De Giorgi and Geometric Measure Theory

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References

1 Introduction

I wish to thank the organizers for inviting me to speak at this event in memory of De Giorgi. With great regret, I am unable to attend due to an illness during the summer. Instead, I have prepared this historical article, which my friend Tullio Zolezzi has kindly agreed to present at the conference. One of my main goals is to outline some of De Giorgi's seminal contributions to geometric measure theory during the 1950s and 1960s, specifically his theory of sets of finite perimeter and contributions to the Plateau (least area) problem in higher dimensions. These contributions have had a profound effect on the field. The great originality and depth of his work remain absolutely amazing until this day. The papers of De Giorgi which I cite in the list of references were written in Italian (except for the joint paper [3] with Bombieri and Giusti). English translations of them are available in the selected papers volume [9].

2 Geometric measure theory.

Geometric measure theory (GMT) is concerned with a theory of k-dimensional measure and integration in euclidean $\mathbb{R}^n$, for any nonnegative integer $k < n$. The name GMT was probably first used by my colleague at Brown University Herbert Federer. His 1969 book [11] with the same title remains a classic in the field. A more detailed historical account of early developments in GMT than is presented here appears in my recent article [17]. This article has not yet been published. However, it is easily accessed by Googling the title "Geometric measure theory at Brown in the 1960s." F. Morgan's Beginner's Guide [18] is an excellent introduction to GMT, including the topics in Sections 4 to 7 of the present survey.

Among the important aspects of GMT are the following:

a) Theory of k-dimensional measures in $\mathbb{R}^n$, for dimensions $k < n$. Among several possible definitions, the one due to Hausdorff is most widely used. Let $H^k(K)$ denote the Hausdorff k-measure of a set $K \subset \mathbb{R}^n$. For $k = n, H^n$ is Lebesgue measure on $\mathbb{R}^n$. If $k < n$, then among the sets K with $0 < H^k(K) < \infty$, there is the important class of sets called k-rectifiable. Roughly speaking, K is k-rectifiable if K differs in arbitrarily small H^k-measure from a finite union of pieces of C^1-submanifolds of $\mathbb{R}^n$.

b) Theories of k-dimensional integration without the usual smoothness assumptions. Included are versions of the classical theorems of Gauss-Green and Stokes. See Sections 3 and 4.

c) Applications to geometric problems in the calculus of variations, and to the higher dimensional Plateau problem in particular. See Sections 5-7.

3 Sets of finite perimeter.

Let us begin with the case $k = n-1$ (co-dimension 1) and De Giorgi's theory of sets of finite perimeter in the fundamental papers [4] [5]. These papers were influenced by related work of R. Caccoppoli.

The classical Gauss-Green (divergence) Theorem says the following. Let $E \subset \mathbb{R}^n$ be a bounded open set with smooth boundary B, and let ζ be any smooth $\mathbb{R}^n$-valued function. Then

$$\int_E \mathrm{div}\zeta(x) dH^n(x) = \int_B \zeta(y) \cdot \nu(y) dH^{n-1}(y), \tag{3.1}$$

where $\nu(y)$ is the exterior unit normal at y. Choosing the exterior (rather than the interior) normal vector amounts to choosing an orientation for B. De Giorgi addressed the question of how to make sense of the right side of (3.1) without any smoothness assumptions on the topological boundary B of the set E. His program was as follows:

(a) Require only that E is a "set of finite perimeter" $P(E)$.

(b) In (3.1), replace B by a set $B_r \subset B$ called the "reduced boundary."

(c) Show that B_r is a k-rectifiable set and that there is an "approximate normal" unit vector $\nu(y)$ at each $y \in B_r$.

As an example, think of a "Swiss cheese" $E \subset \mathbb{R}^2$ with an infinite number of holes. The holes are bounded by curves $C_1, C_2, \cdots$. If the sum of the lengths of these curves is finite, then E has finite perimeter and $B_r = C_1 \cup C_2 \cup \cdots \cup C$ where C is the outer boundary of the cheese.

To define sets of finite perimeter, let $E \subset \mathbb{R}^n$ be a bounded, Lebesgue measurable set. Let 1_E denote its indicator function. In the style of Schwartz distribution theory, think of ζ in (3.1) as any smooth $\mathbb{R}^n$-valued test function.

Let $\Phi = -\text{grad}\, 1_E$ in the Schwartz distribution sense. De Giorgi called E a set of finite perimeter if Φ is a measure. This is equivalent to saying that 1_E is a bounded variation (BV) function of n variables.

An important part of his theory involves approximations of E by sequences E_j of sets with piecewise smooth boundaries B_j, and in particular by polygonal domains with piecewise flat boundaries. The convergence of E_j to E as $j \to \infty$ is in H^n-measure and the perimeter of E_j is $P(E_j) = H^{n-1}(B_j)$. If $P(E_j)$ is bounded, then the corresponding measures Φ_j converge weakly to Φ as $j \to \infty$. Another characterization of the perimeter $P(E)$ is as the lower limit of $P(E_j)$ as $j \to \infty$, among all such sequences E_j of polygonal domains.

In [5] De Giorgi defined in an elegant way the reduced boundary B_r and approximate normal vectors $\nu(y)$. He then showed that (3.1) remains correct, with B replaced by B_r. See [5, Theorem III]. The definitions of B_r and $\nu(y)$ are natural intuitively, but the proof of this version of (3.1) is subtile. For $y \in B_r$, let $I(y, \rho)$ denote the spherical ball in $\mathbb{R}^n$ with center y and radius ρ. Then the ratio of $H^n(E \cap I(y, \rho))$ to $H^n((\mathbb{R}^n \backslash E) \cap I(y, \rho))$ tends to 1 as $\rho \to 0$. Let μ denote the total variation measure of the vector-valued measure Φ. At any point $y \in B_r$, the approximate normal vector $\nu(y)$ is the pointwise derivative of Φ with respect to μ and $\mu(K) = H^{n-1}(K)$ for any Borel set $K \subset B_r$.

These results pertain to measures and integrals in the highest dimensions n and $n-1$. However, a few years later they significantly influenced the theory of integral currents of any dimension $k < n$ (Section 4). For example, the "slicing formula" which De Giorgi used to show that his definition of set of finite perimeter was equivalent to another definition by Caccoppoli anticipated the "coarea" formula in GMT, of which is a particular case.

4 Rectifiable and integral currents.

In 1960 these concepts were introduced by Federer and myself [13]. Only a very concise summary of these ideas is given in this section. A quick summary of basic definitions and notations is included in the Appendix. In 1955, de Rham [22] introduced a definition of currents, in a way quite similar to the definition of Schwartz distributions. A current T of dimension k is a linear functional on a space $\mathcal{D}_k$ of smooth differential forms ω of degree k. We consider only currents of finite mass $M(T)$ and compact support *spt* T.

The boundary of a current T is (by definition) the current ∂T of dimension $k-1$ such that

$$\partial T(\omega) = T(d\omega) \text{ for all } \omega \in \mathcal{D}_{k-1} \tag{4.1}$$

where $d\omega$ denotes the exterior differential of ω.

Example. Let $\mathcal{M}$ be a smooth (class C^1) submanifold of $\mathbb{R}^n$, oriented by a continuously vaying unit tangent k-vector $\tau(y)$ for $y \in \mathcal{M}$. We call a bounded subset $S \subset \mathcal{M}$ a smooth, oriented k-cell if S has piecewise smooth boundary C. To S corresponds the k-dimensional current T_S such that

$$T_S(\omega) = \int_S \omega = \int_S \omega(y) \cdot \tau(y) dH^k(y), \quad \omega \in \mathcal{D}_k. \tag{4.2}$$

Moreover, $M(T_S) = H^k(S)$. If C is given an orientation consistent with that of S, then a k-dimensional version of Stokes formula implies that $T_C = \partial T_S$.

The class of k-rectifiable T is characterized by the following property: for every $\epsilon > 0$ there exists a k dimensional current T_ϵ which is a finite sum of oriented k-cells such that $M(T - T_\epsilon) < \epsilon$.

Any k-rectifiable current has a representation similar to (4.2), involving a k-rectifiable set K and a positive integer valued multiplicity function $\Theta(x)$. See Appendix (A.2). The mass $M(T)$ of a k-rectifiable current is also called the k-area of T.

If T_j is k-rectifiable for $j = 1, 2, \cdots$ and $M(T_j - T)$ tends to 0 as $j \to \infty$, then T is also k-rectifiable. This is called strong convergence of T_j to T. For the study of existence of solutions to geometric calculus of variations problems, some kind of corresponding property involving weakly convergent sequences is needed. See Section 5.

A current T is called integral if both T and its boundary ∂T are rectifiable currents. Let $N(T) = M(T) + M(\partial T)$. The desired property is contained in the following result, called the Closure Theorem [13, Theorem 8.12]. If T_j is a sequence of integral currents such that $N(T_j)$ is bounded, spt T_j is contained in a fixed compact set and T_j tends T weakly as $j \to \infty$, then T is also an integral current.

To connect De Giorgi's sets of finite perimenter (Section 3) with integral currents, let $E \subset \mathbb{R}^n$ have finite perimeter $P(E)$. Denote by U the corresponding current of dimension n, defined as follows. For any smooth differential form η of degree n, $\eta = f dx, \wedge \cdots \wedge dx_n$,

$$U(\eta) = \int_E f(x) dH^n(x).$$

Let $T = \partial U$. Note that $\partial T = \partial(\partial U) = 0$. Both U and T are integral currents, with $M(U) = H^n(E)$ and $M(T) = P(E)$. Let E_j be any sequence of polygonal domains converging in H^n-measure to E with $P(E_j)$ bounded, as in Section 3. Let U_j and $T_j = \partial U_j$ denote the corresponding currents. Then U_j tends to U strongly and T_j tends T weakly as $j \to \infty$.

5 Higher dimensional Plateau problem.

The classical Plateau problem for two dimensional surfaces in $\mathbb{R}^3$ is as follows. Find a surface S^* of least area among all surfaces S with given boundary C. This is a geometric problem in the calculus of variations, which has been studied extensively. During the 1930s, J. Douglas and T. Rado independently gave solutions to a version of the Plateau problem. Their results were widely acclaimed. Douglas received a Fields Medal in 1936 for his work.

Douglas and Rado considered surfaces defined by "parametric representations," which were mappings f from a circular disk $D \subset \mathbb{R}^2$ into $\mathbb{R}^3$, such that the restriction of f to the boundary of D is a parametric representation of the boundary curve C. The Douglas-Rado result was later extended by Douglas and Courant to give a solution to the Plateau problem bounded by a finite number of curves and of prescribed Euler characteristic. However, all of these results depended on conformal parameterizations of surfaces, and are intrinsically two dimensional. They also depend on prescribing in advance the topological type of the surfaces considered.

For these reasons, it became clear by the late 1950s that entirely new formulations and methods were needed to study higher dimensional versions of the Plateau problem, for surfaces of any dimension $k < n$. The first major step was the 1960 paper [19] by E.R. (Peter) Reifenberg. In his formulation, a "surface" is a closed set $S \subset \mathbb{R}^n$ with $H^k(S) < \infty$. A closed set $B \subset S$ is called the boundary if an appropriate relationship in terms of Čech homology groups holds. Reifenberg proved that, given the boundary B, a set S^* which minimizes $H^k(S)$ exists. Moreover, S^* is topologically a k-dimensional spherical ball in a neighborhood of H^* - almost every nonboundary point $x \in S^*$. There were no earlier results to guide Reifenberg in this effort. His methods had to be invented "from scratch" and required amazing ingenuity. Reifenerg also published in 1964 a paper on his important "epiperimetric inequality" [20] and a sequel [21] in which he used the epiperimetric inequality to obtain a regularity result for the higher dimensional Plateau problem.

I first met Peter Reifenberg at the Genova workshop on GMT in August 1962. He visited Brown University in the summer of 1963, and we corresponded regularly during the year which followed. Our all-to-brief friendship ended with his death at the age of 36 during the summer of 1964 in a mountaineering accident. This event was a great loss to mathematics and to GMT in particular. Section 11 of my historical article about GMT [17] is entitled "Remembrances of Leaders in GMT". I included Peter Reifenberg in this list, along with Almgrern, De Giorgi, Federer and L.C. Young.

Oriented Plateau problem. Another formulation (often called the oriented Plateau problem) is in terms of integral currents. In the formulation, a rectifiable current B of dimension $k-1$ with $\partial B = 0$ is given. The problem is to find an integral current T^* which minimizes the mass (or k-area) $M(T)$ among all integral currents T with $\partial T = B$. Since $M(T)$ is weakkly lower semicontinuous, the existence of a minimizing T^* is an immediate consequence of the Closure Theorem mentioned in Section 4. There remained the difficult task of describing regularity properties of T^*. This is the topic of Section 7.

Non-oriented versions. In Reifenberg's paper [19], orientations play no role. Another formulation of a higher dimensional Plateau problem is in terms of Whitney's flat chains with coefficients in a finite group G [15] [23]. When $G = Z_2$ this is called a "nonoriented" Plateau problem. Existence of a mass minimizing flat chain with given boundary follows from arguments similar to those for the oriented Plateau problem. Weak convergence for sequences of integral currents is replaced by convergence in the Whitney flat metric. Yet another formulation which disregards orientations is in terms of Almgren's varifolds.

6 Regularity results.

For the oriented Plateau problem, as formulated in Section 5, there remained the notoriously difficult "regularity problem." Let T be a k-area minimizing integral current. The regularity problem is to prove that spt T-spt ∂T is locally a smooth manifold of dimension k, except at points of a singular set of lower Hausdorff dimension. There are also results about regularity at points in spt ∂T, which we will not discuss.

A result of Federer [10] about mass minimality of complex subvarieties provided a rich class of examples in which the singular set can have Hausdorff dimension $k-2$. For instance, let (z_1, z_2) denote a point in the space C^2 of complex dimension 2, If C^2 is identified with $\mathbb{R}^4$, the equation $z_1 z_2 = 0$ defines two mutually orthogonal planes which intersect at the origin. Their intersection with the unit ball in $\mathbb{R}^4$ with center 0 is area minimizing with 0 as a singular point.

Early results. In the early 1960s, De Giorgi and Reifenberg proved what are called "almost everywhere regularity" results, in which the singular set was shown to have H^k-measure 0. De Giorgi's result [7, Theorem VI] is stated in terms of sets of finite perimeter. Given an open set A and set E of finite perimeter, E is said to have minimal boundary in A if $P(E) \leq P(\tilde{E})$ for any $\tilde{E}$ such that $\tilde{E} - A = E - A$. Let B_r be the reduced boundary of E. De Giorgi's regularity result says that $B_r \cap A$ is locally a smooth hypersurface. It provided the first big regularity result for the higher dimensional Plateau problem. The proof is an amazing "tour de force."

Starting with a locally area minimizing surface, which is not even known to be locally the graph of a function, De Giorgi managed to prove that the surface is smooth near any point at which it is measure theoretically close to some approximate tangent plane. Traditional PDE methods become helpful only at a very late stage of De Giorgi's proof.

In [19], Reifenberg proved the following. Let S^* be a set shown to minimize $H^k(S)$ in his formulation, among all S with the same boundary. Then S^* is topologically a K-dimensional spherical ball in a neighborhood of H^k- almost every non-boundary point $x \in S^*$. In 1964, he proved local smoothness near x using his epiperimetric inequality already mentioned in Section 5.

Almgren's work on regularity. Beginning in the mid 1960s, Almgren was a leading contributor of results on regularity. His results were formulated in terms of what he called varifolds. Almgren's paper [2] represented a major advance. In it he obtained almost everywhere regularity results not only for the Plateau problem in all dimensions, but also for a much broader class of geometric variational problems in which the integrand satisfies a suitable ellipticity condition.

The regularity problem (in its full generality) proved to be incredibly challenging. Almgren wrestled with it for several years. After persistent, courageous efforts he produced a massive manuscript often called his "Big Regularity Paper." It has appeared in book form [1]. The task of reading and trying to assimilate all of the details of this work is a daunting one. In a series of recent papers, De Lellis and Spadaro provide substantially shorter alternatives to many of the arguments in [1].

7 Regularity results for $k = n - 1$ and Bernstein's Theorem.

It seemed at first that $(n-1)$-dimensional area minimizing integral currents might have no singular points. This was proved in [14] for $n = 3$. Closely related to the regularity question in dimension $n-1$ is the question of whether the only cones in $\mathbb{R}^n$ which locally minimize $(n-1)$-area are hyperplanes, which was shown in [14] to be true if $n = 3$. Using this connection, De Giorgi, Almgren and Simons then showed that there are no singular points for $n \leq 7$. However, Bombieri, De Giorgi and Giusti [3] gave an example of a cone in $\mathbb{R}^8$ which provides a seven dimensional area minimizing integral current with a singularity at the vertex. This example (due to Simons) is as follows. Write $\mathbb{R}^8 = \mathbb{R}^4 \times \mathbb{R}^4$ and $x = (x', x'')$ with $x', x'' \in \mathbb{R}^4$. The cone satisfies $|x'| = |x''|$. Its intersection with any ball $\mathcal{B}_r(0)$ in $\mathbb{R}^8$ with center 0 defines a 7 dimensional integral current, which is shown in [3] to be area minimizing. The vertex 0 is a singular point. Federer [12] showed that this example is generic in the following sense: if $k = n - 1$, then the singular set for the oriented Plateau problem can have Hausdorff dimension at most $n - 8$.

Bernstein's Theorem. The classical Bernstein Theorem is as follows. Let f be a smooth, real valued function which satisfies the minimal surface PDE everywhere in $\mathbb{R}^2$. Then f is an affine function (equivalently, the graph of f is a plane.) A GMT proof of this well known result was given in [14]. It relied on the result that locally area minimizing cones in $\mathbb{R}^3$ are planes, mentioned above, and also on the following monotonicity property [13, pp. 518-9]: Let T be an integral current which locally minimizes k-area, and let T_r denote the part of T in the ball $\mathcal{B}_r(x_0)$ with center x_0 and radius r. Then $r^{-k}M(T_r)$ is a nondecreasing function of r. For the classical Bernstein Theorem with

$k = n - 1 = 2$, let f be as above with $f(0) = 0$. Take $x_0 = 0$ and T the oriented graph of f in $\mathbb{R}^3$. Then it is shown in [14] that $r^{-2}M(T_r) = \pi$ is constant and spt T is a plane in $\mathbb{R}^3$.

An interesting question was whether the corresponding result about smooth solutions f to the minimal surface PDE in all of $\mathbb{R}^m$ must be true. This was proved by De Giorgi [8] for $m = 3$. In his proof, he showed that falsity of the Bernstein Theorem for functions on $\mathbb{R}^m$ would imply the existence of non-planar locally area minimizing cones in $\mathbb{R}^m$ of dimension $m - 1$. For $m = 3$, this allowed De Giorgi in [8] to use the same result as in [14] about 2 dimensional locally area minimizing cones in $\mathbb{R}^3$. Making use of similar ideas, Almgren and Simons then proved the Bernstein Theorem for $4 \leq m \leq 7$. However, the Bernstein Theorem is not correct for $m \geq 8$, as was shown in [3].

I was visiting Stanford in the Spring of 1969 when the startling news about this negative result arrived there. D. Gilbarg (an authority on nonlinear PDEs) was perplexed. It was unheard of that a result about PDEs should be true in 7 or fewer variables, but not in more variables. However, Gilbarg wisely observed that the Bernstein Theorem is really a geometric result, not a result about PDEs.

8 Remembrances of De Giorgi.

I first heard about De Giorgi in 1956 or 1957 when the French mathematician C. Pauc urged me to read his new papers [4] [5] on sets of finite perimeter. Pauc was visiting Purdue University, where I was then a faculty member. He and I shared an interest in BV functions on $\mathbb{R}^n$. Upon reading De Giorgi's papers, their importance became clear. Some of his techniques were immediately useful in my own work.

Soon afterward, De Giorgi learned about new work by Reifenberg, Federer and myself on the higher dimensional Plateau problem. In 1961, I received from him two new papers [6] [7]. As mentioned in Section 6, the second of these remarkable papers contained the first regularity results known for the higher dimensional Plateau problem. These seminal papers were published in a SNS Seminario di Matematica series, which was not (I think) widely available.

Genova workshop. In August 1962 J.P. Cecconi hosted a workshop at the

Università di Genova, at which I first met both De Giorgi and Reifenberg. This workshop was easily the most fruitful among many such events which I have attended during my long career. It had an important role in stimulating further work in GMT.As Reifenberg said, the workshop was conducted in a kind of “lingua mista.” Despite some language difficulties, many interesting ideas were circulated and taken home for further study.

Visit to the USA. In 1964 De Giorgi visited Brown and Stanford universities. He came by ship (the Cristoforo Colombo), and I met him in New York. There was a delay of several hours waiting for the passengers to disembark, because of a dock workers strike. During the auto trip from New York to Providence, De Giorgi told me that he had just proved a striking result called the Bernstein theorem for minimal surfaces of dimension 3 in 4 dimensional space. However, there was no mathematics library on the Cristoforo Colombo, and he wished to be certain about the strong maximum principle for elliptic PDEs which he needed in the proof. I assured him that what he needed is OK.

During his stay at Brown, De Giorgi gave a series of lectures on what he called “correnti quasi-normali.” His approach provided an alternative to the one taken by Federer and myself for normal currents. De Giorgi’s goal was to avoid use of a difficult measure theoretic covering theorem of Besicovitch. De Giorgi lectured in English, with occasional assistance from U. D’Ambrosio who was also visiting Brown as a postdoc.

Concluding remarks. After the 1960s De Giorgi’s work and mine took different directions. However, we kept up a lifelong friendship and saw each other from time to time, both in Pisa and elsewhere. Communication became easier as De Giorgi’s English improved and I learned a little Italian. (The other choice was bad French which we mutually decided against early on.) Besides his mathematical work, De Giorgi told me about his trips to Eritrea and his work for Amnesty International. Our last meeting was in 1993 at the 75th birthday conference for Cecconi in Nervi.

Ennio De Giorgi was a mathematician of extraordinary depth and powerful insights. There is a great Italian tradition in the calculus of variations, and among the world leaders in this field during the first part of the 20th century was L. Tonelli. De Giorgi was in every sense a worthy successor to Tonelli. There is a plaque on a wall in the old Università di Pisa building complex concerning Tonelli. While I don’t remember the exact wording, it says in effect that Tonelli was both an excellent mathematician and outstanding citizen. The same can be said about De Giorgi, although his good citizenship was shown perhaps in a different style from Tonelli's.

We miss him very much.

Appendix

This Appendix gives a concise summary of notations and definitions used in Sections 4-7. The textbook [16] gives an introduction to exterior algebra (also called Grassmann algebra) and the calculus of exterior differential forms. A more complete development is given in [11, Chap. 1].

k-vectors

α denotes a k-vector, $k = 1, \cdots, n$.
$\alpha \wedge \beta$ is the exterior product of a k-vector α and ℓ-vector β. Note that $\beta \wedge \alpha = (-1)^{k\ell} \alpha \wedge \beta$.

α is a simple k-vector if $\alpha = v_1 \wedge \cdots \wedge v_k$ with $v_1, \cdots, v_k \in \mathbb{R}^n$. The norm $|\alpha|$ of a simple k-vector α is the k-area of the parallelopiped

$$P = \{x = c_1 v_1 + \cdots + c_k v_k, 0 \leq c_j \leq 1, \text{ for } j = 1, \cdots, k\}.$$

Orientations. Any k-plane π has the form

$$\pi = \{x = x_0 + c_1 v_1 + \cdots + c_k v_k\}$$

with $x_0, v_1, \cdots, v_k \in \mathbb{R}^n, c_1, \cdots, c_k \in \mathbb{R}^1$ and $v_1, \cdots, v_k$ linearly independent. If $\alpha = v_1 \wedge \cdots \wedge v_k$, then $\tau = |\alpha|^{-1}\alpha$ has norm $|\tau| = 1$. This k-vector τ assigns an orientation to π, with $-\tau$ the opposite orientation.

k-covectors and differential forms. A k-covector ω is defined similarly as for k-vectors, with the space $\mathbb{R}^n$ of 1-vectors replaced by its dual space of 1-covectors. The dot product of ω and α is denoted by $\omega \cdot \alpha$.
A differential form ω of degree k is a k-covector valued function on $\mathbb{R}^n$. The norm (or comass) of ω is

$$\|\omega\| = \sup\{\omega(x) \cdot \alpha, x \in \mathbb{R}^n, \alpha \,\text{simple}, |\alpha| = 1\}.$$

Exterior differential calculus and currents. For any smooth k-form ω, the exterior differential is a $(k+1)$-form denoted by $d\omega$. It has the property

$d(d\omega) = 0$. Let $\mathcal{D}_k$ denote the space of all k-forms ω which have compact support and continuous partial derivatives of every order. A current T of dimension k is a linear functional on $\mathcal{D}_k$, which is continuous in the Schwartz topology on $\mathcal{D}_k$. The boundary ∂T is the current of dimension $k-1$ defined by formula:

$$(A.1) \qquad \partial T(\omega) = T(d\omega) \; for \; all \, \omega \in \mathcal{D}_{k-1}.$$

Note that $\partial(\partial T) = 0$.
The mass of T is

$$M(T) = \sup\{T(\omega) : \|\omega\| \leq 1\}.$$

Let $N(T) = M(T) + M(\partial T)$.

The support spt T of a current T is the smallest closed set $\Gamma \subset \mathbb{R}^n$ such that $T(\omega) = 0$ whenever $\omega(x) = 0$ for all x in some open set containing Γ.

Gauss-Green Theorem. The Gauss-Green (or divergence) Theorem (3.1) can be rewritten in the form (A.1) with $k = n-1$. This is explained in [16, Sec. 7.8]. In (2.6) let $T = T_B$, where the smooth boundary B of the set E is oriented by choice of exterior (rather than interior) unit normal vector $\nu(y)$ for $y \in B$. The unit tangent $(n-1)$-vector $\tau(y)$ is adjoint to $\nu(y)$ in the sense that $\nu(y), \tau(y)$ gives positive orientation to $\mathbb{R}^n$. If $\zeta(y)$ in (3.1) is interpreted as a 1-covector, then $\zeta(y)$ is adjoint to the $(n-1)$-covector $\omega(x)$.

The Whitney flat distance between integral currents T_1, T_2 is $W(T_1 - T_2)$, where

$$W(T) = \inf_{Q,R}\{M(Q) + M(R) : T = Q + \partial R, Q, R \, integral\}.$$

Convergence of a sequence T_j to T in this distance means that $W(T_j - T)$ tends to 0 as $j \to \infty$.

Representation formula. For any k-rectifiable current T, the following generalization of (4.2) holds. There exists a bounded k-rectifiable set K, and for H^k-almost all $y \in K$ a multiplicity function $\Theta(y)$ with positive integer values and an approximate unit tangent k-vector $\tau(y)$, such that

$$(A.2) \qquad T(\omega) = \int_K \omega(y) \cdot \tau(y)\Theta(y) dH^k(y), \omega \in \mathcal{D}_k.$$

$$(A.3) \qquad M(T) = \int_K \Theta(y) dH^k(y).$$

In view of (A.3), the mass of a k-rectifiable current T is also called the k-area of T. Implicitly, the definition of k-rectifiable current provides consistent orientations for the approximate tangent k-vectors.

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