculus of variations and optimal control problems with nonconvex integrands, which may have no solution in the traditional sense. A generalized curve solution involves an ordinary curve C, to which is attached a measure-valued function on the set of possible tangent vectors at each point of C. See [Y37] [Y69].

In the seminal paper [Y51], Young defined the notion of generalized parametric surface of dimension $k = 2$. One of his goals was to provide an alternative to the surface area theory formulations of geometric problems in the calculus of variations, which would also apply in dimension $k > 2$. A generalized surface of any dimension k is defined as a nonnegative linear functional on a space $\mathcal{E}_k$ of continuous functions $F(x, \alpha)$, with $x \in \mathbb{R}^n$ and α a k-vector, see Section 8. His approach allows the use of methods based on weak convergence and convex duality arguments. However, Young had a broader vision, including for example a possible Morse theory in terms of generalized surfaces. This was expressed, for example, in the introduction to his paper [Y62]. In modified form, various parts of Young's vision were later achieved by Young himself, and also by others in the framework of integral currents, Whitney-type flat chains and Almgren's varifolds.

Young was my PhD thesis advisor, and he had a profound influence on my mathematical career. We wrote three joint papers on generalized surfaces [FY54, 56a,b], which are mainly of historical interest now. Some results in [FY56b] were precursors of later results about rectifiable and integral currents. Generalized surfaces are discussed further in Section 8 and Appendix C.

vii   Whitney's geometric integration theory.

H. Whitney's book [WH57] was another influential source of ideas for developments in GMT soon afterward. Whitney began by asking what a theory of k-dimensional integration in $\mathbb{R}^n$ should look like. A central role is played by the spaces $\mathcal{P}_k(\mathbb{R}^1)$ of polyhedral chains P of dimension k with real coefficients. Such a polyhedral chain is a finite linear combination of oriented polyhedral convex cells in $\mathbb{R}^n$. Two possible norms on $\mathcal{P}_k(\mathbb{R}^1)$ were considered, called the flat and sharp norms. The flat norm turned out to be particularly useful later. See [FF60][Z62][Fl66], also Sections 3 and 5 below.

A polyhedral convex cell σ is a bounded subset of some k dimensional plane $\pi \subset \mathbb{R}^n$, such that σ is the intersection of finitely many half k-planes of π. Each $P \in \mathcal{P}_k(\mathbb{R}^1)$ is a finite linear combination of polyhedral convex cells σ_i

$$(2.7) \qquad\qquad P = \sum_i a_i \sigma_i$$

with real coefficients a_i.

Whitney was particularly interested in characterizing the dual spaces to $\mathcal{P}_k(\mathbb{R}^1)$ with either flat or sharp norms. Elements of these dual spaces are called cochains, denoted by X in [WH57]. Under either flat or sharp norm, the dual space contains all cochains which correspond to smooth differential forms of degree k. The cochain X_ω associated with such a k-form ω is defined by

$$(2.8) \qquad\qquad X_\omega(P) = \sum_i a_i \int_{\sigma_i} \omega$$

for all $P \in \mathcal{P}_k(\mathbb{R}^1)$.

In [WH57, Chap. 4] it is shown that, under the flat norm, a cochain corresponds to what Whitney called a flat differential form, which is defined pointwise in terms of directional derivatives. Under the sharp norm, the dual space of cochains has a less explicit description [WH57, Chap. 11]. The perspectives in [Rh55] and [WH57] are quite different. In [Rh55]

deRham considered only "cochains" corresponding to smooth differential forms of degree k. These have the role of test functions in deRham's theory. The class of deRham's currents of dimension k is very large, including many currents which have no geometric properties at all. In contrast, Whitney's polyhedral chains correspond to test functions, and the large dual spaces include many cochains which do not correspond to differential forms in the usual sense.

3   Rectifiable and integral currents.

During the academic year 1958-59, Federer and I wrote the paper *Normal and integral currents* [FF60], for which we later received a Steele Prize from the American Mathematical Society. A brief overview of the principal motivations and results of this paper is given in this section, and in Sections 4,6. As already mentioned in Section 2(i), one of the goals of GMT is to provide a theory of k-dimensional measure for subsets of $\mathbb{R}^n$. Another goal is a theory of integration of k-dimensional "surfaces" without traditional smoothness assumptions. In [FF60], this is addressed in a systematic way. The Introduction to [FF60] begins with the following paragraph (written by Federer):

"Long has been the search for a satisfactory analytic and topological formulation of the concept "k dimensional domain of integration in euclidean n-space." Such a notion must partake of the smoothness of differentiable manifolds and of the combinatorial structure of polyhedral chains with integer coefficients. In order to be useful for the calculus of variations, the class of all domains must have certain compactness properties. All these requirements are met by the *integral currents* studied in this paper." In Federer's book [Fe69] Chapter 4, which is titled "Homological integration theory," also gives a systematic treatment of rectifiable and integral currents.

In the following discussion, we refer to Appendix A for notation and definitions. $M(T)$

denotes the mass of a current T, and $f_\#(T)$ the image of T under a Lipschitz function f from $\mathbb{R}^m$ into $\mathbb{R}^n$. Three types of convergence of sequences of currents are of interest: weak, strong and in the Whitney flat distance.

An indication of how we arrived in [FF60] at the class of integral currents is as follows. First of all, we sought a class $\mathcal{C}_k$ of k dimensional currents T, with compact support spt T, and with the following properties:

1. $T_S \in \mathcal{C}_k$ if T_S corresponds to a bounded subset $S \subset \mathcal{M} \subset \mathbb{R}^n$ as in (2.5) and S has "piecewise smooth" boundary. Note that $M(T_S) = H^k(S)$. In particular, we can take $S = \sigma$ where σ is an oriented polyhedral convex cell in some k dimensional plane, as in Section 2 (vii).

2. If $T_1, \ldots, T_m$ are in $\mathcal{C}_k$ and $a_1, \ldots, a_m$ are integers then
$$T = \sum_{i=1}^{m} a_i T_i$$
   is in $\mathcal{C}_k$. In particular, any polyhedral chain P with integer coefficients is in $\mathcal{C}_k$.

3. Any Lipschitz image $f_\#(T)$ of $T \in \mathcal{C}_k$ is also in $\mathcal{C}_k$.

4. If $T_1, T_2, \cdots \in \mathcal{C}_k$ and $T_j \to T$ strongly as $j \to \infty$, then $T \in \mathcal{C}_k$ [Recall that strong convergence means $M(T_j - T) \to 0$ as $j \to \infty$.]

The class of all rectifiable currents of dimension k, as defined in [FF60, Sec. 3], is the smallest class of currents determined by properties (1)-(4). From results [FF60, pp. 500-502] any rectifiable current T has the following representation. There exist a bounded k-rectifiable set K, and for H^k-almost all $x \in K$ there exist $\Theta(x)$ with positive integer values and an approximate tangent vector $\tau(x)$ with $|\tau(x)| = 1$, such that:

(3.1) $$T(\omega) = \int_K \omega(x) \cdot \tau(x) \Theta(x) dH^k(x), \forall \omega \in \mathcal{D}_k,$$

$$(3.2) \qquad M(T) = \int_K \Theta(x) dH^k(x).$$

In view of (3.2), $M(T)$ is also called the k area of the rectifiable current T, and $\Theta(x)$ represents the number of times x is counted. More precisely, K is countably k-rectifiable as defined in [Fe69, Sec 3.2.14], or equivalently in the sense of [FF60, pp. 500-502]. This definition provides consistent orientations for the approximate tangent k-vectors $\tau(x)$

For the study of existence theorems for calculus of variations, some property like (4) is needed in which strong convergence is replaced by weak convergence. As in [FF, Sec. 3], T is called an integral current if both T and its boundary ∂T are rectifiable currents. We take $\mathcal{C}_k$ to be the class of integral currents of dimension k. The desired weak convergence analogue of property (4), called the Closure Theorem, is discussed below. It requires that $N(T_j)$ is bounded, where $N(T) = M(T) + M(\partial T)$.

The need for some such additional condition is seen by considering the case $k = n$. A rectifiable current of dimension n has the form (2.4), where g is an integrable integer valued function with compact support and $M(T_g)$ is the L^1 norm of g. If $g_1, g_2, \ldots$ is a sequence in $L^1(\mathbb{R}^n)$ with bounded L^1 norms, then weak convergence of T_{g_j} as $j \to \infty$ corresponds to convergence of g_j in the Schwartz distribution sense. The limit can be any signed measure on a bounded subset of $\mathbb{R}^n$. The restriction that $M(\partial T_{g_j})$ is also bounded gives a geometric significance to the limit T. In particular, if g_j is the indicator function of a set E_j with "piecewise smooth" boundary which converges in n-dimensional Lebesgue measure to a set E as $j \to \infty$, then E is a set of finite measure in De Giorgi's sense. See Section 2(ii).

Highlights of [FF60].

Among the main results are the following:

(a) **Deformation Theorem** [FF60, Thm. 5.5.]. This essential tool provides polyhedral chain approximations to integral currents in the Whitney flat metric (and hence also

in the sense of weak convergence.) The cells of the approximating integral polyhedral chains belong to the k dimensional skeleton of a cubical grid in $\mathbb{R}^n$.

An immediate consequence of the Deformation Theorem is a result [FF60, Thm. 5.11] which provides a way to characterize homology groups for subsets of $\mathbb{R}^n$. Such subsets need not be smooth manifolds, but are required to have a local Lipschitz neighborhood retract property.

(b) **Isoperimetric inequalities**. Other very useful tools are the isoperimetric inequalities for currents [FF60, Sec. 6]. The proofs rely on the Deformation Theorem. In [FF60, Remark 6.6] the best isoperimetric constant is obtained, by an argument through which the inequality [FF60, Corollary 6.5] was originally discovered. This argument was outlined earlier in an abstract (Appendix B).

(c) **Weak and flat convergence**. In [FF60, Sec. 7] it is shown that weak convergence of a sequence T_j to T as $j \to \infty$ is equivalent to convergence in the Whitney flat distance, provide that $N(T_j)$ is bounded. The Deformation Theorem has an essential role in the proof. The Whitney flat distance also has a key role in the "nonoriented case," discussed in Section 5.

(d) **Closure Theorem** [FF60, Thm. 8.12]. This result says that, if T_j is a sequence of integral currents such that $N(T_j)$ is bounded, spt T_j is contained in a compact subset of $\mathbb{R}^n$ and $T_j \to T$ weakly as $j \to \infty$, then T is also an integral current. The proof of the Closure Theorem given in [FF60] relies on what was called a rectifiable projection property. It also uses deep covering and differentiation theorems of A.S. Besicovitch and A.P. Morse for measures on $\mathbb{R}^n$. Later, B. Solomon [SB84] and B. White [WB89] gave different proofs of the Closure Theorem. A corollary of the Closure Theorem is the following result [FF60, Corollary 8.13]: for any positive constants c, r, the set of

integral currents T such that $N(T) \leq c$ and spt $T \subset B_r(0)$ is weakly compact.

(e) **Strong approximation Theorem**. [FF60, Thm. 8.22] provides the following results, which further justify the idea that (in a measure theoretic sense) any integral current of dimension k nearly coincides with a finite sum of pieces of oriented smooth manifolds. In fact, there exist sequences of integral polyhedral chains T_j and diffeomorphisms f_j converging to the identity map, such that $N[f_{j\#}T_j - T]$ tends to 0 as $j \to \infty$.

(f) **Minimal currents**. [FF60, Sec. 9] is concerned with integral currents which minimize k-area, subject to given boundary conditions. Some of these results are sketched in Section 6.

During the 1950s, I had worked on L.C. Young's generalized surfaces, and I planned to continue doing so after my arrival at Brown in September 1958. A difficult problem was the absence of a generalized surface counterpart to the Closure Theorem for integral currents, except for some results in [FY56b] for $k = 2$, $n = 3$. Late in the autumn of 1958, I found a method which promised to work in any dimensions k and n, and mentioned it to Federer. This technique was similar to that used in [FF60] to prove the Closure Theorem. Quite independently, he had developed other parts of a theory of normal and integral currents. In fact, his paper [Fe55] on surface area theory already contained the basic idea of the Deformation Theorem. Federer soon convinced me of the advantages of the deRham current setting. We then began an intensive joint effort through the rest of the academic year 1958-59 and summer 1959. Federer undertook the task of organizing our results into the systematic and coherent form in which [FF60] appears.

See Appendix B for abstracts which announced early versions of the Closure and Deformation Theorems, and also an isoperimetric inequality.

4 Normal currents, other results.

As in [FF60] a current T is called normal if $N(T) = M(T) + M(\partial T)$ is finite. Weak compactness of $\{T : N(T) \leq c,\ \mathrm{spt}\ T \subset B_r(0)\}$ holds for any positive constants c, r. The Deformation Theorem implies that T is normal if and only if T is the weak limit as $j \to \infty$ of a sequence P_j of polyhedral chains with real coefficients, with $N(P_j)$ bounded and spt $P_j \subset B_r(0)$ for some r.

The case $k = n$ is of interest, due to its connections with bounded variation (BV) functions on $\mathbb{R}^n$. If $k = n$ and T is normal, then as in formula (2.4) $T = T_g$ for some $g \in L^1(\mathbb{R}^n)$ with compact support. The condition $M(\partial T_g) < \infty$ is equivalent to the property that the Schwartz distribution gradient of g is a vector-valued measure with finite norm. The function g is called of BV type. For $n = 1$, BV functions are of bounded variations on $\mathbb{R}^1$ in the usual sense. If a BV function is the indicator function of a bounded set $E \subset \mathbb{R}^n$, then the condition $M(\partial T_g) < \infty$ says that E is a set of finite perimeter in De Giorgi's sense (Section 2(ii)).

Federer made significant contributions to the theory of BV functions on $\mathbb{R}^n$. His results were announced in [Fe68]. The details (including proofs) are included in Theorem 4.5.9 of [Fe69]. The statement of this theorem has 31 parts, which represent a nearly complete theory of BV functions as of 1969. Striking new results announced in [Fe68] include a precise generalization of the classical property for $n = 1$ that a BV function has everywhere left and right limits, which differ only on a countable set. See also [Fe78, pp305-306].

Other results.

During the 1960s, Federer wrote several important papers in addition to his monumental book [Fe69]. His last publication on surface area theory was [Fe61]. In it he used GMT methods to study surfaces of dimension $k \geq 2$, defined by parametric representations and

with finite Lebesgue area. The influential paper [Fe65] has created linkages between Riemannian, complex and algebraic geometry. The technique of slicing for normal currents was introduced in this paper. Another important result in [Fe65] is his proof of mass minimality for complex subvarieties of Kähler manifolds. This led to the subject of calibration theory.

5   Flat chains.

With any integral current T of dimension k is associated an orientation of its approximate tangent spaces. In [Z62] and [Fl66] another formulation was considered in terms of Whitney's flat chains. This formulation does not involve orientations.

Let G be a metric abelian group, for example $G = \mathbb{R}^1$, $G = Z$ (the integers) or $G = Z_p$ (the integers mod a prime p). In [Z62], $p = 2$, which amounts to ignoring orientations. In the discussion here, we let G be a finite group and follow [Fl66].

As in Section 2(vii), let $\mathcal{P}_k(G)$ denote the group of polyhedral chains P of dimension k in $\mathbb{R}^n$, with coefficients in G. Let $M(P)$ denote the k-dimensional area of P, and let

$$(5.1) \qquad\qquad W(P) = \inf_{Q,R}\{M(Q) + M(R)\colon P = Q + \partial R\},$$

where Q, R are also polyhedral chains. The Whitney flat distance between P_1, P_2 in $\mathcal{P}_k(G)$ is $W(P_1 - P_2)$. The elements of the W-completion of $\mathcal{P}_k(G)$ are called flat chains over G and are often denoted by A.

Let $P_1, P_2, \ldots$ be a fundamental sequence in $\mathcal{P}_k(G)$, tending to a flat chain A. Since

$$W(\partial P_i - \partial P_j) \leq W(P_i - P_j),$$

the boundary ∂A is defined as the W-limit of ∂P_j as $j \to \infty$. The mass $M(A)$ of a flat chain A is defined as the lower limit of $M(P_j)$ as $j \to \infty$, taken among all sequences $P_j \in \mathcal{P}_k(G)$ tending to A as $j \to \infty$ [Fl66, Section 3]. Let $N(A) = M(A) + M(\partial A)$.

A flat chain is called rectifiable if A is the strong limit as $j \to \infty$ of $f_{j\#}P_j$, where $P_j \in \mathcal{P}_k(G)$ and f_j is Lipschitz [Fl66, Section 9].

In [Z62] and [Fl66] analogues of the main results of [FF60] about integral currents are proved. The statement and proof of the Deformation Theorem for flat chains are quite similar to that in [FF60, Sec. 5]. However, different arguments were needed to obtain results corresponding to the Closure Theorem and compactness of N-bounded sets mentioned in Section 3(d). The main result of [Fl66] is as follows. Let A_j be rectifiable for $j = 1, 2, \ldots$ with $M(A_j)$ bounded and $W(A_j - A) \to 0$ as $j \to \infty$. Then A is rectifiable. More simply, this result can be restated as follows. If $M(A)$ is finite, then A is rectifiable. The proof uses the Deformation Theorem, a structure theorem for sets of finite Hausdorff measure and the Besicovitch covering theorem for measures on $\mathbb{R}^n$. The compactness property [Fl66, Cor. 7.5] states that, for any positive constant c, r, $\{A \colon N(A) \le c, \text{ spt } A \subset B_r(0)\}$ is W-compact. Another consequence of the Deformation Theorem is an isoperimetric inequality [Fl66, Thm. 7.6].

6 Higher dimensional Plateau problem.

As mentioned at the end of Section 2(iii), by the late 1950s it was clear that entirely new formulations and methods were needed to study the Plateau (least area) problem for surfaces of dimension $k > 2$. The first major step in that direction was Reifenberg's paper [Re60]. In his formulation, a "surface" is a closed set $S \subset \mathbb{R}^n$ with $H^k(S) < \infty$. A closed set $B \subset S$ is called the boundary if an appropriate relationship in terms of Čech homology groups holds. Reifenberg proved that, given the boundary B, a set S^* which minimizes $H^k(S)$ exists. Moreover, S^* is topologically a k-dimensional spherical ball in a neighborhood of H^k - almost every nonboundary point $x \in S^*$. Such points have the property that the lower density does not exceed 1.

There were no earlier results to guide Reifenberg in this effort. His methods had to be invented "from scratch" and required remarkable ingenuity. For example, to prove that S^* is locally a topological k-ball (H^k - almost everywhere), Reifenberg constructed sequences of homeomorphisms from a ball $\sum \subset \mathbb{R}^k$ into $\mathbb{R}^n$, which tend to limits which he showed to also be homeomorphisms [Re60, Lemmas 8 and 9]. These results were immediately of interest in the GMT community, and also to C.B. Morrey. He included Reifenberg's solution to the Plateau problem as the last chapter of his book [MC66].

Oriented Plateau problem.

Another formulation (often called the oriented Plateau problem) is in terms of integral currents. In this formulation, a rectifiable current B of dimension $k-1$ with $\partial B = 0$ is given. The problem is to find an integral current T^* which minimizes the mass (or k-area) $M(T)$ among all integral currents T with $\partial T = B$. Since $M(T)$ is weakly lower semicontinuous, the existence of a minimizing T^* is immediate from the weak compactness property mentioned at the end of Section 3(d). There remained the difficult task of describing regularity properties of T^*. This is the topic of Section 7.

In [FF60, pp. 518-9] the following monotonicity property was proved, which has had quite a useful role in later developments. Let T^* be mass minimizing, and $x \in \operatorname{spt} T^* - \operatorname{spt} B$. For $r > 0$, let T_r^* denote the part of T^* in the ball $\mathcal{B}_r(x)$ with center x and radius r. Then $r^{-k} M(T_r^*)$ is a nondecreasing function of r. Another useful concept introduced in [FF60, Sec. 9] is that of tangent cones to mass minimizing integral currents.

Another formulation of the higher dimensional Plateau problem is in terms of Whitney's flat chains, with coefficients in a finite group G. When $G = Z_2$ this is called a "nonoriented" Plateau problem. Existence of a mass minimizing flat chain A^* with given boundary follows by similar arguments. Another formulation which also disregards orientations is in terms of Almgren's varifolds (Section 8).

7   Regularity results, Bernstein's Theorem.

For the oriented Plateau problem as formulated in Section 6, there remained the notoriously difficult "regularity problem." This is to prove smoothness of spt $T-$ spt ∂T for an integral current T which minimizes k-area, except at points of a singular set of lower Hausdorff dimension. We begin with some examples which show that, for $1 < k < n-1$, the singular set can have Hausdorff dimension $k-2$.

Example. [FF60, Remark 9.15]. Let $k = 2$, $n = 4$ and $T = T_1 + T_2$ with

$$\text{spt } T_1 = \pi_1 \cap \mathcal{B}, \quad \text{spt } T_2 = \pi_2 \cap \mathcal{B}$$

where $\mathcal{B} = \mathcal{B}_1(0)$ is the unit ball in $\mathbb{R}^4$ with center 0 and π_1, π_2 are mutually orthogonal planes in $\mathbb{R}^4$ which intersect at 0. The singular set consists of the single point 0.

The result in [Fe65] about mass minimality of complex subvarieties, mentioned at the end of Section 4, provides a rich class of examples in which spt $T-$spt∂T can have Hausdorff dimension $k-2$. In these examples, $k = 2\ell$, $n = 2m$ and the Kähler manifold is complex m dimensional space C^m, identified with $\mathbb{R}^{2m}$. The subvarieties corresponding to locally area minimizing integral currents are obtained by setting a finite number of homomorphic functions on C^m equal to 0. For instance, the equation $z_1 z_2 = 0$ in C^2 gives an example of the type just mentioned.

A profound difficulty is that it is not known in advance that spt $T-$ spt ∂T is locally the graph of a function, even if the singular set is avoided. Hence, the regularity problem is not just a question about smoothness of solutions to a system of nonlinear PDEs which describe locally necessary conditions for minimum k-area. Entirely new methods had to be developed.

Early results. In the early 1960s, De Giorgi and Reifenberg proved what are called "almost everywhere regularity" results in which the singular set was shown to have zero k-dimensional

Hausdorff measure. De Giorgi's result [DG61b, Thm. VII] is for dimension $k = n - 1$. It is stated in terms of "reduced boundaries" of sets of finite perimeter (also called Caccioppoli sets). Techniques used for this result appeared in a companion paper [DG61a]. By using an argument in [Fl62, Sec. 3] De Giorgi's result implies a corresponding almost everywhere regularity result about integral currents in $\mathbb{R}^n$ which minimize $(n - 1)$-dimensional area. However no regularity results about the oriented Plateau problem were known at that time for $k \leq n - 2$.

As mentioned in Section 6, Reifenberg showed in [Re60] that the minimizing set S in his solution to the Plateau problem is locally a topological k-disk near H^k-almost every nonboundary point $x \in S$. In [Re64b] this result was strengthened to show that these disks are smooth manifolds. The key to the proof is an "epiperimetric inequality" in [Re64a]. This inequality also became a useful tool in later work by other people.

Almgren's work on regularity. Beginning in the 1960's, Almgren was a leading contributor of results on regularity. His paper [AF68] represented a major advance. In it he obtained almost everywhere regularity results not only for the Plateau problem in all dimensions, but for a much broader class of geometric variational problems in which the integrand satisfies a suitable ellipticity condition.

Almegren's results were formulated in terms of varifolds. In [AF68], a surface S is a compact k-rectifiable set, with boundary B a compact (k-1)-rectifiable set. "Boundary" is defined in terms of relative homology groups. In [AF68, Section 4] the excess $E(S)$ for surfaces S lying over a unit disk is defined. It equals the k-area of S less the k-area of the covered disk. A key result is [AF68, Thm. 6.9]. It states that, for an elliptic varitional integrand F, a F-minimal surface S which lies near a k-disk has interior first and second order derivatives bounded by a constant times $E(S)^{\frac{1}{2}}$

Almgren continued to wrestle with the regularity problem for several years. After persis-

tent, courageous efforts he produced a massive manuscript often called his "Big Regularity Paper." In it he showed that singular sets for the higher dimensional oriented Plateau problem indeed have Hausdorff dimension at most $k - 2$. If Σ denotes the singular set, this means that $H^{k-2+\epsilon}(\Sigma) = 0$ for any $\epsilon > 0$. It remained an open question whether $H^{k-2}(\Sigma)$ is finite. The Big Regularity Paper has appeared in book form [AF00]. An essential feature is Almgren's use of multivalued functions, called by him Q-functions, which are interpreted as taking values in the space of integral currents of dimension 0. Almgren then introduced, in very sophisticated ways, nonparametric variational problems related to k-area minimization. In particular, regularity properties of Q-valued functions which minimize Dirichlet's integral are described [AF00, Sec. I.7].

The task of reading and assimilating all of the details of [AF00] seems to be a daunting one. In recent papers [DLS11) and [DLS13a], DeLellis and Spadaro began their study of Almgren's multiple valued functions and their links to integral currents. These papers are followed by [DLS13b,c,d], which provide substantially shorter alternatives to many of the arguments in [AF00].

Simon's paper [SL93] is another major contribution to the regularity problem. In it, more precise information about the structure of singular sets is obtained. A notion of stratification of singular sets by tangent cone type is used, originally introduced by Almgren in [AF00]. Among the results obtained is the $(k - 2)$-rectifiability of the interior singular set in the nonoriented (mod 2) case, and the local finiteness of the H^{k-2}-measure of the "top dimensional part" of this singular set.

Regularity results for $k = n - 1$.

It seemed at first that $(n - 1)$-dimensional area minimizing integral currents might have no singular points This was proved in [Fl62] for $n = 3$. Closely related to the regularity question in dimension $n - 1$ is the question of whether the only cones in $\mathbb{R}^n$ which locally minimize

$(n-1)$-area are hyperplanes. Using this connection, De Giorgi [DG65], Almgren [AF66] and Simons [SJ68] showed that there are no singular points for $n \leq 7$. However, Bombieri, De Giorgi and Giusti [BDG69] gave an example of a cone in $\mathbb{R}^8$ which provides a seven dimensional area minimizing integral current with a singularity at the vertex. This example (due to Simons) is as follows. Write $\mathbb{R}^8 = \mathbb{R}^4 \times \mathbb{R}^4$ and $x = (x', x'')$ with $x', x'' \in \mathbb{R}^4$. The cone satisfies $|x'| = |x''|$. Its intersection with any ball $\mathcal{B}_r(0)$ in $\mathbb{R}^8$ with center 0 defines a 7 dimensional integral current, which is shown in [BDG69] to be area minimizing. The vertex 0 is a singular point. Federer [Fe70] showed that this example is generic in the following sense: if $k = n-1$, then the singular set for the oriented Plateau problem can have Hausdorff dimension at most $n - 8$. For the nonoriented Plateau problem, he also showed in [Fe70] that the singular set has Hausdorff dimension $k - 2$, for arbitrary n and $k < n$. The method of [Fe70], now called Federer dimensional reduction, has proven to have wide applicability, not only to other area minimizing problems but also to energy minimizing harmonic maps and other systems of elliptic PDEs.

Boundary regularity. Let T minimize $(n - 1)$-area among all integral currents with $\partial T = B$, where spt B is a smooth oriented $(n - 2)$-dimensional manifold. In Allard's 1968 PhD thesis, he considered the regularity of spt T near points of spt B. The main results were announced in [AW69]. Regularity of spt T is proved near any boundary point x where the density of the total variation measure is $1/2$. Sufficient geometric conditions for boundary regularity to hold at all points of spt B were also given. Later, Hardt and Simon [HS79] obtained regularity at all boundary points without these geometric conditions on spt B. The restriction $n \leq 7$ for everywhere interior regularity is not needed for these boundary regularity results.

Bernstein's Theorem

The classical Bernstein Theorem is as follows. Let f be a smooth, real valued function which satisfies the minimal surface PDE for all $x \in \mathbb{R}^2$. Then f is an affine function (equivalently, the graph of f is a plane.) A GMT proof of this well known result was given in [Fl62]. It used the monotonicity property (Section 6) and the result [Fl62, Lemma 2.2] that cones in $\mathbb{R}^3$ which locally minimize area must be planes. An interesting question was whether the corresponding result about smooth solutions f to the minimal surface PDE in all of $\mathbb{R}^m$ must be true. This was proved by De Giorgi [DG65] for $m = 3$. In his proof, he showed that falsity of the Bernstein Theorem for functions on $\mathbb{R}^m$ would imply the existence of non-planar locally area minimizing cones in $\mathbb{R}^m$ of dimension $m - 1$. For $m = 3$, this allowed De Giorgi to use the same result as in [Fl62] about 2 dimensional locally area minimizing cones in $\mathbb{R}^3$. Making use of the same idea, Almgren and Simons then proved the Bernstein Theorem for $4 \leq m \leq 7$. However, the Bernstein Theorem is not correct for $m \geq 8$.

I was visiting Stanford in the Spring of 1969 when the startling news about this negative result arrived there. D. Gilbarg (an authority on nonlinear PDEs) was perplexed. It was unheard of that a result about PDEs should be true in 7 or fewer variables, but not in more variables. However, Gilbarg wisely observed that the Bernstein Theorem is really a geometric result, not a result about PDEs.

8 Geometric variational problems, generalized surfaces, varifolds.

A real-valued function $F = F(x, \alpha)$ is called a geometric variational integrand if F is continuous and satisfies the homogeneity condition

(8.1) $$F(x, c\alpha) = cF(x, \alpha), \text{ if } c \geq 0.$$

In (8.1), $x \in \mathbb{R}^n$ and α is any k-vector (not necessarily simple). Given F we can consider the problem of minimizing

$$(8.2) \qquad J(T, F) = \int_K F(x, \tau(x)) \Theta(x) dH^k(x)$$

among all integral currents T with given boundary $B = \partial T$. In (8.2), K, $\tau(x)$, $\Theta(x)$ are as in (3.1). In particular if $F(x, \alpha) = |\alpha|$ then by (3.2) $J(T, F) = M(T)$ is the mass (or k-area) of T.

If $J(T, F) \geq kM(T)$ for some $k > 0$ and J is a lower semicontinuous function of T under weak convergence, then the Closure Theorem implies that the minimum is attained. However, lower semicontinuity requires that $F(x, \cdot)$ has some additional property.

As in [Fe69, Sec. 5.1.2], F is called semi-elliptic if the following holds. For fixed x, let $F_x(\alpha) = F(x, \alpha)$. Then for any oriented polyhedral convex cell σ,

$$(8.3) \qquad J(T_\sigma, F_x) \leq J(T, F_x)$$

for any integral current T such that $\partial T = \partial T_\sigma$, where T_σ is the integral current corresponding to σ. Semi-ellipticity holds if $F(x, \cdot)$ is a convex function. By [Fe69, Thm. 5.1.5] semiellipticity implies weak lower semicontinuity.

When F is not semielliptic then we can seek what Young called a "generalized solution" to the problem of minimizing $J(T, F)$ for given F, subject to given boundary conditions. His work on generalized curve solutions for one dimensional problems [Y69] and two dimensional nonparametric problems in the calculus of variations [Y48a,b] suggest a solution in which (8.2) is modified such that $F(x, \tau(x))$ is replaced by an integral of $F(x, \cdot)$ with respect to a measure on the Grassmanian G_k^n of possible oriented tangent k-planes at x. This kind of result has not yet been proved. However, some related results and conjectures are given in Appendix C.

As mentioned in Section 2(vi), Young defined a generalized surface to be a nonnegative

linear functional L on the space of all parametric integrands F. In particular, if T is any rectifiable current, then corresponding generalized surface $L(T)$ satisfies $L(T)(F) = J(T, F)$ as in (8.2). See Appendix C.

Young's original method for proving the existence of a minimizing generalized curve proceeded as follows. Consider a minimizing sequence of ordinary curves, of bounded lengths and given endpoints defined parametrically by mappings from an interval $I \subset \mathbb{R}^1$ into $\mathbb{R}^n$. By choosing arc length as the parameter, a subsequence converges to a generalized curve solution to the minimum problem.

Unfortunately, this idea is too simplistic in the context of finding generalized surface solutions to multidimensional versions of this problem. Let T_j be a minimizing sequence for the problem of minimizing $J(T, F)$ subject to $\partial T = B$. Let $L_j = L(T_j)$ be the corresponding generalized surfaces. If T_j tends weakly to T and L_j tends weakly to L as $j \to \infty$, then L can be regarded as a generalized surface solution to the minimum problem. Weak convergence of L_j to L does not exclude the possibility that L has a "singular part" L_s, which has no geometric interpretation. This is discussed in Appendix C.

Varifolds. By the Riesz representation theorem, with any generalized surface L is associated a measure on $\mathbb{R}^n \times G_k^n$. There is a formal similarity with Almgren's theory of varifolds. A varifold is defined as a measure $\mathbb{R}^n \times \tilde{G}_k^n$, where $\tilde{G}_k^n$ is defined similarly to G_n^k without considering orientations. Almgren envisioned varifold theory as a way to study a wide range of problems in mathematics and its applications in the physical sciences and biology. It has become an important tool in GMT. See, for instance, [AF66] [SL83]. Taylor [T74] gave varifold solutions to some nonelliptic variational problems which arise in crystallography.

In [AF68] existence and almost everywhere regularity results were given for a varifold version of the problem of minimizing $J(T, F)$, with given boundary conditions. Boundaries were defined in terms of singular homology groups [AF68, pp 334-335]. Another goal was the

study of varifolds which are minimal (not necessarily area minimizing) in the sense that first order necessary conditions for minimum k-area are satisfied. Allard's paper [AW72] was an important contribution in that direction. He considered the first variation δV of an integral varifold V, which can be represented in terms of mean curvature and exterior normals at the boundary if V is a smooth manifold. In [AW72, Section 6] a compactness result about integral varifolds was proved. In this result, the bound on $M(\partial T)$ for the corresponding compactness result about integral currents (Section 3(d)) is replaced by a bound on the norm of δV.

Despite the formal similarity in the definitions of Young's' generalized surfaces and Almgren's varifolds few connections between these two theories seem to have been made.

See end of Appendix C for further comments.

Part II - Remembrances

9 Brown University Mathematics Department in the 1960s.

When I came to Brown in 1958, teaching loads in the Mathematics Department were high (3 courses per semester). The Department was located in an old house at 65 College Street. Office furnishing were austere. Telephone service was state of the art for the year 1908. However, none of that mattered very much. Most importantly for me, Herb Federer was at Brown. The winter of 1958-59 was when we did most of the work which resulted in our **Normal and integral currents** paper [FF60]. Nearly all Math Department faculty were young, and an atmosphere of excitement about mathematics abounded. My algebraist colleague Dave Buchsbaum and I talked often about our work. I still remember the name of one of his nice theorems "Every regular local ring is a unique factorization domain" (but don't ask me to explain it). Brown students, at both undergraduate and graduate levels

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were good. One of my courses in 1958-59 was real analysis, which I had never taught before. Fred Almgren and Bill Ziemer were among the students in this course.

10   Graduate students and visitors in GMT.

During the 1960's there were six PhD students in geometric measure theory at Brown. Their names and year of completion of the PhD are as follows:

Federer's PhD students: Frederick Almgren (1962), John Brothers (1964), Lawrence Ernst (1970) and Robert Hardt (1971).

Fleming's PhD students: William Ziemer (1961) and William Allard (1968).

The fact that this number of PhDs is not larger can be attributed partly to the relatively small size of Brown's Math PhD program compared to state universities such as Berkeley and Michigan, and also to the demanding nature of research in GMT. In the 1960s, the students who chose to work in GMT were entering a field which was just being invented. There was a chance to contribute something really new, not just to add a few more bricks to a long standing mathematical edifice. The number of "mathematical descendents" of our small program in GMT is much larger. As of March 2014, the Mathematics Genealogy Project website listed 156 students and "grandstudents" of the six former Brown PhD students listed above.

Both Federer and I encouraged regular discussions with students. A great deal of learning happened in one-on-one conversations with faculty, other students and former students in GMT. The task of mastering a difficult mathematical field is challenging, and at times discouraging. I told Bill Ziemer to start by reading both [FF60] and Whitney's book [WH57]. An unnamed source told me (much later) that while struggling with this assignment, Bill occasionally wondered whether some other career (such as construction worker) might be

better than life as a mathematician. However, he overcame the difficulties admirably. His thesis was published as [Z62]. It was the beginning of the theory of flat chains over a finite coefficient group, with applications to the nonoriented Plateau problem.

Bill Allard's thesis topic was boundary regularity for the oriented Plateau problem [AW68]. A short version of the results appeared in [AW69]. This topic seemed scary to me, with either total success or nothing as possible outcomes. I hinted that he might try something safer, but Bill didn't agree to this. I also remember his attitude that "there is no such thing as a dumb question." That is an efficient way to learn the math which you need.

All of our graduate students had duties as teaching assistants. Several of them also gave Federer and me substantial help with book projects. During the writing of the first edition of my textbook [Fl77], several students read various chapters. John Brothers carefully read the entire manuscript and furnished many improvements to it. This book (in its second edition) is still in print nearly 50 years later.

Our students also read parts of the manuscript for Federer's book [Fe69], and Allard read all of it. The introduction to [Fe69] says: "William K. Allard read the whole manuscript with great care and contributed significantly, by many valuable queries and comments, to the accuracy of the final version."

Some remembrances of Fred Almgren are included in Section 11.

Visitors. Among visitors to Brown in the field of GMT were E.R. (Peter) Reifenberg in the summer of 1963 and Ennio De Giorgi in the spring semester of 1964. I had met both of them in August 1962 at a workshop in Genoa, Italy. That workshop was unusually lively and productive. Someone described the language of the workshop as "lingua mista" – a mixture of English, Italian and bad French.

Other visitors included J. Marstrand, J. Michael and Bill Ziemer (summer 1963). Ubiritan D'Ambrosio came as a postdoc starting in January 1964, and later returned to Brazil for

a distinguished career in math education. D'Ambrosio was fluent in Italian, and was quite helpful during DeGiorgi's seminar talks at Brown. Harold Parks was a junior faculty member at Brown during the 1970s.

11 Remembrances of leaders in GMT (Almgren, De Giorgi, Federer, Reifenberg, Young.)

The focus of this section is on my personal remembrances of these five deceased leaders in GMT. The parts about Almgren and De Giorgi are rather brief. Readers may consult more detailed scientific obituary articles and selected papers volumes cited in the references. Young was my PhD advisor and later mentor. The part about him is written from that perspective. Federer initiated the offer to me of a faculty position at Brown, which led to my arrival in 1958. I was privileged to be part of the excitement in GMT in the years which followed. I am deeply indebted to both Young and Federer. My remembrances of Reifenberg are from the all too brief period 1962-1964 of our friendship. His untimely death in 1964 was a great loss to mathematics.

Frederick J. Almgren, Jr. (1933-1997)

Fred Almgren was an Engineering undergraduate at Princeton, followed by three years as a US Navy pilot. He then came to Brown as a graduate student in 1958, and completed the PhD in 1962 under Herbert Federer's supervision. Fred then moved to Princeton, where he remained until his death in 1997.

Dana Mackenzie's obituary article [MD97] is a moving account of Fred's professional and personal life. In 1998, an issue of the Journal of Geometric Analysis was dedicated to Fred Almgren. This issue includes papers by Brian White [WB98] which summarizes Almgren's mathematical contributions, and by Frank Morgan [MF98] with recollections of his life. Reference [AF99] contains selected papers by Almgren.

Fred Almgren was a person with boundless energy and good cheer. He had many interests, and was an avid sailor. His enthusiasm for mathematics and ability to locate beautiful problems which were "ready to be solved" attracted many students. See list of names at the end of [WB98]. Besides Almgren's profound contributions to GMT mentioned in earlier sections of this article, he did (with coauthors) important work on such other topics as curvature driven flows, liquid crystals, energy minimizing maps and rearrangements. Almgren was one of the founders of the Geometry Supercomputer Project in Minneapolis.

The following statement by me is excerpted from [MF98]:

Fred Almgren and I arrived at Brown at the same time (fall 1958), he as a beginning graduate student and I as a new assistant professor. Fred took my real analysis course. While it was clear from the start that Fred had an excellent intuition and original ideas, he was not yet trained to think like a mathematician.

Fred's PhD thesis was a brilliant one. In his excellent article, Brian White mentioned the curious episode in which the Brown Graduate School hesitated to accept it, on the grounds that the thesis had already been accepted by the journal Topology. A very firm stand by Herb Federer persuaded the Dean to withdraw his objection.

My thesis advisor L.C. Young had expressed the need for a kind of Morse theory in terms of multivariable calculus of variations. Soon after the thesis, Fred provided such a theory in terms of what he called varifolds. Varifolds are very similar to Young's generalized surfaces, but the name varifold is much more appealing.

A lot was happening in geometric measure theory during the years 1958-62 when Fred was at Brown. He and I ate lunch regularly in the cafeteria. During these lunches, Fred found out more or less all I knew and of course I learned a great deal from him in return. It was clear even then that the regularity problem for varifolds which minimize k-dimensional area (or some other geometric variational integral) was going to prove extremely difficult in

codimension more than one. I have the greatest admiration for Fred's determination and persistence in wrestling with these regularity problems through many years, culminating in his massive three-volume regularity proof.

After Fred left Brown we saw each other only occasionally. One such occasion was in summer 1965 when we were both visiting De Giorgi at the Scuola Normale in Pisa. A photo of Almgren, De Giorgi and me taken during this visit appears on page 2 of [MF00]. I still have very pleasant memories of excursions with Fred to Lucca and Siena, which are interesting towns nearby. He knew how to enjoy life during the times when he was not immersed in mathematics.

We were very pleased to have Fred as an honored guest at the 1988 Brown Commencement, when he received a Distinguished Graduate School Alumnus Award. Each year this award is given to two or three of Brown's most distinguished former PhD graduates.

Ennio De Giorgi (1928-1996)

De Giorgi studied in Rome with M. Picone. After one year 1958-59 at the Università di Messina, he moved to the Scuola Normale Superiore, (SNS) in Pisa, and remained there for the rest of his life. Besides his many contributions to GMT, De Giorgi is renowned for his work on PDEs (including the De Giorgi-Nash a priori estimates in the 1950s), gamma convergence and other topics. Among the honors which he received as the prestigious Wolf Prize, awarded in 1990. The book of Selected Papers [DG06] includes a biography of De Giorgi in Chapter 1 and an account of his scientific contributions in Chapter 2. Chapter 3 contains a selection of his papers, with translations from Italian into English. [AL99] is an obituary article, written in Italian.

In Pisa De Giorgi lived in simple accommodations in a residence along the Arno River which belonged to the SNS. He had a wide circle of friends, and he enjoyed good food, conversation and mountain walks.

Beginning in 1988, De Giorgi spent long periods in Lecce, which was his boyhood home and still the home of his extended family. He established mathematical ties in Lecce, and the Mathematics Department at Università di Lecce is now named after him.

Besides his contributions to mathematics, De Giorgi was deeply involved with charitable and human rights issues. He was a devout Christian, with nuanced views about relationships between science and faith.

The following passage was included in my lecture at a conference in De Giorgi's memory, held at the SNS in October 1997. The attendance at this conference was very large, which is an indication of De Giorgi's iconic status among mathematical analysts in Italy.

1. **Geometric measure theory.** During the 1950s and 1960s both De Giorgi and I were working in what is now called geometric measure theory. These remembrances concern mostly some memories of De Giorgi and his brilliant work during that time period. Geometric measure theory provides class of objects, which I will call in an imprecise way "surfaces" of arbitrary dimension k in some euclidean space. They were called "generalized surfaces" by L.C. Young, "varifolds" by F. Almgren and "integral currents" by H. Federer and myself. For De Giorgi, the objects were portions of the reduced boundary of a set of finite perimeter, in codimension 1, and later a particular class of what is called "correnti quasinormali" in arbitrary codimension. Of course, the objects are not really smooth surfaces in a classical sense, but it happens that they coincide approximately (in a suitable measure theoretic sense) with finite unions of surfaces of class C^1. The theory provides compactness of sequences of surfaces with bounded k dimensional area and boundaries with bounded $(k-1)$-dimensional area. Another important property is that versions of the classical theorems of Gauss-Green and Stokes remain true.

   Geometric multidimensional problems of the calculus of variations provided an im-

portant motivation for geometric measure theory. A famous example is the Plateau problem, which is to find a k-dimensional surface with least k-dimensional area, among all surfaces with the same boundary. Geometric measure theory provides immediately the existence of an area minimizing surface. However, the problem of regularity of area minimizing surfaces turned out to be quite complicated. The most which can be expected is regularity except at points of some lower dimensional singular set. In codimension 1, the singular set is empty in low dimensions. However, the famous 1969 Bombieri-De Giorgi-Giusti paper (which will be mentioned again later) shows that this is false in higher dimensions.

2. **Sets of finite perimeter.** I first heard about De Giorgi in 1956 or 1957 when the French mathematician C. Pauc urged me to read De Giorgi's important new papers in the Annali di Matematica and Richerche di Matematica, on sets of finite perimeter (also called at that time Caccioppoli sets). From the Annali paper I first learned about the "slicing formula" which equates the total gradient variation of a function and an integral of the areas of level sets. This formula was used by De Giorgi to show that his definition of set of finite perimeter was equivalent to another definition of Caccioppoli. The slicing formula anticipated the so-called coarea formula, of which it is a particular case.

3. **First main regularity theorem.** In 1961 De Giorgi published two seminal papers in a Seminario di Matematica della Sculoa Normale Superiore di Pisa series, which was not I think widely available. This work provided the first big regularity result for the Plateau problem in codimension 1. The proof of this result is an amazing "tour de force." Starting with a locally area minimizing surface, which is not even known to be locally the graph of a function, De Giorgi managed to prove that the surface is smooth near any point at which it is measure theoretically close to some approximate tangent

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plane.

4. **Genova workshop.** In August 1962 J.P. Cecconi hosted a workshop at the Università di Genova, at which I first met De Giorgi. In addition to several other Italian mathematicians, E. Reifenberg, from England also attended. Reifenberg had recently written an important 1960 Acta Mathematica paper on the Plateau problem. This workshop had a fundamental role in stimulating further work in geometric measure theory. As Reifenberg said, it was conducted in a kind of "lingua mista." Despite some language difficulties, many interesting ideas were circulated and taken home for further study.

5. **Visit to the USA.** In 1964 De Giorgi visited Brown and Stanford universities. He came by ship (the Cristoforo Colombo), and I met him in New York. There was a delay of several hours waiting for the passengers to disembark, because of a dock workers strike. During the auto trip from New York to Providence, De Giorgi told me that he had just proved a striking result call the Bernstein theorem for minimal surfaces of dimension 3 in 4 dimensional space. However, there was no mathematics library on the Cristoforo Colombo, and he wished to be certain about the strong maximum principle for elliptic PDEs which he needed in the proof. I assured him that what he needed is OK. We will return to the Bernstein problem in a moment.

During his stay at Brown, DeGirogi gave a series of lectures on what he called "correnti quasi-normali." His approach provided an alternative to the one taken by Federer and myself for normal currents. De Giorgi's method has the advantage that no use was made of a difficult measure theoretic covering theorem of Besicovitch.

6. **Minimal cones and the Bernstein problem.** In 1969, Bombieri, De Giorgi and Giusti published a truly remarkable paper on area minimizing cones and the Bernstein problem. The results were unexpected and at least for some analysts contrary to

intuition. Speaking only of the Bernstein problem, the question is as follows. Let f be a smooth function of m variables which satisfies the minimal surface PDE in all of m-dimensional Euclidean space. Must f be a linear function? This was known for a long time to be true if $m = 2$. It was proved by geometric measure theory methods by De Giorgi for $m = 3$, then by Almgren for $m = 4$ and by J. Simons for $m = 5, 6, 7$. However, Bombieri, De Giorgi and Giusti showed that the result is false for $m \geq 8$.

7. **Further remarks.** After the 1960s De Giorgi's work and mine took different directions. However, we kept up a lifelong friendship and saw each other from time to time, both in Pisa and elsewhere. Communication became easier as De Giorgi's English improved and I learned a little Italian. (The other choice was bad French which we mutually decided against early on.) Besides his mathematical work, De Giorgi told me about his trips to Eritrea and his work for Amnesty International. Our last meeting was in 1993 at the 75th birthday conference for Cecconi in Nervi.

Ennio De Giorgi was a mathematician of extraordinary depth and powerful insights. There is a great Italian tradition in the calculus of variations, and among the world leaders in the first part of the 20th century was L. Tonelli. De Giorgi was in every sense a worthy successor to Tonelli. There is a plaque on a wall in the old Università di Pisa building complex concerning Tonelli. While I don't remember the exact wording, it says in effect that Tonelli was both an excellent mathematician and outstanding citizen. The same can be said about De Giorgi, although his good citizenship was shown perhaps in a different style from Tonelli's.

We miss him very much.

Herbert Federer (1920-2010)

These remembrances of Herbert Federer's career are based to a considerable extent on references [P12] and [FZ14]. Many of Federer's contributions to GMT have been discussed in earlier sections, and will not be repeated here. There is an overview of his contributions to surface area theory, written by W.P. Ziemer, in [P12] and also in [FZ14].

Federer was born in Vienna, Austria in 1920 and immigrated to the United States in 1938. His education in the US began at a teachers college which later became the University of California, Santa Barbara. His exceptional mathematical talent was quickly recognized, and he soon transferred to the University of California, Berkeley. He received his PhD in mathematics in 1944, under the supervision of A.P. Morse. During 1944-1945, Federer served in the U.S. Army at the Ballistics Research Laboratory in Aberdeen, MD. In the fall of 1945, he joined the Department of Mathematics at Brown University, where he remained until his retirement in 1985. Another major event in Federer's life was his marriage to Leila Raines, in 1949. Mathematics and his family were Herb's two great loves. He was devoted to Leila and their three children.

Among the honors which Federer received was a Steele Prize from the American Mathematical Society (AMS). He was a fellow of the American Academy of Arts and Sciences, and a member of the National Academy of Sciences. At the 1977 summer AMS meeting, he was the Colloquium Lecturer.

Federer is remembered for his many deep and original contributions to the fields of surface area and geometric measure theory (GMT). It is difficult to imagine that the rapid growth of GMT (beginning in the 1950s), as well as its subsequent influence on other areas of mathematics and applications, could have happened without his groundbreaking efforts.

A characteristic of Federer's work was his dedication to learning many different kinds of mathematics. When he became interested in a new subject (e.g., algebraic topology, differ-

ential geometry or algebraic geometry), he would first spend many weeks reading classical and modern books on it. He would then teach a graduate course on the topic and produce a large collection of lecture notes. One of his principal points of advice to graduate students was indicated by the only sign on his office door. It was a long, vertically stacked series of small stickers that said "Read, Read, Read, ..."

Federer set very high standards for his mathematical work and expected high-quality research from his students; he supervised the PhD theses of 10 of them. In addition to Federer's own work, the contributions of his many mathematical descendents (PhD students, "grandstudents" and "greatgrandstudents") continue to have a major impact on GMT. While some students found Federer's courses daunting, he was very welcoming to anyone who was deeply committed to mathematics and who took the trouble to get to know him. The sections of [P12] written by Allard, Hardt and Ziemer, are eloquent testimonials to the great regard and esteem which former students have for Federer.

Federer was fair-minded and very careful to give proper credit to the work of other people. He was also generous with his time when serious mathematical issues were at stake. He was the referee for John Nash's 1956 *Annals of Mathematics* paper "The imbedding problem for Riemannian manifolds," which involved a collaborative effort between author and referee over a period of several months. In the final accepted version, Nash stated, "I am profoundly indebted to H. Federer, to whom may be traced most of the improvements over the first chaotic formulation of this work." This paper provided the solution to one of the most daunting and longstanding mathematical challenges of its time.

Despite the many advances in GMT after it was published in 1969, Federer's authoritative book *Geometric Measure Theory* [Fe69] is still a classic in the field. In [P12], Bob Hardt said: "Forty years after the book's publication, the richness of its ideas continue to make it both a profound and indispensable work. Federer once told me that, despite more than a

decade of his work, the book was destined to become obsolete in the next 20 years. He was wrong. The book was just like his car, a Plymouth Fury wagon purchased in the early 1970s that he somehow managed to keep going for almost the rest of his life. Today [May 2012], the book *Geometric Measure Theory* is still running fine and continues to provide thrilling rides for the youngest generation of geometric measure theorists."

I first met Herb Federer at the summer 1957 AMS Meeting at Penn State. Afterwards, he suggested to the Mathematics Department at Brown that I might be offered an assistant professorship. An offer was made, which I accepted. Upon our arrival in Providence in September 1958, my wife Flo and I were warmly welcomed by Herb and Leila Federer. The academic year 1958-1959 was the most satisfying time of my career. Our joint work on normal and integral currents was done then. This involved many blackboard sessions at Brown, as well as evening phone calls at home (no Skype in those days). Herb undertook the task of organizing our results into a systematic coherent form, which appeared as [FF60].

In later years, after our research paths had taken different directions, Federer and I didn't often discuss mathematics. However, we always brought each other up to date about family news.

E.R. (Peter) Reifenberg (1928-1964)

Peter Reifenberg was a student of A.S. Besicovitch at Cambridge University. After a postdoctoral position at Berkeley, he joined the Mathematics Department at the University of Bristol. During the 1950s, he wrote a series of papers on surface area theory, including [Re55]. He then produced the remarkable Acta Mathematica paper [Re60], which was the first major work on the higher dimensional Plateau problem. There followed a paper [Re64a] on his important "epiperimetric inequality" and the sequel [Re64b] which used the epiperimetric inequality to obtain the regularity result for the Plateau problem mentioned in Section 7.

I first met Peter in Italy during August 1962. He and his wife Penny met my wife Flo and me at Milan airport. We followed them in a rented car to Genoa to participate in a very productive workshop on GMT organized by J.P. Cecconi. We met Ennio De Giorgi for the first time at this workshop. After the workshop, Flo and I met the Reifenbergs again at Zermatt, just after they had climbed the Matterhorn.

My friendship with Peter continued through mail correspondence and his visit to Brown during the summer of 1963. During the summer of 1964 we got news of his death while climbing with Penny in the Dolomites. It was caused by falling rock, and not at all due to carelessness on Peter's part. A few months after his death, Penny Reifenberg sent to me his handwritten notes concerning regularity for the Plateau problem. I shared them with Fred Almgren. Those notes were too fragmentary to determine what new results Peter Reifenberg had obtained.

Peter was fearless in his approach to mathematics. He could be undiplomatic, which sometimes led to worthwhile mathematical outcomes. On one hot, humid Friday afternoon during Peter's 1963 visit to Brown, I outlined at the blackboard a possible method to prove the main result of [Fl66]. Peter's skepticism was probably intended as friendly advice, but was expressed in a negative way. I worked intensively over the following weekend, and had verified the essential details of my argument by the following Monday.

Peter Reifenberg was surely destined for a brilliant future in mathematics. His premature death was a great loss to GMT, as well as to his family and friends.

Laurence C. Young (1905-2000)

These remembrances of Laurence Young are, in part, adapted from the obituary article [FW04] and from remarks which I made in May 2005. The occasion was a Mathfest at the University of Wisconsin in honor of Young's 90th birthday.

Young came from a mathematical family. Both parents, William H. and Grace C. Young, were distinguished English mathematicians, whose research was at the forefront of real analysis in the early 20th century. L.C. Young began his study of mathematics at Trinity College, Cambridge in 1925, and became a Fellow in 1931. He interspersed his studies at Cambridge with extended stays in Munich, where the great Greek mathematician Carathéodory became a mentor. From 1938 to 1948, Young was Professor and Head of the Mathematics Department at the University of Capetown, South Africa. He then moved to the University of Wisconsin Mathematics Department where he remained until his retirement in 1976.

Starting in the 1930's L.C. Young made major contributions to several areas of mathematical analysis, especially in integration theory and the calculus of variations. His imagination and vision provided a key to bringing the calculus of variations to its present form, with applications in geometric measure theory, control theory, partial differential equations and material science.

Young initiated in 1963 the "Wisconsin Talent Search" for discovering talented students among Wisconsin's high schools. This program still thrives today. In addition, he was completely fluent in at least four languages and had a working knowledge of several others. He was an accomplished pianist and a champion chess player.

In the calculus of variations, Young introduced two radically new ideas which had a fundamental and lasting impact. The first of these was the notion of generalized curve, which he introduced in the 1930s. The second idea was that of generalized surface as a linear functional, introduced in [Y51]. I regard this paper as seminal for the development of what came to be known soon after as geometric measure theory.

In 1948, Young extended the idea of generalized curve to nonparametric double integral problems in the calculus of variations [Y48a,b]. He obtained "generalized solutions" in the form of a pair of functions f, μ on a region D in $\mathbb{R}^2$. For $(u, v) \in D$, $f(u, v)$ is real

valued and $\mu(u, v)$ is a probability measure on a space of possible gradient vectors (called a Young measure). Young measures later provided the basis for the study of minimum energy configurations in solid mechanics. The Tartar-Murat method of compensated compactness makes essential use of Young measures. Subsequently, they were also applied to problems in such diverse areas as hyperbolic PDEs, microstructures and phase transitions.

Reference [FW04] gives a more thorough review of Young's life and mathematical career, with a complete list of his publications. During the 1940s and early 1950s, he worked on surface area theory and two-dimensional generalized surfaces defined in terms of parametric representations. He considered an alternative to the Lebesgue definition of area, similar to the definition given by Besicovitch [B45].

Young continued his work on generalized surfaces into the 1960s. Reference [Y59] concerns what he called a partial area formula and its applications. This formula is very similar to Federer's coarea formula. In [Y62] Young studied variational principles using what he called a theory of contours, related to slicing for integral currents.

During the 1970s, Young developed his own approach to stochastic integration. This work is in a spirit similar to his early work on Stieltjes integrals during the 1930s. It is elementary that the Stieltjes integral $\int f \, dg$ over a compact interval I on the real line exists, provided that one of the functions of f, g is continuous and the other is of bounded variation. Young replaced this asymmetric condition on f, g by more symmetric conditions which also guarantee that the Stieltjes integral exists. For instance f can have bounded pth power variation and g bounded qth power variation, where $p^{-1} + q^{-1} > 1$.

I first heard about Young's generalized surfaces in a seminar lecture which he gave in the spring of 1950. Later that year he became my PhD thesis advisor. He was not a "hands on" kind of thesis advisor, providing detailed suggestions in regular meetings with me. However, at a more fundamental level, Young's guidance was excellent. In the autumn of 1950, he

mentioned to me an interesting research problem. Young intended to work on this problem himself, and later (jokingly) said that "I stole his problem." This problem was the basis for my PhD thesis, which later appeared in revised form as [FY54]. In the spring of 1951, I gave a first draft of my thesis to Young. He pronounced it "unreadable," which was certainly true. The process of revision extended into the hot, humid summer of 1951. My wife, Flo, typed the thesis, without the aid of modern text processing technology. Our departure to begin my new job at the RAND Corporation in Santa Monica, California was delayed until late August.

Young later invited me as a visitor to the University of Wisconsin Mathematics Department and Mathematics Research Center for extended periods, during 1953-54, summer 1955 and academic year 1962-63. Flo and I always remember the warm hospitality which the Young family showed us on many occasions at their home on the shore of Lake Mendota. The kindness of his wife Elizabeth Young when our oldest son Randy was born in February 1954 was particularly appreciated.

Both Young and I attended a conference on partial differential equations at Berkeley in June 1955. My family were returning by car to the Midwest, to begin my new job at Purdue in September. After the conference, Young rode with Flo, one year old Randy and me as far as Denver. He enjoyed the chance to see more of the American West during this auto trip including rustic overnight accommodations at Lake Tahoe. Afterward, Young commented that, in former times, an English gentleman usually traveled accompanied by a trunk to hold the clothing which he was expected to have available.

Appendix A

Notations and Definitions

I. $\mathbb{R}^n$ is euclidean n-dimensional space, with elements denoted by x or y.

$$\mathcal{B}_r(x) = \{y \in \mathbb{R}^n : |x - y| \leq r\}$$

is the ball with center x and radius r.

For $0 \leq k \leq n$, $H^k(K)$ is the Hausforff k-dimensional measure of a set $K \subset \mathbb{R}^n$

π denotes a k-dimensional plane in $\mathbb{R}^n$, if $1 \leq k \leq n - 1$.

II. **Exterior algebra** (also called Grassmann algebra). We use the notations and definitions in the textbook [Fl77]. For a more compete development, see [Fe69, Chap. 1].

α denotes a k-vector, $k = 1, \ldots, n$.

$\alpha \wedge \beta$ is the exterior product of a k-vector α and ℓ-vector β. Note that $\beta \wedge \alpha = (-1)^{k\ell}\alpha \wedge \beta$.

α is a simple k-vector if $\alpha = v_1 \wedge \cdots \wedge v_k$ with $v_1, \cdots, v_k \in \mathbb{R}^n$.

The norm $|\alpha|$ of a simple k-vector α is the k-area of the parallelopiped

$$P = \{x = c_1 v_1 + \ldots + c_k v_k, 0 \leq c_j \leq 1, \text{ for } j = 1, \ldots, k\}.$$

Orientations. Any k-plane π has the form

$$\pi = \{x = x_0 + c_1 v_1 + \ldots + c_k v_k\}$$

with $x_0, v_1, \cdots, v_k \in \mathbb{R}^n$, $c_1, \cdots, c_k \in \mathbb{R}^1$ and $v_1, \cdots, v_k$ linearly independent. If $\alpha = v_1 \wedge \cdots \wedge v_k$, then $\tau = |\alpha|^{-1}\alpha$ has norm $|\tau| = 1$. This k-vector τ assigns an orientation to π, with $-\tau$ the opposite orientation.

G^n_k is the set of all simple k-vectors τ with $|\tau| = 1$. Alternatively, one can think of G^n_k as the set of all oriented k-planes π which contain the point $x_0 = 0$. G^n_k is called the oriented Grassmannian.

k-**covectors and differential forms**. A k-covector ω is defined similarly as for k-vectors, with the space $\mathbb{R}^n$ of 1-vectors replaced by its dual space of 1-covectors. The dot product of ω and α is denoted by $\omega \cdot \alpha$.

A differential form ω of degree k is a k-covector valued function on $\mathbb{R}^n$. The norm (or comass) of ω is

$$\| \omega \| = \sup\{\omega(x) \cdot \alpha, x \in \mathbb{R}^n, \alpha \in G_k^n\}.$$

III. **Exterior differential calculus and currents**. For any smooth k-form ω, the exterior differential is a $(k+1)$-form denoted by $d\omega$. It has the property $d(d\omega) = 0$. Let $\mathcal{D}_k$ denote the space of all k-forms ω which have compact support and continuous partial derivatives of every order. A current T of dimension k is a linear functional on $\mathcal{D}_k$, which is continuous in the Schwartz topology on $\mathcal{D}_k$. The boundary ∂T is the current of dimension $k - 1$ defined by formula (2.6):

$$\partial T(\omega) = T(d\omega) \text{ for all } \omega \in \mathcal{D}_{k-1}.$$

Note that $\partial(\partial T) = 0$.

The mass of T is

$$M(T) = \sup\{T(\omega) : \| \omega \| \leq 1\}.$$

Let $N(T) = M(T) + M(\partial T)$.

The support spt T of a current T is the smallest closed set $\Gamma \subset \mathbb{R}^n$ such that $T(\omega) = 0$ whenever $\omega(x) = 0$ for all x in some open set containing Γ.

A smooth function f from $\mathbb{R}^m$ into $\mathbb{R}^n$ induces a mapping $f^{\#}$ of differential forms. The corresponding mapping $f_{\#}$ of currents is defined by

$$f_{\#}(T)(\omega) = T(f^{\#}\omega), \omega \in \mathcal{D}_k.$$

If $N(T)$ is finite, then $f_{\#}(T)$ is defined for f locally Lipschitz by approximating f uniformly on compact sets by smooth mappings. See [FF60, Defn. 3.5].

Types of convergence for sequences of currents. A sequence of currents $T_1, T_2, \cdots$ converges to T weakly if

$$T(\omega) = \lim_{j \to \infty} T_j(\omega), \text{ for all } \omega \in \mathcal{D}_k.$$

Strong convergence of T_j to T means that $M(T_j - T) \to 0$ as $j \to \infty$.

Convergence of T_j to T in the Whitney flat distance is defined in the way indicated below in Part IV.

Gauss-Green Theorem. The Gauss-Green (or divergence) Theorem (2.1) can be rewritten in the form (2.6) with $k = n - 1$. This is explained in [Fl77, Sec. 7.8]. In (2.6) let $T = T_B$, where the smooth boundary B of the set E is oriented by choice of exterior (rather than interior) unit normal vector $\nu(y)$ for $y \in B$. The unit tangent $(n-1)$-vector $\tau(y)$ is adjoint to $\nu(y)$ in the sense that $\nu(y), \tau(y)$ gives positive orientation to $\mathbb{R}^n$. The 1-vector $\zeta(y)$ in (2.1) is adjoint to the $(n-1)$-vector $\omega(y)$, where ω is the differential form in (2.6).

IV. **Flat chains and currents**. Let G denote a group, which will be either the real numbers $\mathbb{R}^1$, the integers Z or Z_p (the integers modulo a prime p).

$\mathcal{P}_k(G)$ is the set of polyhedral chains P of dimension k, with coefficients in G. Flat chains are W-limits of polyhedral chains (Section 4). The Whitney distance between flat chains A_1, A_2 is $W(A_z - A_2)$, where

$$W(A) = \inf_{Q,R}\{M(Q) + M(R) \colon A = Q + \partial R\}$$

and Q, R are flat chains for dimensions $k, k+1$ with coefficient group G.

For $G = \mathbb{R}^1$, flat chains with $N(A) = M(A) + M(\partial A)$ finite correspond to normal currents. If $G = Z$, such flat chains correspond to integral currents.

Appendix B: Abstracts

The following abstracts appeared in the Notices American Math. Society, Volume 6 (1959). They announce preliminary versions of results in [FF60].

Abstract No. 557-73, page 280, W.H. Fleming, *Weak limits of chains.* Let Ω_k denote the space of all k forms with continuous bounded coefficients in euclidean N-space $R_N(k \leq N)$. A Lipschitzian mapping g from a Borel set A in R_k (A not necessarily bounded) defines in the usual way by integration a linear functional c on Ω_k, provided the classical k-area integral of g over A is finite. We call c a σ-Lipschitz k-chain of finite mass.

Theorem. Suppose $c = $ weak $\lim c_n$, where for $n = 1, 2, \ldots, c_n$ is a polyhedral k-chain in R_N with integer coefficients such that the sum of the elementary k-mass of c_n and the elementary $(k-1)$-mass of the boundary bc_n remains bounded,. Then c is a σ-Lipschitz k-chain of finite mass. This result can be restated in terms of L.C. Young's generalized surfaces. (Received March 3, 1959).

Abstract No. 559-126, page 515, Herbert Federer, *A functional isoperimetric inequality.* It is shown that if f is a real-valued Lipschitzian function on Euclidean n-space with compact support and $q = n/(n-1)$, then $(\int |f|^q)^{1/q} \leq n^{-1} \alpha^{-1/n} \cdot \int |\mathrm{grad} f|$, where the integrals are taken with respect to Lebesgue measure over n-space and α is the measure of the unit ball. Through regularization this inequality may be extended to those Schwartz distributions whose partial derivatives are representable by Randon measures; such distributions are shown to be representable by q'th power summable functions. (Received May 18, 1959).

Abstract No. 559-126, Page 515: Herbert Federer, *An approximation theorem concerning currents of finite mass.* For each infinitely differentiable k-form w of Euclidean n-space E_n, associating the k-covector $w(x)$ with x in E_n let $\parallel w \parallel$ be the supremum of $|w(x)|$. For each k-dimensional current T, let $\parallel T \parallel$ be the supremum of $|T(w)|$ where $\parallel w \parallel \leq 1$; call T "integral" if $\parallel T \parallel + \parallel \partial T \parallel < \infty$ and $T, \partial T$ are Lipschitzian images of Borel subsets of E_k,

E_{k-1}. Let $C(e)$ be the usual cubical subdivision of E_n with side length e.

Theorem: There is a constant Q_n such that if $\parallel T \parallel + \parallel \partial T \parallel < \infty$ and $e > 0$, then there exist currents U, V, W for which $T = U + \partial V + W$, U is a chain of $C(e)$, $\parallel U \parallel \le Q_n (\parallel T \parallel + e \parallel \partial T \parallel)$, $\parallel \partial U \parallel \le Q_n \parallel \partial T \parallel$, $\parallel V \parallel \le Q_n e \parallel T \parallel$, $\parallel W \parallel \le Q_n e \parallel \partial T \parallel$ and the supports of $U, \partial U, V, W$ are within e of the supports of $T, \partial T, T, \partial T$; if T is integral, so are U, V, W; if ∂T is polyhedral, so is W.

Applications: (1) A converse of W.H. Fleming's theorem 557-73,. (2) The integral homology groups of any compact Lipschitz neighborhood retract X in E_n are isomorphic with the groups obtained using integral currents with support in X; each homology class contains a T with minimal $\parallel T \parallel$. (3) Each integral k-cycle T bounds an integral current V for which $\parallel V \parallel^{k/(k+1)} \le Q_n \parallel T \parallel$. (Received June 10, 1959).

Appendix C
Generalized Surfaces

This Appendix is concerned with L.C. Young's generalized surfaces, and their role as solutions to geometric problems in the calculus of variations. It includes a concise summary of concepts and results as well as several conjectures. The appendix continues the discussion in Section 8.

Let $\mathcal{E}_k$ denote the space of all continuous functions $F = F(x, \alpha)$, where $x \in \mathbb{R}^n$ and α is a simple k-vector, such that F satisfies the homogeneity condition (8.1). Such a function F is called a geometric variational integrand. Equivalently, F can be regarded as an element of $C(\mathbb{R}^n \times G_k^n)$, where G_k^n is the oriented Grassmanian (Appendix A).

In [Y51] Young defined a generalized surface of dimension k as any nonnegative linear functional L on $\mathcal{E}_k$, such that L has compact support. The restriction of L to the linear subspace $\mathcal{D}_k \subset \mathcal{E}_k$ is a current of dimension k, denoted by $T(L)$. The boundary bL is the $(k-1)$-dimensional current $\partial T(L)$.

Decompositions of generalized surface measures. By the Riesz representation theorem, a generalized surface L corresponds to a nonnegative measure m_L on $\mathbb{R}^n \times G_k^n$ with compact support. The mass $M(L)$ is defined to be

$$M(L) = m_L(\mathbb{R}^n \times G_k^n) = L(F_0),$$

where $F_0(\alpha) = |\alpha|$. Of interest are decompositions of m_L in terms of a measure μ on $\mathbb{R}^n$ and a family of measures λ_x on G_k^n, such that for all $F \in \mathcal{E}_k$

(C.1)
$$L(F) = \int_{\mathbb{R}^n} \Lambda_x(F) d\mu(x)$$

(C.2)
$$\Lambda_x(F) = \int_{G_k^n} F(x, \alpha) d\lambda_x(\alpha).$$

Let μ_L denote the projection of m_L onto $\mathbb{R}^n$, namely $\mu_L(A) = m_L(A \times G_k^n)$ for every Borel set $A \subset \mathbb{R}^n$. One can think of $\mu_L(A)$ as the mass (or k-area) of the part of L in A. The decomposition (C.1)-(C.2) is not unique. However, it becomes unique if we choose $\mu = \mu_L$. In that case, $\lambda_x(G_k^n) = 1$ for μ_L-almost all x. In the terminology of [FY56a, Sec. 2], Λ_x is a microsurface. If $F(x, \alpha) = \omega(x) \cdot \alpha$, then $\Lambda_x(F) = \omega(x) \cdot \bar{\lambda}_x$, where

(C.3)
$$\bar{\lambda}_x = \int_{G_k^n} \alpha d\lambda_x(\alpha).$$

With any current T of finite mass $M(T)$ is associated a measure μ_T on $\mathbb{R}^n$, called the total variation measure. See [FF60, Sec. 2.4], where μ_T is denoted by $\| T \|$. Now let T be rectifiable, with associated rectifiable set K, integer valued multiplicity function $\Theta(x)$ and approximate tangent k-vector $\tau(x) \in G_k^n$ for H^k-almost all $x \in K$. See (3.1). For $x \notin K$, let $\Theta(x) = 0$. Then for any Borel set $E \subset \mathbb{R}^n$

(C.4)
$$\mu_T(E) = \int_E \Theta(x) dH^k(x).$$

Let $L(T)$ be the generalized surface defined by $L(T)(F) = J(T, F)$ for all $F \in \mathcal{E}_k$. See (8.2). Thus,

(C.5)
$$L(T)(F) = \int_{\mathbb{R}^n} F(x, \tau(x)) \Theta(x) dH^k(x).$$

In the decomposition (C.1)-(C.2) of $L(T)$, we take $\mu = \mu_T = \mu_{L(T)}$ and λ_x is the Dirac measure at $\tau(x)$.

Generalized surface solutions to geometric variational problems. Let F_1 be a geometric variational integrand, which satisfies for some constant $k_1 > 0$

(C.6) $$F(x, \alpha) \geq k_1 |\alpha|.$$

Given a boundary B, which is an integral $(n-1)$-dimensional current with $\partial B = 0$, we consider the following problem: minimize $L(T)(F_1) = J(T, F_1)$ subject to the boundary condition $\partial T = B$. It can be shown that the infimum of $J(T, F_1)$ among all such T equals the infimum with the additional restriction spt $T \subset \mathcal{B}_r(0)$ for some fixed r (sufficiently large). Let

(C.7) $$\Gamma_B = \{L(T): T \text{ an integral current, } \partial T = B, \text{ spt } T \subset \mathcal{B}_r(0)\}$$

and $c\ell\Gamma_B$ the weak closure of Γ_B. Since the semi-ellipticity condition (8.3) is not assumed, $L(T)(F_1) = J(T, F_1)$ may not have a minimum on Γ_B. We look instead for a generalized surface solution L in $c\ell\Gamma_B$. Condition (C.6) implies that $M(T) \leq k_1^{-1} J(T, F_1)$. Hence, for the minimization problem we can consider only T such that $M(T) \leq b_1$, for some b_1. Since $L(F_1)$ is a weakly continuous function of L,

(C.8) $$\inf_{\Gamma_B} J(T, F_1) = \inf_{\Gamma_B} L(T)(F_1) = \min_{c\ell\Gamma_B} L(F_1).$$

Denote the right side of (C.8) by ℓ_0, and choose $L \in c\ell\Gamma_B$ such that $L(F_1) = \ell_0$. Let T_j be a sequence of integral currents such that $L_j = L(T_j)$ is in Γ_B and tends weakly to L as $j \to \infty$. By the Closure Theorem, for a subsequence of j, T_j tends to an integral current limit T. Note that $T(L) = T$ and $\partial T = B$.

Lemma 1 $\mu_T \leq \mu_L \leq c_1 \mu_T$ *for some* $c_1 > 1$.

A proof of Lemma 1 is sketched later in this Appendix. Lemma 1 implies that μ_L is absolutely continuous with respect to μ_T, and that the Radon-Nikodym derivative $\psi(x)$ satisfies $1 \leq \psi(x) \leq c_1$. Let $K, \Theta(x), \tau(x)$ be as above, with $\Theta(x) = 0$ for $x \notin K$. We then have:

Theorem 1 *Let $L \in cl\Gamma_B$ with $L(F_1) = \ell_0$ and $L = L(T)$ as above. Let $\mu = \mu_T$. Then the decomposition (C.1)-(C-2) has the form*

$$(C.9) \qquad L(F) = \int_{\mathbb{R}^n} \Lambda_x(F)\Theta(x)dH^k(x)$$

and the measure λ_x satisfies H^k-almost everywhere in K

$$(C.10) \qquad \lambda_x(G_k^n) = \psi(x), \quad \bar{\lambda}_x = \tau(x).$$

The function ψ can also be defined pointwise using covering and differentiation theorems of Besicovitch and A.P. Morse. See [FF60, Sec. 8.7] [Fe69, Secs. 2.8 and 2.9].

For L to give the minimum in (C.8), Λ_x should also be a solution of what we will call a local minimization problem. To explain this idea, we first introduce what are called shape measures.

Shape measures. We first define the shape measure λ_P of an integral polyhedral chain $P = \sigma_1 + \ldots + \sigma_m$, where $\sigma_1 \ldots, \sigma_m$ are oriented polyhedral convex k-cells in $\mathbb{R}^n$ (Section 2(vii)). Assume that $\partial P = \partial \sigma_0$, where σ_0 is another oriented polyhedral convex k-cell. Let $a_i = M(\sigma_i)$ and τ_i the unit tangent k-vector to σ_i for $i = 0, 1, \ldots, m$. Then $M(P) = \sum_{i=1}^m a_i$. Let $\lambda_i = a_i/a_0$ and δ_i the Dirac measure at τ_i. The shape measure of P is

$$(C.11) \qquad \lambda_P = \sum_{i=1}^m \lambda_i \delta_i.$$

The condition $\partial P = \partial \sigma_0$ implies

$$(C.12) \qquad \tau_0 = \sum_{i=1}^m \lambda_i \tau_i = \bar{\lambda}_P,$$

which agrees with (C.3) with λ_x replaced by λ_P. To see that (C.12) holds, let $\alpha_i = a_i\tau_i$. In (2.6), let $T = P$ and $\omega(x) = \omega$ any covector (not depending on x.) Then $\alpha_0 = \alpha_1 + \ldots + \alpha_m$, which is equivalent to (C.12). Since $M(\sigma_0) \leq M(P)$,

$$|\lambda_P| = \sum_{i=1}^{m} \lambda_i \geq 1.$$

We call τ_0 the base tangent k-vector of the shape measure λ_P.

Note that λ_P is invariant under translations and rotations in $\mathbb{R}^n$, and also under changes of scale with x replaced by $cx(c > 0)$.

Definition. A nonnegative measure λ on G_k^n is a shape measure with base tangent k-vector τ if there exists a sequence P_j of integral polyhedral chains with base σ_{0j} such that λ_{P_j} tends weakly to λ and τ_{0j} tends to τ as $j \to \infty$.

For any shape measure λ, we have (as in (C.12)) $\bar{\lambda} = \tau$ with $\bar{\lambda}$ as in (C.3). Let

$$\mathcal{A}(\tau) = \{\text{all shape measures } \lambda : \bar{\lambda} = \tau\}$$

$$\mathcal{A}_1(\tau) = \{\text{all nonnegative measures } \lambda : \bar{\lambda} = \tau\}$$

Evidently, $\mathcal{A}(\tau) \subset \mathcal{A}_1(\tau)$. For $k = 1$ and $k = n - 1$ it can be shown that $\mathcal{A}(\tau) = \mathcal{A}_1(\tau)$. To do this, it suffices to show that $\lambda \in \mathcal{A}(\tau)$ if λ is any "molecular" measure:

$$\lambda = \sum_{i=1}^{m} \lambda_i \delta_i,$$

with δ_i the Dirac measure at some τ_i and with $\bar{\lambda} = \tau$. For $k = 1$, $\tau = v$ is a vector in $\mathbb{R}^n$ with $|v| = 1$. By induction on m, $\lambda = \lambda_P$ for some polygon P with $\partial P = \partial \sigma_0$ and σ_0 is a line segment with direction v. Hence $\lambda \in \mathcal{A}(v)$.

For $k = n - 1$, one can show that $\mathcal{A}(\tau) = \mathcal{A}_1(\tau)$ by an argument involving induction m, similar to one used in [Y48a, p. 101] and [Y51, p. 71] for $k = 2$, $n = 3$. Note that any $(n - 1)$-vector α can be identified with its adjoint vector $v \in \mathbb{R}^n$. See [Fl77, Sec. 7.8], also Appendix A. In particular, every $(n - 1)$-vector α is simple.

It is not clear (at least to me) whether $\mathcal{A}(\tau) = \mathcal{A}_1(\tau)$ when $1 < k < n-1$. Perhaps there are counter examples.

Local minimum problem. For $\lambda \in \mathcal{A}(\tau)$, let Λ denote the corresponding microsurface:

(C.13)
$$\Lambda(F) = \int_{G_k^n} F(\alpha) d\lambda(\alpha), \text{ for all } F \in C(G_k^n).$$

The local minimum problem is: given F_1 find Λ which minimizes $\Lambda(F_1)$ among all $\lambda \in \mathcal{A}(\tau)$. If $F_1(\alpha) \geq k_1|\alpha|$ as in (C.6), then the minimum is attained.

Conjecture. Let $L \in c\ell\Gamma_\beta$ be minimizing in (C.8). Then $\lambda_x \in \mathcal{A}(\tau(x))$ and Λ_x is a solution to the local minimum problem for H^k - almost all $x \in K$, with $F_1 = F_1(x, \alpha)$.

The following Theorem 2 implies that for the local minimum problem, it suffices to consider measures λ which are "molecular" in the sense that spt λ is a finite set. A measure $\lambda \in \mathcal{A}_1(\tau)$ is called irreducible if there does not exist $\lambda' \in \mathcal{A}_1(\tau)$ with $0 \leq \lambda' \leq \lambda$ and $\lambda^s = \lambda - \lambda' \neq 0$.

Given $\tau_1, \ldots, \tau_m$ let $< \tau_1, \ldots, \tau_m >$ denote the set of all convex combinations

$$\theta = \sum_{i=1}^{m} \theta_i \tau_i, \quad \theta_i \geq 0, \quad \sum_{i=1}^{m} \theta_i = 1.$$

Theorem 2 *Let $k = 1$ or $k = n - 1$. Then:*

(a) $\mathcal{A}(\tau) = \mathcal{A}_1(\tau)$.

(b) *There exists a measure λ which solves the local minimum problem, such that* spt λ *is a finite set with m elements $\tau_1, \ldots, \tau_m$ where $m \leq n + 2$.*

(c) *λ is irreducible and $0 \notin \langle \tau_1, \ldots, \tau_m \rangle$.*

A proof of Theorem 2 is sketched below.

Example. (L.C. Young). Let $k = 1, n = 2$ with $e_1 = (1, 0)$, $e_2 = (0, 1)$ the standard basis for $\mathbb{R}^2$. Let σ be the line segment from 0 to e_1, with B corresponding to the initial endpoint

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0 and final endpoint e_1. Let $K = [0, 1] \times \{0\}$, $\Theta(x) = 1$, $\tau(x) = e_1$ for $x \in K$, and

$$
\begin{aligned}
\lambda_x &= \lambda = 2^{\frac{1}{2}}(\delta^+ + \delta^-) \\
v^{\pm} &= 2^{-\frac{1}{2}}(e_1 \pm e_2).
\end{aligned}
$$

where $\delta^{\pm}$ is the Dirac measure a $v^{\pm}$ This generalized curve L is the weak limit as $j \to \infty$ of L_j, where $L_j = L(P_j)$ and P_j is a "sawtooth shaped" polygon with endpoints 0 and e_1 and with j teeth. Each tooth is an isocoles right triangle with hypotenuse on K. Let

$$
F_1(x, v) = g(x \cdot e_2)h(v),
$$

where $g(0) = 0$, $g(u) > 0$ for $u \neq 0$, $h(v^{\pm}) = 0$, and $h(v) > 0$ for $v \neq v^+$ or v^- ($|v| = 1$). Then $L(F_1) = 0$ and hence L is minimizing.

If F_1 is a convex function of α, then in Theorem 2 we can take $m = 1$. Since F_1 has the homogeneity property (8.1), (C.12) implies

$$
a_0 F_1(\tau_0) = F_1(\alpha_0) \leq \sum_{i=1}^{m} F(\alpha_i) = \sum_{i=1}^{m} a_i(F(\tau_i)).
$$

for any polyhedral chain with $\partial P = \partial \sigma_0$. The measure $\lambda = a_0 \delta_0$ minimizes $\Lambda(F_1)$, where δ_0 is the Dirac measure at τ_0.

Sketch of proof of Lemma 1.

(a) For the left hand inequality, it suffices to show that, for every nonnegative continuous function $g \in C(\mathbb{R}^n)$, $\langle g, \mu_T \rangle \leq \langle g, \mu_L \rangle$, where $\langle g, \mu \rangle$ is the integral of g with respect to μ. From [Fe60, Sec. 2], or from (3.1),

$$
\langle g, \mu_T \rangle = \sup_{\omega} \{ T(\omega) : |\omega(x)| \leq g(x) \ \forall x \in \mathbb{R}^n \}.
$$

Since $T = T(L)$,

$$
T(\omega) = L(F_\omega) \leq \langle g, \mu_L \rangle,
$$

where $F_\omega(x, \alpha) = \omega(x) \cdot \alpha$ and

$$|F_\omega(x, \alpha)| \leq |\omega(x) \cdot \alpha| \leq g(x)|\alpha|.$$

(b) To prove the right hand inequality, it suffices to show that $\mu_L(I) \leq c_1 \mu_T(I)$ for every n-dimensional cube I such that $\mu_L(frI) = 0$ with frI the topological boundary of I. It then follows that $\mu_L(A) \leq c_1 \mu_T(A)$ for every open set $A \subset \mathbb{R}^n$, and hence for every Borel set.

Weak convergence of T_j to T and $\partial T_j = \partial T = B$ imply convergence in the Whitney metric. Hence $T_j - T = \partial R_j$ where R_j is an integral current of dimension $k + 1$ and $M(R_j)$ tends to 0 as $j \to \infty$. For $\rho > 0$, let I_ρ denote the ρ-neighborhood of I. Since I is convex, it is a set of positive reach in the sense of [Fe59]. The theory of slicing of integral currents can be applied [Fe65] [Fe69, Sec. 4.3]. The part $T \cap E$ of a current T of finite mass is defined in [Fe60, Sec. 2]. If T is rectifiable, then $T \cap E$ has the representation (3.1) with K replaced by $K \cap E$, and $M(T \cap E) = \mu_T(E)$.

Let $R_j(\rho)$, $T_j(\rho)$, $T(\rho)$ denote the parts of R_j, T_j, T in I_ρ respectively. Fix $\delta > 0$ and choose $\rho_j \in (0, \delta)$ such that

$$\partial R_j(\rho_j) = T_j(\rho_j) - T(\rho_j) - Q_j,$$

where $M(Q_j)$ tends to 0 as $j \to \infty$. Here, an inequality due to Eilenberg [FF60, Cor. 3.10] is also used. Let

$$\begin{aligned} \tilde{T}_j &= T_j - T_j(\rho_j) + Q_j + T(\rho_j) \\ &= T_j - \partial R_j(\rho_j) \end{aligned}$$

Since $\partial(\partial R_j(\rho_j)) = 0$, $\partial \tilde{T}_j = \partial T_j = B$. Since T_j is a minimizing sequence in Γ_B,

$$J(T_j, F_1) = J(T_j(\rho_j), F_1) + J(T_j - T_j(\rho_j), F_1)$$

which tends to ℓ_0 as $j \to \infty$. Moreover,

$$\ell_0 \leq J(\tilde{T}_j, F_1) \ \leq \ J(T(\rho_j), F_1)$$
$$+ \ J(T_j - T_j(\rho_j), F_1) + \epsilon_j$$

where $\epsilon_j \to 0$ as $j \to \infty$. We subtract to obtain

(C.14) $$J(T_j(\rho_j), F_1) \leq J(T(\rho_j), F_1) + \epsilon_j.$$

Since F_1 satisfies (C.6)

$$k_1 M(T_j(\rho_j)) \leq \| F_1 \| M(T(\rho_j)) + \epsilon_j$$

with $\| F_1 \|$ the sup norm in $C(B_r \times G_k^n)$, where spt $T_j \subset B_r = B_r(0)$. Hence

$$\mu_{T_j}(I) \leq c_1 \mu_T(I_\delta) + \epsilon_j$$

with $c_1 = k_1^{-1} \| F_1 \|$. Since $L_j = L(T_j)$, $L_j \to L$ as $j \to \infty$ and $\mu_L(frI) = 0$,

$$\mu_L(I) \leq c_1 \mu_T(I_\rho).$$

Let $\delta \to 0$ to obtain $\mu_L(I) \leq c_1 \mu_T(I)$. $\qquad\qquad \square$

Proof of Theorem 2 (sketch). Part (a) was discussed before the statement of Theorem 2. Part (b) can be proved by the following argument, borrowed from the theory of relaxed optimal controls [Bk74, Sec. 4.3] [BM13, Sec. 3.2]. The minimum of $\Lambda(F_1)$ on $\mathcal{A}_1(\tau)$ equals the infimum of $\Lambda(F_1)$ over molecular measures $\lambda \in \mathcal{A}_1(\tau)$. For any such λ,

(C.15) $$c\lambda = \sum_{i=1}^{m} \theta_i \delta_i$$

with δ_i the Dirac measure at τ_i and

$$c\lambda_i = \theta_i, \quad c = |\lambda|_1^{-1} \quad |\lambda|_1 = \sum_{i=1}^{m} \lambda_i.$$

Moreover,

$$c\tau = \sum_{i=1}^{m} \theta_i \tau_i$$

$$c\Lambda(F_1) = \sum_{i=1}^{m} \theta_i F_1(\tau_i).$$

Hence

$$(c\tau, c\Lambda(F_1)) \in \langle \xi_1, \ldots, \xi_m \rangle,$$

where $\xi_i = (\tau_i, F_1(\tau_i))$. Recall that for $k = n - 1$, the k-vector τ_i can be identified with its adjoint vector in $\mathbb{R}^n$. Hence, ξ_i is in a Euclidean space of dimension $n + 1$. By a theorem of Carathéodory [Fl77, p. 24], there exists a subset J of $\{\xi_1, \ldots, \xi_m\}$ with $m \leq n + 2$ such that

$$(c\tau, c\Lambda(F_1)) \in \langle J \rangle.$$

The set of all molecular measures $\lambda \in \mathcal{A}_1(\tau)$ with $m \leq n+2$ and with $|\lambda|_1 \leq k^{-1} \| F_1 \|$ is weakly compact. Hence, the infimum of $\Lambda(F_1)$ among all molecular measures in $\mathcal{A}_1(\tau)$ is the same as the minimum of $\Lambda(F_1)$ among such measures satisfying $m \leq n + 2$.

To prove part (c), the irreducibility of λ is immediate from the definition and condition (C.6). Suppose that $0 \in \langle \tau_1, \ldots, \tau_m \rangle$. Then

$$0 = \sum_{i=1}^{m} \zeta_i \tau_i$$

with $\zeta_i \geq 0$ and $\zeta_1 + \ldots + \zeta_m = 1$. Let

$$\lambda^s = \beta \sum_{i=1}^{m} \zeta_i \delta_i, \beta > 0.$$

For β small enough, $\lambda = \lambda^s + \lambda'$ with $\lambda^s \neq 0$ and $\lambda' \in \mathcal{A}_1(\tau)$. This contradicts the irreducibility of λ. $\qquad\square$

Singular generalized surfaces. We conclude this Appendix with some concepts and results from early work on generalized surfaces. The notion of singular generalized

surface had an important role in [Y51] and also [FY56a,b]; L is called a singular generalized surface if the current $T(L) = 0$. This implies that the boundary $bL = 0$ if L is singular. It can be shown that L is singular if and only if, in (C.3), $\bar{\lambda}_x = 0$ for μ-almost all x.

Example. Let T be a rectifiable current, with associated rectifiable set K, $\Theta(x) = 1$ and approximate tangent k-vectors $\tau(x)$. Then $-T$ has the opposite orientation $-\tau(x)$, and $T + (-T) = 0$. The generalized surface $L = L(T) + L(-T)$ is singular. In (C.1) -(C.2) we have

$$\mu_L(E) = 2\mu_T(E) = 2H^k(K \cap E)$$

for any Borel set E, and

$$\lambda_x = \frac{1}{2} \left(\delta_x^+ + \delta_x^- \right)$$

where $\delta_x^\pm$ is the Dirac measure at $\pm\tau(x)$.

An important class of singular generalized surfaces is obtained as follows. Let U_j, R_j be a sequence of integral currents such that $U_j = \partial R_j$, $M(U_j)$ tends to a positive limit a and $M(R_j)$ tends to 0 as $j \to \infty$. For a subsequence of j, the generalized surface $L(U_j)$ tends weakly to a limit L^s as $j \to \infty$ with $M(L^s) = a$. L^s is a singular generalized surface.

For $k = 2$, $n = 3$ one can think (for instance) of R_j as a thin, tentacle-like body. Another possibility is that R_j is composed of thin platelets, or of many small bubbles lying close to some 2-dimensional surface.

Singular parts. Let L be a generalized surface with boundary $bL = B \neq 0$. A generalized surface $L^s \neq 0$ is called a singular part of L if $L^s \leq L$. Then

(C.16) $$L = L^s + L'$$

where L' is also a generalized surface and $bL = bL'$. It is not difficult to show that either L has no singular part or else there is a singular part of largest mass $M(L^s)$.

The following questions are interesting. Suppose that $L \in cl\Gamma_B$, where B is an integral current. As in a previous discussion, L is the weak limit of $L_j = L(T_j)$, where T_j is an integral current tending to T as $j \to \infty$ and $\partial T_j = \partial T = B$. Let L^s, L' be as in (C.16) with $M(L^s)$ maximum among singular parts of L.

Question 1. Does L' have a representation of the form (C.9)-(C.10)with ψ a μ_T-integrable function on $\mathbb{R}^n$?

Question 2. If the answer to Question 1 is "yes," is $L' \in cl\Gamma_B$?

A proof that Question 2 has a positive answer would require some kind of "chattering lemma" in the terminology of relaxed optimal control theory [Y69] [Bk74] [BM13].

"Abstract" minimization problems. A different approach to generalized solutions of geometric variational problems was considered in [FY54] [FY56 a,b]. In this approach, all generalized surfaces L with given boundary B are considered. Let

(C.17) $$\Delta_B = \{\text{all } L : bL = B\}.$$

The "abstract" minimization problem is to find a generalized surface which minimizes $L(F_1)$ among all generalized surfaces in Δ_B.

If B is an integral current, let $\langle \Gamma_B \rangle$ be the weak convex closure of Γ_B. Since $L(F_1)$ is a linear, weakly continuous function of L, the minimum in (C.8) is also the minimum of $L(F_1)$ on $\langle \Gamma_B \rangle$. It is no less than the minimum of $L(F_1)$ on Δ_B, since $\langle \Gamma_B \rangle \subset \Delta_B$.

Conjecture. If $k = n - 1$, then $\langle \Gamma_B \rangle = \Delta_B$.

In [FY56b] this conjecture was proved to be correct if $n = 3$ and B is an "elementary boundary," consisting of a finite number of closed oriented curves $C_1, \ldots, C_m$. However,

for $k = 2$, $n = 4$ an example was given in [FY56b, Thm. 1] in which $\langle \Gamma_B \rangle$ is a proper subset of Δ_B. This example is based on a Klein bottle, embedded as a polyhedral 2-manifold in $\mathbb{R}^4$.

When $\langle \Gamma_B \rangle = \Delta_B$, the minimum of $L(F_1)$ on Δ_B is attained at an extreme point L of Δ_B, with $L \in c\ell\Gamma_B$. Further information about such a minimizing extreme point L is contained in [FY56b, Thm. 7]. When restated in the language of currents, it says that $T(L)$ is an integral current. Included in this statement is a precursor of the Closure Theorem in [FF60], for $k = 2$, $n = 3$.

By considering this "abstract" version of the minimum problem, functional analysis methods are available to derive necessary and sufficient conditions that L should minimize $L(F_1)$ on Δ_B. See, in particular, the use of the Hahn-Banach Theorem in [FY54, Sec. 5].

"Nonoriented" versions of geometric variational problems. If F_1 satisfies the symmetry condition

(C.18) $$F(x, \alpha) = F(x, -\alpha),$$

then it is of interest to consider a "nonoriented" version of the problem of minimizing $L(F_1)$ among all $L \in c\ell\Gamma_B$. This version can be formulated in terms of varifolds. Let

$$\tilde{\mathcal{E}}_k = \{F \in \mathcal{E}_k \colon F \text{ satisfies (C.18)}\}$$

The restriction of a generalized surface L to $\tilde{\mathcal{E}}_k$ defines a varifold V, in Almgren's terminology. Let $\tilde{G}_k^n$ denote the "unoriented" Grassmannian, in which simple k-vectors α and $-\alpha$ are identified. The Riesz representation theorem gives a corresponding measure m_V on $\mathbb{R}^n \times \tilde{G}_k^n$, with compact support. The measure m_V can be decomposed in a way similar to (C.1)-(C.2). However functions F in $\mathcal{D}_k$, of the form $F(x, \alpha) = \omega(x) \cdot \alpha$, do not belong to $\tilde{\mathcal{E}}_k$, and (C.3) does not have a varifold counterpart.

We offer only a few comments and suggestions about how the method outlined above could be adapted to the nonoriented version. It is natural to replace integral currents by rectifiable flat chains A over the group Z_2 with $N(A) = M(A) + M(\partial A)$ finite (Section 5). Associated with A should be a rectifiable set K, unoriented approximate tangent k-vectors $\tau(x)$ and a measure μ_A such that $\mu_A(E) = H^k(K \cap E)$ for every Borel set $E \subset \mathbb{R}^n$. In analogy with (C.5), define the varifold $V(A)$ by

(C.19)
$$V(A)(F) = J(A, F)$$

$$= \int_{\mathbb{R}^n} F(x, \tau(x))\Theta(x)dH^k(x),$$

for every $F \in \tilde{\mathcal{E}}_k$, where $\Theta(x) = 1$ for $x \in K$, $\Theta(x) = 0$ for $x \notin K$.

As in (C.7), given a rectifiable flat chain B with $\partial B = 0$, let

(C.20) $\qquad \mathcal{F}_B = \{V(A) \colon A \text{ a flat chain over } Z_2, \partial A = B,\ \text{spt } A \subset \mathcal{B}_r(0)\}.$

As in (C.8), the infimum of $V(A)(F_1)$ over $\mathcal{F}_B$ is attained at some varifold $V \in cl\mathcal{F}_B$, where as before cl denotes weak closure. The counterpart of Lemma 1 should be true, by an argument similar to that sketched above. Note the use of slicing and the Eilenberg inequality in [Fl66, Theorems 5.6 and 5.7]. This provides a representation for the minimizing varifold V similar to (C.9), with $\Theta(x) = 1$ for $x \in K$ and $\Theta(x) = 0$ otherwise.

Shape measures can be defined in a way similar to the oriented case. However, the set $\tilde{\mathcal{A}}(\tau)$ of shape measures corresponding to base tangent k-vector τ must be defined differently, since the condition $\bar{\lambda} = \tau$ cannot be used in the nonoriented case. A suitable analogue of Theorem 2 also needs to be formulated.

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