Reprinted from JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS Vol. 51, No. 2, August 1975

Polygonal Approximation of Plane Convex Bodies*

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1. INTRODUCTION

In this paper we develop an asymptotic analysis of problems of approximating plane convex bodies by polygons with n faces. We examine criteria for best approximation such as the Hausdorff metric on compact convex sets and also measures of deviation defined in terms of area and perimeter differences. The main results give sharp estimates of the order of convergence of best approximations of a convex body by circumscribed and inscribed polygons with n faces. Further, asymptotic characterizations of best approximations are obtained and two methods are given for the construction of asymptotically efficient approximations.

Our primary motivation for considering these problems comes from the area of mathematical pattern analysis. Here we are concerned with set patterns in the plane that possess the structure of convexity. A fundamental problem in pattern analysis is the feature-selection problem, which is concerned with providing concise and precise pattern representations in terms of simple "features" of the patterns. Simple features of set patterns are derived from a binary feature logic, which identifies a single feature with a closed half-space $F = \{(x, y) \in R^2 \colon ax + by \leqslant c\}$. Such a binary feature F assigns the value 1 or 0 to a set $K \subseteq R^2$ according as $K \subseteq F$ or $K \nsubseteq F$. Conjunctions of such features describe convex subsets of R^2. In particular, a compact convex subset K of R^2 can be identified with a (possibly infinite) conjunction

$$K \sim \bigcap_{\alpha \in A} F_\alpha$$

for an appropriate index set A. A concise approximate representation of K is provided by a finite conjunction

$$K^* \sim \bigcap_{\nu=1}^{n} F_{\alpha_\nu}, \qquad \alpha_\nu \in A.$$

* This work was supported in part by NSF Grant GP-40521.

Such finite conjunctions are identified with convex polygons (polytopes), with n or fewer faces, that circumscribe K.

An approximate representation such as K^* is concise when n is small. It is precise if by an appropriate measure of deviation $D(\cdot, \cdot)$ the value of $D(K, K^*)$ is small. The feature-selection problem is concerned with attaining these goals, i.e., with selecting a finite set of features $\{F_{\alpha_\nu}: \alpha_\nu \in A, \nu = 1,..., n\}$, where the features F_{α_ν} are chosen to make $D(K, K^*)$ as small as possible. The problem translates immediately into one of best approximation of a compact convex set by a circumscribed polygon.

The problem may be viewed in another light. A convex set can be interpreted as a system of linear inequalities and a convex polygon as a finite system of linear inequalities (see Poritsky [6]). Thus problems of best approximation of convex sets by (circumscribed or inscribed) polygons admit interpretations as problems of optimal reduction or compression of large systems of linear inequalities.

Finally, the approximation problems bear an inherent interest within the theory of convex sets *per se*. Often functionals of a convex set, such as volume (area) or surface area (perimeter), are defined as limits as n increases of their values for circumscribed or inscribed polyhedra with n faces (see Valentine [7]). The asymptotic analysis that we develop is directly related to questions of the behavior of these limits.

In the literature on convex sets there are results which are related to our own. Dowker [2] considers the area deviation between a plane convex set and (i) the minimal-area circumscribed polygon with n faces and (ii) the maximal-area inscribed polygon with n faces. He shows that the area deviations are convex functions of n. The elegant development of Eggleston [3] extends Dowker's results to perimeter deviations. Eggleston also shows that deviations measured by a modified Hausdorff metric need not be convex functions of n.

The paper by Carlsson and Grenander [1] contains asymptotic analysis of the area deviation of approximations by circumscribed polygons. The results there are related both to a design problem in the statistical estimation of areas of convex sets and to a pattern representation problem like the one described above. The stochastic flavor of their arguments distinguishes their paper from this one.

In the next section we review some preliminary results on convex sets and formulate the approximation problems. Through the use of support-function representations of the sets and their polygonal approximations, the problems are translated into ones of optimal function approximation by trigonometric splines. Section 3 states eight theorems which summarize the principal results of the asymptotic analysis. These results describe (i) order of convergence of best approximations, (ii) characterization of best approximations, and (iii) methods of constructing "good", i.e., asymptotically efficient, approximations. Section 4 lays the background for the proofs of these results. We relate some general results of McClure [4] on problems of interval segmentation, from which the results of Section 3 will follow immediately. Then Section 5 proves the main theorems for circumscribed polygonal approximations by verifying the simple hypotheses of the general results on segmentation in Section 4. Finally, Section 6 presents the similar analysis for inscribed polygons.

Valentine's book [7] is a valuable reference for the properties of convex sets that we introduce in the next section.

2. Preliminaries: Convex Sets

Symbols $x, y, \ldots$, denote points in the plane R^2. Points in R^2 may be described by their components as $x = (x_1, x_2)$. Distances between points in R^2 are defined by the standard Euclidean norm and, where topological considerations enter, we are concerned with the topology of R^2 induced by the Euclidean metric.

A set $K \subseteq R^2$ is *convex* if for all pairs $x, y \in K$ the line segment $\alpha x + (1 - \alpha) y$, for $0 \leqslant \alpha \leqslant 1$, is contained in K. A *convex body* is a convex set with nonempty interior, relative to the topology of R^2. In the sequel it is implicit that all the sets we consider are *compact*, that is, closed and bounded in R^2; by "convex set" and "convex body" we will mean "compact convex set" and "compact convex body", respectively.

Let $\mathscr{K}$ denote the set of convex subsets of R^2:

$$\mathscr{K} = \{K \subseteq R^2\colon K \text{ is compact and convex}\}. \tag{2.1}$$

Useful representations of members of $\mathscr{K}$ are provided by their support functions. In some settings it is natural to define support functions on R^2 or on the unit circle in R^2. For the approximation problems, it is convenient to identify the domain of a support function with the interval $[0, 2\pi)$. For $K \in \mathscr{K}$ define the associated *support function* s_K by

$$s_K(\theta) = \max_{x \in K}(x_1 \cos\theta + x_2 \sin\theta), \tag{2.2}$$

for $0 \leqslant \theta < 2\pi$. Support functions are continuous and 2π-periodic, when their domain is extended.

The geometrical interpretation of a support function is illustrated in Fig. 1. The figure also depicts a support line of K. Since K is compact there is at least one boundary point x in K for which $s_K(\theta) = x_1 \cos\theta + x_2 \sin\theta$. The line ℓ that passes through this point and is orthogonal to $(\cos\theta, \sin\theta)$ is termed a *support line* of K in the direction θ. There is a support line passing

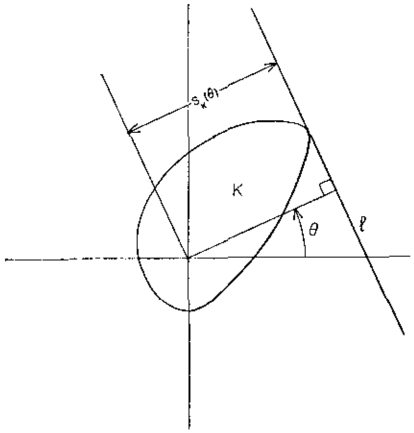

FIGURE 1

through every boundary point of a set $K \in \mathscr{K}$, and, as noted in Section 1, a member of $\mathscr{K}$ can be regarded as an intersection of half-spaces determined by its support lines.

Define $\mathscr{S}$ as the subset of $C[0, 2\pi)$ whose members are support functions of sets in $\mathscr{K}$:

$$\mathscr{S} = \{s_K: K \in \mathscr{K}\}.$$

The correspondence between $\mathscr{K}$ and $\mathscr{S}$ is one-to-one, so we can identify $\mathscr{K}$ with $\mathscr{S}$. This identification, together with its order-preserving and topological properties, is what allows our reduction of set approximation problems to function approximation problems.

There is also an interesting algebraic property of the identification, which we do not exploit in our problems. The mapping (2.2) between $\mathscr{K}$ and $\mathscr{S}$ preserves operations of addition and scalar multiplication. The sum of two sets K_1 and K_2 in $\mathscr{K}$ is defined by

$$K_1 + K_2 = \{x + y: x \in K_1, y \in K_2\};$$

$K_1 + K_2$ is again in $\mathscr{K}$ and

$$s_{K_1+K_2} = s_{K_1} + s_{K_2}.$$

For any nonnegative value λ and a set K in $\mathscr{K}$, the set λK is defined by

$$\lambda K = \{\lambda x: x \in K\};$$

λK is in $\mathscr{K}$ and

$$s_{\lambda K} = \lambda s_K.$$

The *order-preserving property* of the correspondence between $\mathscr{K}$ and $\mathscr{S}$ is more important in our discussion of circumscribed and inscribed figures. If K_1 and K_2 are in $\mathscr{K}$ and $K_1 \subseteq K_2$, then $s_{K_1} \leqslant s_{K_2}$. This follows immediately from (2.2).

There are several interesting measures of deviation or "distances" between pairs of sets in $\mathscr{K}$. It is convenient for us to describe these for pairs of sets that are ordered by inclusion, since the definitions take particularly simple forms in this case and since the approximations we treat satisfy this constraint. The definitions have natural extensions for sets that are not ordered (see Eggleston [3]).

Let C_r denote the circle in R^2 centered at the origin, with radius $r \geqslant 0$. For any K in $\mathscr{K}$ defined $K(r) = K + C_r$. The Hausdorff metric D_∞ between sets K_1 and K_2 in $\mathscr{K}$ is defined by

$$D_\infty(K_1, K_2) = \min_{r \geqslant 0}\{r : K_2 \subseteq K_1(r)\}, \tag{2.3}$$

where

$$K_1 \subseteq K_2 .$$

Since $s_{K(r)} = s_K + r$, we obtain

$$D_\infty(K_1, K_2) = \max_{[0,2\pi)}(s_{K_2}(\theta) - s_{K_1}(\theta)) \tag{2.4}$$

when $K_1 \subseteq K_2$. $\mathscr{K}$ with the Hausdorff metric is isometric to $\mathscr{S}$ with the max–norm,

$$D_\infty(K_1, K_2) = \| s_{K_2} - s_{K_1} \|_\infty .$$

Let $m(K)$ denote the area of a set K in $\mathscr{K}$. The ***area deviation*** D_A between sets K_1 and K_2 in $\mathscr{K}$ is defined as

$$D_A(K_1, K_2) = m(K_2) - m(K_1), \qquad \text{when} \qquad K_1 \subseteq K_2 . \tag{2.5}$$

D_A is expressed in terms of support functions through the equation [7]

$$m(K) = \tfrac{1}{2}\int_0^{2\pi} [s_K{}^2(\theta) - \dot{s}_K{}^2(\theta)]\, d\theta.$$

We obtain

$$D_A(K_1, K_2) = \tfrac{1}{2}\int_0^{2\pi} \{[s_{K_2}^2(\theta) - s_{K_1}^2(\theta)] - [\dot{s}_{K_2}^2(\theta) - \dot{s}_{K_1}^2(\theta)]\}\, d\theta \tag{2.6}$$

when $K_1 \subseteq K_2$. Equation (2.7) does not describe a norm on $\mathscr{S}$, but it will relate to a weighted integral norm on $\mathscr{S}$ when we consider polygonal approximations.

A measure of deviation with the same geometric appeal as D_A is described by the perimeter $\ell(K)$ of a convex body in $\mathscr{K}$. The ***perimeter deviation*** D_ℓ between bodies K_1 and K_2 in $\mathscr{K}$ is defined as

$$D_\ell(K_1, K_2) = \ell(K_2) - \ell(K_1), \qquad \text{when} \qquad K_1 \subseteq K_2. \tag{2.7}$$

The perimeter $\ell(K)$ is given by [7]

$$\ell(K) = \int_0^{2\pi} s_K(\theta)\, d\theta,$$

so D_ℓ is expressed as

$$D_\ell(K_1, K_2) = \int_0^{2\pi} [s_{K_2}(\theta) - s_{K_1}(\theta)]\, d\theta \tag{2.8}$$

when $K_1 \subseteq K_2$. Thus the perimeter deviation is identified with the L_1 norm on $\mathscr{S}$.

Equations (2.4) and (2.8) suggest consideration of deviation measures on $\mathscr{K}$ induced by the L_p norms on $\mathscr{S}$. Define the ***p-norm deviation*** D_p on $\mathscr{K}$ by

$$D_p(K_1, K_2) = \| s_{K_2} - s_{K_1} \|_p, \qquad 1 \leqslant p \leqslant \infty, \tag{2.9}$$

where $\|\cdot\|_p$ is the L_p norm on $\mathscr{S}$. When $K_1 \subseteq K_2$, $D_\ell(K_1, K_2) = D_1(K_1, K_2)$. The metrics D_p for $1 < p < \infty$ do not admit simple geometric interpretations.

The analysis of these distances on $\mathscr{K}$ and their counterparts on $\mathscr{S}$ depends on a special structure of support functions. One part of a representation theorem proven by Vitale [8] shows that $\mathscr{S}$ is identical to the class of functions on $[0, 2\pi)$ that admit representations

$$s(\theta) = a \cos\theta + b \sin\theta + \int_0^\theta \sin(\theta - \lambda)\, R(d\lambda), \tag{2.10}$$

where a and b are constants and R is a finite measure on $[0, 2\pi)$ satisfying

$$\int_0^{2\pi-} \cos\theta\, R(d\theta) = \int_0^{2\pi-} \sin\theta\, R(d\theta) = 0.^1$$

R is appropriately termed the ***radial distribution*** because, when it is absolutely continuous with respect to Lebesgue measure and $R(d\theta) = r(\theta)\, d\theta$,

[1] Equation (2.10) gives us explicitly the identification between support functions and measures on the unit circle that was first described by Minkowski. For a discussion of Minkowski's problem and its development see H. Busemann, "Convex Surfaces," p. 60, Wiley, New York, 1958.

then $r(\theta)$ is the familiar radius of curvature function or *radial density*. Indeed, if K is in $\mathscr{K}$ and s_K is twice differentiable, then

$$s_K(\theta) = a \cos \theta + b \sin \theta + \int_0^\theta \sin(\theta - \lambda)\, r_K(\lambda)\, d\lambda; \tag{2.11}$$

$$r_K(\theta) = \ddot{s}_K(\theta) + s_K(\theta) \tag{2.12}$$

describes the radius of curvature of the boundary of K at the boundary point $x \in K$, where

$$s_K(\theta) = x_1 \cos \theta + x_2 \sin \theta.$$

The representation (2.10) for an n-sided polygon P_n reduces to a discrete sum of the form

$$s_{P_n}(\theta) = a \cos \theta + b \sin \theta + \sum_{\nu=1}^{n} r_\nu \sin(\theta - \lambda_\nu)_+ , \qquad 0 \leqslant \theta < 2\pi \tag{2.13}$$

where

$$(\theta - \lambda)_+ = \begin{cases} 0, & \theta < \lambda \\ \theta - \lambda, & \theta \geqslant \lambda. \end{cases}$$

The discrete values λ_ν, $\nu = 1,\ldots, n$, are the angles between the horizontal axis and outward normals to the faces of P_n. We observe that s_{P_n} is a trigonometric spline function associated with the second-order differential operator

$$L = \frac{d^2}{d\theta^2} + I; \tag{2.14}$$

s_{P_n} is continuous on $[0, 2\pi)$, $\dot{s}_{P_n}$ is continuous except at the points λ_ν, and L annihilates s_{P_n} on open intervals between the λ_ν.

Our asymptotic expressions for distances between a convex body K in $\mathscr{K}$ and circumscribed or inscribed polygons will depend on the radial density r_K and thus on the regularity of s_K. In this direction, we define

$$\mathscr{S}^2 = \{s \in \mathscr{S} : \dot{s} \text{ and } \ddot{s} \text{ are continuous on } [0, 2\pi)\}. \tag{2.15}$$

Radial densities of sets K with support functions in $\mathscr{S}^2$ exist and are continuous on $[0, 2\pi)$.

Further define

$$\mathscr{P}_n = \{P \in \mathscr{K} : P \text{ is a polygon with } n \text{ or fewer faces}\}. \tag{2.16}$$

$\mathscr{P}_n$ is identified with the trigonometric spline support functions described above.

The first results in Section 3 give sharp estimates for the asymptotic behavior of

$$\inf_{\substack{P_n \in \mathscr{P}_n \\ K \subseteq P_n}} D(K, P_n) \qquad \text{and} \qquad \inf_{\substack{P_n \in \mathscr{P}_n \\ P_n \subseteq K}} D(P_n, K)$$

for K in $\mathscr{K}$ and $D = D_\infty$, D_A, D_ℓ and D_p. Characterizations of best approximating polygons follow from these asymptotic estimates.

3. Main Results

All of the results in this section follow from the general results on interval segmentation stated in Section 4 and from properties of convex bodies and their support functions developed in Sections 2, 5, and 6. The first theorem describes rate of convergence of polygons that circumscribe a convex body K.

THEOREM 1. *If K is a convex body in $\mathscr{K}$ with support function s_K in $\mathscr{S}^2$* (2.15) *and radial density r_K* (2.12), *then*

$$\text{(i)} \quad \lim_{n\to\infty} n^2 [\inf_{\substack{P_n \in \mathscr{P}_n \\ K \subseteq P_n}} D_\infty(K, P_n)] = \tfrac{1}{8} \left(\int_0^{2\pi} [r_K(\theta)]^{1/2} \, d\theta \right)^2,$$

$$\text{(ii)} \quad \lim_{n\to\infty} n^2 [\inf_{\substack{P_n \in \mathscr{P}_n \\ K \subseteq P_n}} D_A(K, P_n)] = \tfrac{1}{24} \left(\int_0^{2\pi} [r_K(\theta)]^{2/3} \, d\theta \right)^3,$$

$$\text{(iii)} \quad \lim_{n\to\infty} n^2 [\inf_{\substack{P_n \in \mathscr{P}_n \\ K \subseteq P_n}} D_\ell(K, P_n)] = \tfrac{1}{12} \left(\int_0^{2\pi} [r_K(\theta)]^{1/3} \, d\theta \right)^3,$$

$$\text{(iv)} \quad \lim_{n\to\infty} n^2 [\inf_{\substack{P_n \in \mathscr{P}_n \\ K \subseteq P_n}} D_p(K, P_n)] = \tfrac{1}{2} [B(p)]^{1/p} \left(\int_0^{2\pi} [r_K(\theta)]^{p/(2p+1)} \, d\theta \right)^{(2p+1)/p}$$

for $1 \leqslant p < \infty$, *where*

$$B(p) = \int_0^1 [x(1-x)]^p \, dx.$$

Thus all of the deviations considered are of order n^{-2}, and sharp estimates of rates of convergence are provided by the integral expressions in Theorem 1. The other results are concerned with characterizing best approximations and prescribing methods of constructing good approximations.

Each face of a polygon that circumscribes a set K is a segment of a support line of K. A circumscribing polygon is uniquely determined by specifying the directions of these support lines, say $0 \leqslant \theta_1 < \theta_2 < \cdots < \theta_n < 2\pi$ (see Fig. 2). This specification of a polygon in terms of the directions of its

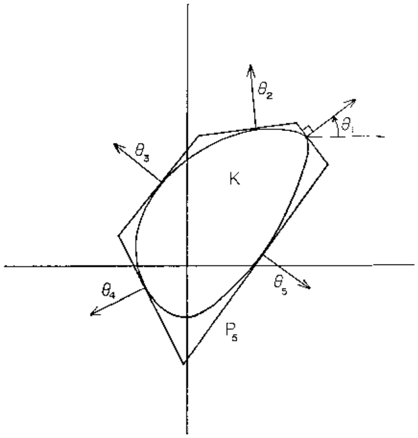

FIGURE 2

faces is equivalent to specifying the spline support function (2.13) of the polygon in terms of the locations of its knots. The condition that a polygon P_n circumscribes a convex body K is equivalent to the conditions that s_{P_n} interpolate the values of s_K at the knots of s_{P_n} and $s_{P_n} \geqslant s_K$.

It is easily argued that best approximations of K exist in $\mathscr{P}_n$ under all of the measures of deviation we consider. The deviations $D(K, P_n)$, where P_n circumscribes K, depend continuously on the directions $(\theta_1, \ldots, \theta_n)$ of the faces of P_n. Thus the minimum of $D(K, P_n)$ is attained for some values $0 \leqslant \theta_1^* \leqslant \theta_2^* \leqslant \cdots \leqslant \theta_n^* < 2\pi$. Let P_n^* denote the circumscribing polygon with n (or fewer) faces in directions $\theta_1^*, \ldots, \theta_n^*$; then

$$D(K, P_n^*) = \inf_{\substack{P_n \in \mathscr{P}_n \\ K \subseteq P_n}} D(K, P_n). \tag{3.1}$$

As in problems of variable-knot approximation, there is no guarantee that a best approximation P_n^* will be unique, but unique asymptotic characterizations can be obtained in terms of the distributions of the values $(\theta_1^*, \ldots, \theta_n^*)$. Define the empirical distribution functions

$$G_n^*(\theta) = n^{-1} \operatorname{card}\{\theta_\nu^*: \theta_\nu^* \leqslant \theta\} \qquad \text{for } 0 \leqslant \theta \leqslant 2\pi; \tag{3.2}$$

card denotes the cardinality of the indicated set. Note that θ_ν^* depends implicitly on n; in order to avoid cumbersome notation, we do not explicitly denote this dependence. G_n^* describes the distribution of optimal face directions.

The limiting behavior of G_n^* is stated in terms of distributions related to the integral expressions of Theorem 1. G_n^* depends on the particular measure of deviation D with respect to which $P_n{}^*$ is a best approximation. We define the following distribution functions on $[0, 2\pi]$ associated with a fixed convex body K, having continuous radial density r_K:

$$G_\infty(\theta) = \left(\int_0^\theta [r_K(\tau)]^{1/2}\, d\tau\right) \left(\int_0^{2\pi} [r_K(\tau)]^{1/2}\, d\tau\right)^{-1}, \tag{3.3}$$

$$G_A(\theta) = \left(\int_0^\theta [r_K(\tau)]^{2/3}\, d\tau\right) \left(\int_0^{2\pi} [r_K(\tau)]^{2/3}\, d\tau\right)^{-1}, \tag{3.4}$$

$$G_\ell(\theta) = \left(\int_0^\theta [r_K(\tau)]^{1/3}\, d\tau\right) \left(\int_0^{2\pi} [r_K(\tau)]^{1/3}\, d\tau\right)^{-1}, \tag{3.5}$$

$$G_p(\theta) = \left(\int_0^\theta [r_K(\tau)]^{p/(2p+1)}\, d\tau\right) \left(\int_0^{2\pi} [r_K(\tau)]^{p/(2p+1)}\, d\tau\right)^{-1} \tag{3.6}$$

for $0 \leqslant \theta \leqslant 2\pi$.

THEOREM 2. *Let K be a convex body in $\mathscr{K}$ with support function s_K in $\mathscr{S}^2$ and radial density r_K. Let $P_n{}^*$ be a best circumscribing approximation of K in $\mathscr{P}_n$ relative to the measure of deviation D* (3.1) *and define $G_n{}^*$ by* (3.2).

(i) *If $D = D_\infty$, then*

$$\lim_{n\to\infty} G_n{}^*(\theta) = G_\infty(\theta);$$

(ii) *if $D = D_A$, then*

$$\lim_{n\to\infty} G_n{}^*(\theta) = G_A(\theta);$$

(iii) *if $D = D_\ell$, then*

$$\lim_{n\to\infty} G_n{}^*(\theta) = G_\ell(\theta); \quad \textit{and}$$

(iv) *if $D = D_p$, then*

$$\lim_{n\to\infty} G_n{}^*(\theta) = G_p(\theta),$$

for all θ in $[0, 2\pi]$.

One construction of asymptotically efficient approximations is based on establishing sufficiency for asymptotic optimality of the distribution characterizations in Theorem 2. We construct polygons from a specified distribution of face angles.

Let f be a positive, bounded, piecewise continuous function on $[0, 2\pi]$, which is normalized so that $\int_0^{2\pi} f(\tau)\, d\tau = 1$. Let F denote its integral;

$F(\theta) = \int_0^\theta f(\tau)\, d\tau$. For any $n \geqslant 3$, define an n-point partition of $[0, 2\pi]$ by inversion of F at equally spaced ordinates;

$$\theta_\nu = F^{-1}[\nu/(n+1)], \qquad \text{for } \nu = 1,..., n. \tag{3.7}$$

Let $P_n{}^F$ be the unique polygon in $\mathscr{P}_n$ that circumscribes K and whose faces are in the directions θ_ν of (3.7).

The following theorem describes the convergence of $P_n{}^F$ to K and states sufficient conditions for constructing asymptotically efficient approximations.

THEOREM 3. *Let K be a convex body in $\mathscr{K}$ with support function s_K in $\mathscr{S}^2$ and radial density r_K. Let $P_n{}^F$ be the polygon in $\mathscr{P}_n$ that circumscribes K and whose faces are in the directions θ_ν of* (3.7). *Suppose $f = dF/d\theta$ is strictly positive, bounded, and piecewise continuous on* $[0, 2\pi]$. *Under these conditions*

$$\text{(i)} \quad \lim_{n\to\infty} n^2 D_\infty(K, P_n{}^F) = \operatorname*{ess\,sup}_{[0,2\pi]}(\tfrac{1}{8} f^{-2} r_K),$$

$$\text{(ii)} \quad \lim_{n\to\infty} n^2 D_A(K, P_n{}^F) = \frac{1}{24} \int_0^{2\pi} [r_K(\theta)]^2\, [f(\theta)]^{-2}\, d\theta,$$

$$\text{(iii)} \quad \lim_{n\to\infty} n^2 D_\ell(K, P_n{}^F) = \frac{1}{12} \int_0^{2\pi} r_K(\theta)\, [f(\theta)]^{-2}\, d\theta, \quad \textit{and}$$

$$\text{(iv)} \quad \lim_{n\to\infty} n^2 D_p(K, P_n{}^F) = \tfrac{1}{2}[B(p)]^{1/p} \left(\int_0^{2\pi} [r_K(\theta)]^p\, [f(\theta)]^{-2p}\, d\theta\right)^{1/p}.$$

In particular, if (i) *r_K is strictly positive on* $[0, 2\pi]$, (ii) *$F = G_\infty$, G_A, G_ℓ or G_p* ((3.3)–(3.6)) *according as $D = D_\infty$, D_A, D_ℓ, or D_p, and* (iii) *$P_n{}^*$ is an optimal circumscribing approximation of K in $\mathscr{P}_n$ relative to D, then*

$$\lim_{n\to\infty} D(K, P_n{}^*)/D(K, P_n{}^F) = 1.$$

The approximations $P_n{}^F$ are asymptotically efficient.

An alternative to this "density approach" for constructing good many-sided approximations is based on a notion of balancing local errors of approximation. Consider a decomposition of K and a circumscribed polygon P_n as depicted in Fig. 3. Compare this to Fig. 2. We have partitioned the combined figure by drawing rays from an interior point of K to points of contact between K and each face of P_n. Denote the components of the partitioned sets K and P_n by $K^{(\nu)}$ and $P_n^{(\nu)}$;

$$K = \bigcup_{\nu=1}^{n} K^{(\nu)} \qquad \text{and} \qquad P_n = \bigcup_{\nu=1}^{n} P_n^{(\nu)}.$$

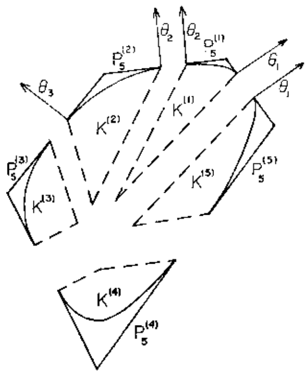

FIGURE 3

$P_n^{(\nu)}$ circumscribes $K^{(\nu)}$, and the support functions of $P_n^{(\nu)}$ and $K^{(\nu)}$ coincide with the support functions of P_n and K on $[\theta_\nu, \theta_{\nu+1}]$, where θ_{n+1} is identified with θ_1.

The deviations $D(K, P_n)$ are easily related to the values of $D(K^{(\nu)}, P_n^{(\nu)})$. Indeed,

$$D_\infty(K, P_n) = \max_{1 \leqslant \nu \leqslant n} D_\infty(K^{(\nu)}, P_n^{(\nu)}),$$

$$D_A(K, P_n) = \sum_{\nu=1}^{n} D_A(K^{(\nu)}, P_n^{(\nu)}),$$

$$D_\ell(K, P_n) = \sum_{\nu=1}^{n} D_\ell(K^{(\nu)}, P_n^{(\nu)}),$$

and

$$D_p{}^p(K, P_n) = \sum_{\nu=1}^{n} D_p{}^p(K^{(\nu)}, P_n^{(\nu)}).$$

From remarks in Section 4, it follows that optimal approximations in the Hausdorff metric balance the values of the local error $D_\infty(K^{(\nu)}, P_n^{(\nu)})$; that is, if $P_n{}^*$ minimizes $D_\infty(K, P_n)$ over polygons $P_n \in \mathscr{P}_n$, $K \subseteq P_n$, then

$$D_\infty(K^{(\nu)}, P_n^{*(\nu)}) = D_\infty(K, P_n{}^*) \qquad \text{for } \nu = 1,..., n.$$

Polygons which balance the local errors measured by the other deviations also yield asymptotically efficient approximations.

Because of the continuous dependence of $D(K, P_n)$ on the face angles $(\theta_1, ..., \theta_n)$ of P_n and from the remarks on this point in Section 4, it follows that we can define a polygon $\bar{P}_n$ in $\mathscr{P}_n$ by the error-balancing condition

$$D(K^{(\nu)}, \bar{P}_n^{(\nu)}) = D(K^{(\nu+1)}, \bar{P}_n^{(\nu+1)}) \qquad \text{for } \nu = 1,..., n-1, \tag{3.8}$$

where $D = D_A$, D_ℓ, or D_p, and $\bar{P}_n$ circumscribes K.

The following theorem describes the convergence of $\bar{P}_n$.

THEOREM 4. *Let K be a convex body in $\mathscr{K}$ with support function s_K in $\mathscr{S}^2$ and radial density r_K. Let D denote any of the measures of deviation D_A, D_ℓ, or D_p. Let $P_n{}^*$ be a best circumscribing approximation of K in $\mathscr{P}_n$ (relative to D), and let $\bar{P}_n$ in $\mathscr{P}_n$ circumscribe K and satisfy* (3.8). *Then*

$$\lim_{n\to\infty} D(K, P_n{}^*)/D(K, \bar{P}_n) = 1.$$

The approximations $\bar{P}_n$ are asymptotically efficient.

Analogous results are obtained for the problem of approximating a convex body by an inscribed polygon. These differ slightly in detail and development from the theorems above.

THEOREM 5. *If K is a convex body in $\mathscr{K}$ with support function s_K in $\mathscr{S}^2$ and radial density r_K, then*

(i) $$\lim_{n\to\infty} n^2[\inf_{\substack{P_n\in\mathscr{P}_n\\ P_n\subseteq K}} D_\infty(P_n, K)] = \tfrac{1}{8}\left(\int_0^{2\pi} [r_K(\theta)]^{1/2}\, d\theta\right)^2,$$

(ii) $$\lim_{n\to\infty} n^2[\inf_{\substack{P_n\in\mathscr{P}_n\\ P_n\subseteq K}} D_A(P_n, K)] = \tfrac{1}{12}\left(\int_0^{2\pi} [r_K(\theta)]^{2/3}\, d\theta\right)^3,$$

(iii) $$\lim_{n\to\infty} n^2[\inf_{\substack{P_n\in\mathscr{P}_n\\ P_n\subseteq K}} D_\ell(P_n, K)] = \tfrac{1}{24}\left(\int_0^{2\pi} [r_K(\theta)]^{1/3}\, d\theta\right)^3, \quad \textit{and}$$

(iv) $$\lim_{n\to\infty} n^2[\inf_{\substack{P_n\in\mathscr{P}_n\\ P_n\subseteq K}} D_p(P_n, K)]$$
$$= [1/8(2p+1)^{1/p}]\left(\int_0^{2\pi} [r_K(\theta)]^{p/(2p+1)}\, d\theta\right)^{(2p+1)/p}$$

Again, all the deviations considered are of order n^{-2}, and the integral expressions of Theorem 5 give precise estimates of the asymptotic deviations.

Inscribed polygons are not as neatly characterized as are the circumscribed polygons in terms of directions of their faces. There need not exist a polygon P_n that inscribes a set K and has faces in specified directions. Therefore, it is convenient to characterize inscribed polygons in terms of their vertices, which lie on the boundary of K (see Fig. 4). We can parametrize the vertices of the inscribed polygon in terms of the directions of support lines of K at the respective vertices; say these directions are

$$0 \leqslant \theta_1 < \theta_2 < \cdots < \theta_n < 2\pi.$$

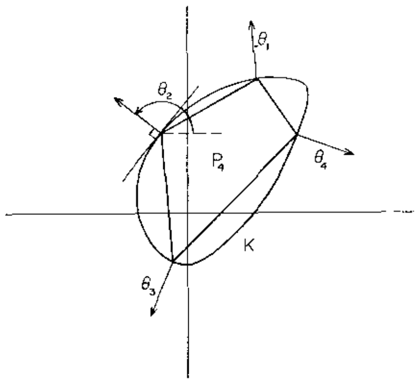

FIGURE 4

The specification of an inscribed polygon in terms of the angles $\theta_1, \ldots, \theta_n$ at its vertices is equivalent to the specification of its spline support function (2.13) in terms of the points where it interpolates values of s_K and $\dot{s}_K$. If P_n inscribes K then $s_{P_n} \leqslant s_K$, $s_{P_n}(\theta_\nu) = s_K(\theta_\nu)$, $\dot{s}_{P_n}(\theta_\nu) = \dot{s}_K(\theta_\nu)$ and s_{P_n} has exactly one knot η_ν in each open interval $(\theta_\nu, \theta_{\nu+1})$. The second-order interpolation of s_K by s_{P_n} at the points θ_ν follows from the regularity of s_{P_n} between knots and the properties

$$s_{P_n} \leqslant s_K \quad \text{and} \qquad s_{P_n}(\theta_\nu) = s_K(\theta_\nu).$$

As for the case of circumscribed polygons, best approximations by inscribed polygons always exist. For any measure of deviation D, the values $D(P_n, K)$ depend continuously on the parameters $(\theta_1, \ldots, \theta_n)$ of P_n. The minimum of $D(P_n, K)$ is attained for some values $\theta \leqslant \theta_1^* \leqslant \cdots \leqslant \theta_n^* \leqslant 2\pi$. If P_n^* denotes the inscribing polygon with n (or fewer) vertices associated with the parameters $(\theta_1^*, \ldots, \theta_n^*)$, then

$$D(P_n^*, K) = \inf_{\substack{P_n \in \mathscr{P}_n \\ P_n \subseteq K}} D(P_n, K). \tag{3.9}$$

The best approximations P_n^* are characterized in terms of the asymptotic distribution of the parameters $(\theta_1^*, \ldots, \theta_n^*)$. In analogy to the previous development, define the empirical distribution of these vertex parameters on $[0, 2\pi]$:

$$V_n^*(\theta) = n^{-1} \operatorname{card}\{\theta_\nu^*: \theta_\nu^* \leqslant \theta\} \qquad \text{for } 0 \leqslant \theta \leqslant 2\pi; \tag{3.10}$$

card means "cardinality." Recall that the values θ_ν^* depend implicitly on n.

The necessary condition on the limiting behavior of V_n^* parallels the result of Theorem 2.

THEOREM 6. *Let K be a convex body in $\mathscr{K}$ with support function s_K in $\mathscr{S}^2$ and radial density r_K. Let P_n^* be a best inscribing approximation of K in $\mathscr{P}_n$ relative to the measure of deviation D* (3.9) *and define V_n^* by* (3.10). *G_∞, G_A, G_ℓ, and G_p are defined by Eqs.* (3.3)–(3.6).

(i) *If $D = D_\infty$, then*

$$\lim_{n\to\infty} V_n^*(\theta) = G_\infty(\theta);$$

(ii) *if $D = D_A$, then*

$$\lim_{n\to\infty} V_n^*(\theta) = G_A(\theta);$$

(iii) *if $D = D_\ell$, then*

$$\lim_{n\to\infty} V_n^*(\theta) = G_\ell(\theta); \quad \textit{and}$$

(iv) *if $D = D_p$, then*

$$\lim_{n\to\infty} V_n^*(\theta) = G_p(\theta),$$

for all θ in $[0, 2\pi]$.

Constructions of asymptotically efficient approximations follow from a theorem on the sufficiency of the limiting distribution characterizations in Theorem 6.

Let f be positive, bounded, and piecewise continuous on $[0, 2\pi]$ and let $\int_0^{2\pi} f(\tau)\, d\tau = 1$. Define $F(\theta) = \int_0^\theta f(\tau)\, d\tau$ and partition $[0, 2\pi]$ at points $\theta_1, \ldots, \theta_n$ by setting

$$\theta_\nu = F^{-1}[\nu/(n+1)] \qquad \text{for } \nu = 1, \ldots, n. \tag{3.11}$$

Then let P_n^F be the polygon in $\mathscr{P}_n$ that inscribes K and has vertices at the points on the boundary of K where the support lines are in the directions θ_ν. The convergence of P_n^F to K is described in the next theorem.

THEOREM 7. *Let K be a convex body in $\mathscr{K}$ with support function s_K in $\mathscr{S}^2$ and radial density r_K. Let P_n^F be the polygon in $\mathscr{P}_n$ that inscribes K and whose vertices are identified with the parameters θ_ν of* (3.11). *Suppose $f = dF/d\theta$*

is strictly positive, bounded, and piecewise continuous on $[0, 2\pi]$. *Under these conditions*

$$\text{(i)} \quad \lim_{n\to\infty} n^2 D_\infty(P_n{}^F, K) = \operatorname*{ess\,sup}_{[0,2\pi]}(\tfrac{1}{8} f^{-2} r_K),$$

$$\text{(ii)} \quad \lim_{n\to\infty} n^2 D_A(P_n{}^F, K) = \tfrac{1}{12} \int_0^{2\pi} [r_K(\theta)]^2 \, [f(\theta)]^{-2} \, d\theta,$$

$$\text{(iii)} \quad \lim_{n\to\infty} n^2 D_\ell(P_n{}^F, K) = \tfrac{1}{24} \int_0^{2\pi} r_K(\theta) \, [f(\theta)]^{-2} \, d\theta, \quad \textit{and}$$

$$\text{(iv)} \quad \lim_{n\to\infty} n^2 D_p(P_n{}^F, K) = \frac{1}{8(2p+1)^{1/p}} \left(\int_0^{2\pi} [r_K(\theta)]^p \, [f(\theta)]^{-2p} \, d\theta \right)^{1/p}.$$

In particular, if (i) r_K *is strictly positive on* $[0, 2\pi]$, (ii) $F = G_\infty$, G_A, G_ℓ, *or* G_p ((3.3)–(3.6)) *according as* $D = D_\infty$, D_A, D_ℓ, *or* D_p, *and* (iii) $P_n{}^*$ *is an optimal inscribing approximation of* K *in* $\mathscr{P}_n$ *relative to* D, *then*

$$\lim_{n\to\infty} D(P_n{}^*, K)/D(P_n{}^F, K) = 1.$$

The approximations $P_n{}^F$ *are asymptotically efficient.*

Finally, the "balanced-error" approach to constructing asymptotically efficient approximations carries over to the present context.

Consider a decomposition of K and an inscribed polygon P_n as depicted in Fig. 5. Compare this to Fig. 4; see also Figs. 2 and 3. Rays have been drawn from an interior point of P_n to its vertices. Let $K^{(\nu)}$ and $P_n^{(\nu)}$ denote the separate components of the two partitioned sets. $P_n^{(\nu)}$ inscribes $K^{(\nu)}$, and the support functions of $P_n^{(\nu)}$ and $K^{(\nu)}$ on $[\theta_\nu, \theta_{\nu+1}]$ coincide with those of P_n and K, respectively.

As related above, a best approximation in the Hausdorff metric is a balanced-error approximation; that is, if $P_n{}^*$ minimizes $D(P_n, K)$ among polygons $P_n \in \mathscr{P}_n$, $P_n \subseteq K$, then $D(P_n^{*(\nu)}, K^{(\nu)})$ is independent of ν. We can

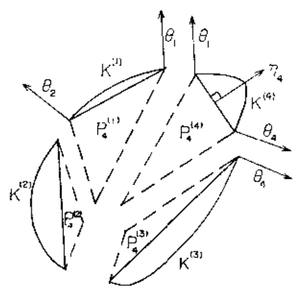

FIGURE 5

define inscribing polygons by this balancing condition for any of our measures of deviation. Let $\bar{P}_n$ inscribe K and satisfy

$$D(\bar{P}_n^{(\nu)}, K^{(\nu)}) = D(\bar{P}_n^{(\nu+1)}, K^{(\nu+1)}) \qquad \text{for } \nu = 1,..., n-1, \tag{3.12}$$

where $D = D_A$, D_ℓ, or D_p.

The last theorem describes the convergence of $\bar{P}_n$.

THEOREM 8. *Let K be a convex body in $\mathscr{K}$ with support function s_K in $\mathscr{S}^2$ and radial density r_K. Let D denote any of the measures of deviation D_A, D_ℓ, or D_p. Let $P_n{}^*$ be a best inscribing approximation of K in $\mathscr{P}_n$ (relative to D) and let $\bar{P}_n$ in $\mathscr{P}_n$ inscribe K and satisfy* (3.12). *Then*

$$\lim_{n\to\infty} D(P_n{}^*, K)/D(\bar{P}_n, K) = 1.$$

The approximations $\bar{P}_n$ are asymptotically efficient.

Remark 1. All of these theorems have direct translations into results on the convergence, characterization, and construction of best function approximations by variable-knot interpolating splines. These translations can be inferred from the isometries we have noted between $\mathscr{K}$ and $\mathscr{S}$ and from the analysis of Sections 5 and 6.

Remark 2. In addition to the convergence results stated in these theorems, one can easily deduce bounds on the deviations $D(K, P_n)$ for fixed sets K and P_n. These are reflected in the remainder expressions of Sections 5 and 6.

4. Interval Segmentation

The partitions of sets K and P_n introduced in Section 3 and depicted in Figs. 3 and 5 allow us to decompose the global measures of deviation $D(K, P_n)$ into expressions that reflect "local" deviations between components $K^{(\nu)}$ and $P_n^{(\nu)}$. Consider again the case of a circumscribing approximation P_n of K and introduce the partitions

$$P_n = \bigcup_{\nu=1}^{n} P_n^{(\nu)} \qquad \text{and} \qquad K = \bigcup_{\nu=1}^{n} K^{(\nu)}$$

described after Theorem 3. For decompositions of the area, perimeter, and p-norm deviations we obtain the expressions

$$D_A(K, P_n) = \sum_{\nu=1}^{n} D_A(K^{(\nu)}, P_n^{(\nu)}), \tag{4.1}$$

$$D_\ell(K, P_n) = \sum_{\nu=1}^{n} D_\ell(K^{(\nu)}, P_n^{(\nu)}), \tag{4.2}$$

and

$$D_p{}^p(K, P_n) = \sum_{\nu=1}^{n} D_p{}^p(K^{(\nu)}, P_n^{(\nu)}). \tag{4.3}$$

These global deviations are expressed as *sums* of local deviations. The expression for the Hausdorff metric takes a different form:

$$D_\infty(K, P_n) = \max_{1 \leqslant \nu \leqslant n} D_\infty(K^{(\nu)}, P_n^{(\nu)}). \tag{4.4}$$

Similar expressions are obtained for the decomposed inscribed figures; the order of the arguments of the deviations is reversed to be consistent with our earlier definitions.

This process of partitioning sets is equivalent to a process of partitioning or segmenting the interval $[0, 2\pi)$ at the points $(\theta_1, ..., \theta_n)$ that parametrize the circumscribed (inscribed) polygon. Indeed, from the correspondence between support functions observed in Section 3, each of the local deviations $D(K^{(\nu)}, P_n^{(\nu)})$ in (4.1)–(4.4) only depends on s_K and s_{P_n} on the interval $[\theta_\nu, \theta_{\nu+1}]$. From the relationships between s_K and s_{P_n}, e.g., $s_K \leqslant s_{P_n}$, $s_K(\theta_\nu) = s_{P_n}(\theta_\nu)$, and $Ls_{P_n} = 0$, we can express s_{P_n} in terms of s_K to say further that the local deviation $D(K^{(\nu)}, P_n^{(\nu)})$ is some functional $e(\cdot, \cdot, \cdot)$ of s_K and the interval $[\theta_\nu, \theta_{\nu+1}]$.

These observations fit the polygon-approximation problems into a general framework of interval segmentation problems described in McClure [4]. Other references to earlier work and to additional applications of these problems and methods are described in [4]; the references there by Sacks and Ylvisaker motivate some of the general results, and their work also represents a very nice application of this approach to asymptotic analysis. We will give separate consideration to additive functionals like (4.1)–(4.3) and to functionals of the distinct form (4.4).

Let f be a real-valued function on an interval $[a, b]$ and let T_n denote a partition of $[a, b]$ of the form $T_n = (t_0, t_1, ..., t_n, t_{n+1})$, where

$$a = t_0 < t_1 < \cdots < t_n < t_{n+1} = b.$$

Consider a functional $E(f, T_n)$ that admits a decomposition of the following additive form, relative to T_n:

$$E(f, T_n) = \sum_{\nu=0}^{n} e(f; t_\nu, t_{\nu+1}). \tag{4.5}$$

This compares with (4.1)–(4.3) when we associate f with s_K, $[a, b]$ with $[0, 2\pi]$, and T_n with $(\theta_1, ..., \theta_n)$.

In the approximation problems and the theorems for them cited in Section 3 we are concerned with $E(f, T_n)$ for a prescribed partition T_n and also with

$$E_n(f) = \inf_{T_n} E(f, T_n). \tag{4.6}$$

$E_n(f)$ identifies with the minimum deviation $D(K, P_n{}^*)$ (Eqs. (3.1) and (3.9)) between a convex body K and a best approximation of K in $\mathscr{P}_n$.

Three assumptions on the terms $e(f; t_\nu, t_{\nu+1})$ that contribute to $E(f, T_n)$ suffice to obtain results like those in Section 3. These will follow in the present specific context from the assumption that s_K is in $\mathscr{S}^2$. In other problems these are commonly satisfied by imposing regularity conditions on f.

A1. For any (α, β) satisfying $a \leqslant \alpha < \beta \leqslant b$, $e(f; \alpha, \beta) \geqslant 0$. Further, $e(f; \cdot, \cdot)$ is subadditive over contiguous subintervals of $[a, b]$; that is, if $a \leqslant \alpha < \beta < \gamma \leqslant b$ then

$$e(f; \alpha, \beta) + e(f; \beta, \gamma) \leqslant e(f; \alpha, \gamma).$$

A2. There is a function Jf on $[a, b]$, associated with f, and a constant $m > 1$ such that (i) Jf is nonnegative and piecewise continuous on $[a, b]$, admitting at worst a finite number of jump discontinuities, and (ii)

$$\lim_{h \downarrow 0} e(f; \alpha, \alpha + h)/h^m = Jf(\alpha+).$$

This limit is uniform in that the difference $| Jf(\alpha+) - e(f; \alpha, \alpha + h)/h^m |$ can be made uniformly small when $(\alpha, \alpha + h)$ is contained in an interval where Jf is continuous.

A3. $e(f; \alpha, \beta)$ depends continuously on (α, β).

Assumption A1 implies that $E(f, T)$ is nonnegative. The subadditivity assumption is equivalent to the assertion that finer partitions of $[a, b]$ reduce $E(f, T)$.

In applications of the general results that follow it is usually easy to verify assumptions A1 and A3. More work is involved in verifying A2 since the form of Jf and the value of m must be deduced. This is what we develop in Sections 5 and 6.

We now relate several results concerning the asymptotic behavior of $E_n(f)$ (4.6) and $E(f, T_n)$ (4.5). Only the first lemma is proved here. The other results are developed in [4]. We offer a new proof of the first result since it is based on somewhat weaker assumptions than those used in [4].

LEMMA 1. *If assumptions* A1–A3 *hold for* $e(f;\alpha,\beta)$, *then*

$$\liminf_{n\to\infty} n^{m-1}E_n(f) \geqslant \left(\int_a^b (Jf(s))^{1/m}\,ds\right)^m. \tag{4.7}$$

Proof. Fix f and let Jf denote its image in $PC^+[a,b]$ (nonnegative, piecewise continuous functions on $[a,b]$).

By the continuity assumption A3, the functional $E(f,T_n)$ is continuous in the variables $(t_1,\ldots,t_n)$ over the compact region $a\leqslant t_1\leqslant\cdots\leqslant t_n\leqslant b$. Thus the minimum of $E(f,T_n)$ is attained; there is an optimal n-point partition $T_n{}^*$ satisfying

$$E_n(f) = E(f, T_n{}^*).$$

Denote $T_n{}^* = (t_0, t_1,\ldots, t_n, t_{n+1})$, where the dependence of t_ν on n is not explicitly noted, but will be implicit in context.

We first prove that $E_n(f)$ is of order n^{1-m} by bounding it above by a particular value $E(f,\cdot)$ of this order. Let $\{\tau_1,\ldots,\tau_k\}$ be the fixed discontinuity points of Jf in (a,b). Let U_{n-k} denote the uniform partition of $[a,b]$ with the $n-k$ interior points $u_\nu = a+\nu(b-a)/(n-k+1)$, for $\nu=1,\ldots,n-k$. The partition $S_n = U_{n-k}\cup\{\tau_1,\ldots,\tau_k\}$ comprises no more than n interior points of $[a,b]$. By the optimality of $T_n{}^*$, therefore,

$$E_n(f)\leqslant E(f,S_n).$$

Straightforward calculation, based on assumption A2, shows

$$\lim_{n\to\infty} n^{m-1}E(f,S_n) = \int_a^b [Jf(s)]\,(b-a)^{m-1}\,ds,$$

and thus

$$\limsup_{n\to\infty} n^{m-1}E_n(f) \leqslant \int_a^b [Jf(s)]\,(b-a)^{m-1}\,ds, \tag{4.8}$$

or $E_n(f) = O(n^{1-m})$.

From this order of convergence, we can conclude that certain distinguished subintervals of the partition $T_n{}^*$ become arbitrarily small as n increases. Denote

$$F^+ = \{t\in[a,b]\colon Jf(t)>0\}$$

and

$$F_c = \{t\in[a,b]\colon Jf \text{ is continuous at } t\}.$$

Let $h_n(t)$ be the length of the interval $[t_\nu, t_{\nu+1})$ of $T_n{}^*$ that contains t. If $t\in F^+\cap F_c$, then $\lim_{n\to\infty} h_n(t) = 0$. Otherwise, there is a positive value

h_0 such that for some τ, $\tau \leqslant t \leqslant \tau + h_0$, Jf is continuous and positive on $[\tau, \tau + h_0)$ and, for a subsequence of the partitions $T_n{}^*$,

$$t_\nu \leqslant \tau \leqslant t \leqslant \tau + h_0 \leqslant t_{\nu+1},$$

But, by A1 and A2, then

$$e(f; t_\nu, t_{\nu+1}) \geqslant e(f; \tau, \tau + h_0) > 0.$$

In turn, this would imply

$$n^{m-1} E_n(f) \geqslant n^{m-1} e(f; t_\nu, t_{\nu+1}) \geqslant n^{m-1} e(f; \tau, \tau + h_0).$$

The right side diverges as n increases, contradicting (4.8).

We can now separate $E(f, T_n{}^*)$ into parts where assumption A2 can or cannot be invoked. Define the set $K_n{}^*$ associated with $T_n{}^*$ by

$$K_n{}^* = \bigcup_{\nu=0}^{n} \{[t_\nu, t_{\nu+1}): \{t_\nu, t_{\nu+1}\} \subseteq T_n{}^*, [t_\nu, t_{\nu+1}) \subseteq F^+ \cap F_c\}.$$

By the definition of $K_n{}^*$, the asymptotic expression in assumption A2 can be used on the subintervals of $T_n{}^*$ that comprise $K_n{}^*$. Now on $K_n{}^*$ define a step-function approximation $J_n{}^*$ of Jf by

$$J_n{}^*(t) = \frac{e(f; t_\nu, t_{\nu+1})}{(t_{\nu+1} - t_\nu)^m} \qquad \text{for } t_\nu \leqslant t < t_{\nu+1}, \quad [t_\nu, t_{\nu+1}) \subseteq K_n{}^*,$$

and let $J_n{}^*(t) = 0$ for $t \notin K_n{}^*$. By the argument that $h_n(t) \to 0$ for $t \in F^+ \cap F_c$, it follows that every t in $F^+ \cap F_c$ is eventually in a set $K_n{}^*$, for n sufficiently large. The indicator functions $I_{K_n{}^*}$ of $K_n{}^*$ converge pointwise to $I_{F^+ \cap F_c}$. Also, from A2, $\lim_{n\to\infty} J_n{}^*(t) = Jf(t)$ for $t \in F^+ \cap F_c$. Since $F^+ \cap F_c$ differs from F^+ by a set of measure zero, together these observations yield

$$\lim_{n\to\infty} I_{K_n{}^*}(t)\, J_n{}^*(t) = Jf(t) \quad \text{a.e.} \tag{4.9}$$

Also, the uniform convergence assumption in A2 implies that the functions $J_n{}^*$ are uniformly bounded on $[a, b]$.

Now separate the terms in the sum $E(f, T_n{}^*)$ for which $[t_\nu, t_{\nu+1}) \subseteq K_n{}^*$, and let the summation symbol $\sum^*$ denote the sum over the values ν, $0 \leqslant \nu \leqslant n$, for which $[t_\nu, t_{\nu+1}) \subseteq K_n{}^*$:

$$E_n(f) = E(f, T_n{}^*) \geqslant \sum{}^* e(f; t_\nu, t_{\nu+1}),$$

by A1. Apply assumption A2 to the terms in the reduced sum to write

$$E_n(f) \geqslant \sum^* J_n{}^*(t_\nu)\, h_\nu{}^m,$$

where $h_\nu = t_{\nu+1} - t_\nu$. By Hölder's inequality,

$$\sum^* [J_n{}^*(t_\nu)]^{1/m}\, h_\nu \leqslant \left(\sum^* J_n{}^*(t_\nu)\, h_\nu{}^m\right)^{1/m} \left(\sum^* 1\right)^{(m-1)/m}$$

$$\leqslant \left(\sum^* J_n{}^*(t_\nu)\, h_\nu{}^m\right)^{1/m} n^{(m-1)/m}.$$

Equivalently,

$$\sum^* J_n{}^*(t_\nu)\, h_\nu{}^m \geqslant n^{1-m} \left(\sum^* [J_n{}^*(t_\nu)]^{1/m}\, h_\nu\right)^m = n^{1-m} \left(\int_a^b I_{K_n{}^*}(s)\, [J_n{}^*(s)]^{1/m}\, ds\right)^m.$$

Thus,

$$n^{m-1} E_n(f) \geqslant \left(\int_a^b I_{K_n{}^*}(s)\, [J_n{}^*(s)]^{1/m}\, ds\right)^m.$$

By the bounded convergence theorem and (4.9),

$$\liminf_{n\to\infty} n^{m-1} E_n(f) \geqslant \left(\int_a^b (Jf(s))^{1/m}\, ds\right)^m;$$

the proof is complete.

We can actually conclude that

$$\lim_{n\to\infty} n^{m-1} E_n(f) = \left(\int_a^b (Jf(s))^{1/m}\, ds\right)^m.$$

This is established as the main convergence result in [4]. An easier way to draw this conclusion is based on a construction of n-point partitions T_n for which $E(f, T_n)$ converges to a limit arbitrarily close to the lower bound (4.7). Such partitions are described by density functions on $[a, b]$.

Let g be a strictly positive, bounded, piecewise-continuous function on $[a, b]$, which is normalized so that $\int_a^b g(s)\, ds = 1$. Let G denote its integral, $G(t) = \int_a^t g(s)\, ds$. For any integer $n \geqslant 1$, define an n-point partition $T_n(g)$ of $[a, b]$ by

$$T_n(g) = \{t_\nu \in [a, b] \colon G(t_\nu) = \nu/(n+1),\ \nu = 0, 1, \ldots, n+1\}. \qquad (4.10)$$

The points t_ν of $T_n(g)$ are uniquely defined through inversion of the distribution function G.

The following result is quoted without proof from [4].

LEMMA 2. *Suppose assumptions* A1–A3 *hold for* $e(f; \alpha, \beta)$. *Let* $\{\tau_1, \ldots, \tau_n\}$ *be the discontinuity points of* Jf *in* (a, b). *Let* g *be a positive, bounded, piecewice-continuous density on* $[a, b]$ *with associated distribution* G. *For the partitions* $T_n = T_{n-k}(g) \cup \{\tau_1, \ldots, \tau_k\}$ *defined by* (4.10)

$$\lim_{n\to\infty} n^{m-1} E(f, T_n) = \int_a^b Jf(s)\,[g(s)]^{1-m}\,ds. \tag{4.11}$$

By a simple variational argument on g in (4.11) the minimum value of the limit is obtained to prove that the lower bound in (4.7) is attained.

COROLLARY 2.1. *If assumptions* A1–A3 *hold for* $e(f; \alpha, \beta)$, *then*

$$\lim_{n\to\infty} n^{m-1} E_n(f) = \left(\int_a^b (Jf(s))^{1/m}\,ds\right)^m. \tag{4.12}$$

When Jf is strictly positive on $[a, b]$, Lemma 2 provides a method for constructing asymptotically efficient partitions.

COROLLARY 2.2. *Suppose assumptions* A1–A3 *hold for* $e(f; \alpha, \beta)$ *and that* Jf *is strictly positive on* $[a, b]$. *Let* $\{\tau_1, \ldots, \tau_k\}$ *be the discontinuity points of* Jf *in* (a, b) *and set*

$$g_f(t) = (Jf(t))^{1/m} \left(\int_a^b (Jf(s))^{1/m}\,ds\right)^{-1}.$$

The partitions $T_n = T_{n-k}(g_f) \cup \{\tau_1, \ldots, \tau_k\}$ *are asymptotically efficient; that is,*

$$\lim_{n\to\infty} n^{m-1} E(f, T_n) = \left(\int_a^b (Jf(s))^{1/m}\,ds\right)^m.$$

The result follows from substitution of g_f for g in (4.11).

This last result provides sufficient conditions for specifying asymptotically efficient partitions in terms of a distribution function defined by Jf. A converse result that characterizes the optimal partitions $T_n{}^*$ introduced in Lemma 1 is also proven in [4].

Let $\{T_n{}^*\}$ be a sequence of optimal partitions; $E_n(f) = E(f, T_n{}^*)$. Define the empirical distribution $G_n{}^*$ of the points in $T_n{}^*$ by

$$G_n{}^*(t) = (n+2)^{-1}\,\mathrm{card}\{t_\nu \in T_n{}^*\colon t_\nu \leqslant t\} \qquad \text{for } a \leqslant t \leqslant b. \tag{4.13}$$

Define also the distribution function G_f by

$$G_f(t) = \left(\int_a^t (Jf(s))^{1/m}\,ds\right)\left(\int_a^b (Jf(s))^{1/m}\,ds\right)^{-1} \qquad \text{for } a \leqslant t \leqslant b. \tag{4.14}$$

The optimal partitions $T_n{}^*$ are characterized by the following result.

LEMMA 3. *Suppose assumptions* A1–A3 *hold for* $e(f; \alpha, \beta)$. *Let* $T_n{}^*$ *be an optimal n-point partition;* $E_n(f) = E(f, T_n{}^*)$. *Define* $G_n{}^*$ *and* G_f *by* (4.13) *and* (4.14). *If* Jf *is not identically zero on* $[a, b]$, *then*

$$\lim_{n\to\infty} G_n{}^*(t) = G_f(t)$$

for all t *in* $[a, b]$.

The proof in [4] appeals to Helly's selection theorem on compactness of proper distribution functions.

In addition to such density and distribution descriptions of optimal and efficient partitions as Lemmas 2 and 3 provide, other results in [4] relate optimal partitions to so-called "balanced-error partitions." A "balanced-error partition" is one for which the separate local terms $e(f; t_\nu, t_{\nu+1})$ contributing to $E(f, T)$ assume the same value. From Lemma 3 it can be shown that optimal partitions tend to balance these terms as n increases (see [4]). In the reverse direction, we show that partitions prescribed by this balancing condition are asymptotically efficient.

When assumptions A1 and A3 hold for $e(f; \alpha, \beta)$, it is easily argued [4] that an n-point partition $\bar{T}_n$ exists that satisfies

$$\begin{aligned}\bar{T}_n = \{t_\nu \in [a, b]: t_0 = a,\ t_{n+1} = b, \text{ and } e(f; t_{\nu-1}, t_\nu) = e(f; t_\nu, t_{\nu+1})\\ \text{for } \nu = 1,..., n\}.\end{aligned} \tag{4.15}$$

For such a partition, $E(f, \bar{T}_n) = (n + 1)\, e(f; t_\nu, t_{\nu+1})$. In [4] the following result is proved.

LEMMA 4. *Suppose assumptions* A1–A3 *hold for* $e(f; \alpha, \beta)$. *The partitions* $\bar{T}_n$ *defined by* (4.15) *are asymptotically efficient; that is,*

$$\lim_{n\to\infty} n^{m-1} E(f, \bar{T}_n) = \left(\int_a^b (Jf(s))^{1/m}\, ds\right)^m.$$

These results provide powerful tools for the asymptotic analysis of functionals expressed in the additive form (4.5). Analogous results, which are easier to prove, hold for the forms which arise in the consideration of sup–norm approximations. In this case, global error functionals admit descriptions in the form

$$\mathscr{E}(f, T_n) = \max_{0\leqslant\nu\leqslant n} e(f; t_\nu, t_{\nu+1}), \tag{4.16}$$

and optimization with respect to the partition $T_n = (t_0, t_1, ..., t_n, t_{n+1})$ points to consideration of the value

$$\mathscr{E}_n(f) = \min_{T_n} \max_{0\leqslant\nu\leqslant n} e(f; t_\nu, t_{\nu+1}).$$

Compare (4.16) and (4.4). The results of [4] concerning $\mathscr{E}(f, T_n)$ are related here.

We impose the following assumptions on the "local" terms $e(f; \alpha, \beta)$ in (4.16).

B1. For any (α, β) satisfying $a \leqslant \alpha < \beta \leqslant b$, $e(f; \alpha, \beta) \geqslant 0$. Further, if $a \leqslant \alpha < \beta < \gamma \leqslant b$, then

$$\max\{e(f; \alpha, \beta), e(f; \beta, \gamma)\} \leqslant e(f; \alpha, \gamma).$$

B2. Same as A2, but replace the condition $m > 1$ with $m > 0$.

B3. Same as A3.

Under these assumptions, optimal n-point partitions $T_n{}^*$ exist; that is,

$$\mathscr{E}_n(f) = \mathscr{E}(f, T_n{}^*).$$

Further, the min–max description of $\mathscr{E}_n(f)$ implies that optimal partitions are balanced partitions. If $T_n{}^* = (t_0, t_1, ..., t_n, t_{n+1})$, then

$$\mathscr{E}(f, T_n{}^*) = e(f; t_\nu, t_{\nu+1}) \qquad \text{for } \nu = 0, ..., n.$$

In analogy to the lemmas above, we obtain the following results.

LEMMA 5. *If assumptions* B1–B3 *hold for* $e(f; \alpha, \beta)$, *then*

$$\lim_{n\to\infty} n^m \mathscr{E}_n(f) = \left(\int_a^b (Jf(s))^{1/m} \, ds \right)^m.$$

LEMMA 6. *Suppose assumptions* B1–B3 *hold for* $e(f; \alpha, \beta)$. *Let* $T_n{}^*$ *be an optimal n-point partition;* $\mathscr{E}_n(f) = \mathscr{E}(f, T_n{}^*)$. *Define* $G_n{}^*$ *and* G_f *by* (4.13) *and* (4.14). *If* Jf *is not identically zero on* $[a, b]$, *then*

$$\lim_{n\to\infty} G_n{}^*(t) = G_f(t)$$

for all t in $[a, b]$.

Finally, there is an analog of Lemma 2 that describes the efficiency of partitions defined by a density.

LEMMA 7. *Suppose assumptions* B1–B3 *hold for* $e(f; \alpha, \beta)$. *Let* $\{\tau_1, ..., \tau_k\}$ *be the discontinuity points of* Jf *in* (a, b). *Let g be a positive, bounded, piecewise-continuous density on* $[a, b]$ *with associated distribution G. For the partitions* $T_n = T_{n-k}(g) \cup \{\tau_1, ..., \tau_k\}$ *defined by* (4.10),

$$\lim_{n\to\infty} n^m \mathscr{E}(f, T_n) = \operatorname*{ess\,sup}_{[a,b]} g^{-m} Jf.$$

COROLLARY 7.1. *Suppose assumptions* B1–B3 *hold for* $e(f; \alpha, \beta)$ *and that* Jf *is strictly positive on* $[a, b]$. *Let* $\{\tau_1, \ldots, \tau_k\}$ *be the discontinuity points of* Jf *in* (a, b) *and set*

$$g_f(t) = (Jf(t))^{1/m} \left(\int_a^b (Jf(s))^{1/m}\, ds\right)^{-1}.$$

The partitions $T_n = T_{n-k}(g_f) \cup \{\tau_1, \ldots, \tau_k\}$ *are asymptotically efficient; that is*

$$\lim_{n\to\infty} n^m \mathscr{E}(f, T_n) = \left(\int_a^b (Jf(s))^{1/m}\, ds\right)^m.$$

The nature of these results and the assumptions on which they are based point the direction for our analyses of the convex-set approximation problems in the following two sections.

5. Circumscribed Polygons

Let K be a fixed convex body with support function s_K in $\mathscr{S}^2$. In this section P_n will refer to a member of $\mathscr{P}_n$ that circumscribes K. Recall the parametric representation $(\theta_1, \ldots, \theta_n)$ of P_n described before Theorem 2 in Section 3 and the decompositions of K and P_n into components $K^{(\nu)}$ and $P_n^{(\nu)}$ described after Theorem 3.

Three conditions completely characterize the spline support function s_{P_n} in terms of the support function s_K:

$$s_K(\theta) \leqslant s_{P_n}(\theta), \qquad 0 \leqslant \theta < 2\pi \tag{5.1}$$

$$s_{P_n}(\theta_\nu) = s_K(\theta_\nu), \qquad \nu = 1, \ldots, n \tag{5.2}$$

$$Ls_{P_n}(\theta) = 0, \qquad \theta \neq \theta_\nu . \tag{5.3}$$

(See (2.14).) These conditions are easily applied, in turn, to deduce the asymptotic behavior of $D(K, P_n)$.

We refer to Eqs. (4.1)–(4.4) and the discussion immediately following these. In a manner consistent with the notation of Section 4, define

$$\begin{aligned} e_\infty(s_K; \theta_\nu, \theta_{\nu+1}) &= D_\infty(K^{(\nu)}, P_n^{(\nu)}),\\ e_A(s_K; \theta_\nu, \theta_{\nu+1}) &= D_A(K^{(\nu)}, P_n^{(\nu)}),\\ e_\ell(s_K; \theta_\nu, \theta_{\nu+1}) &= D_\ell(K^{(\nu)}, P_n^{(\nu)}),\\ \text{and}\quad e_p(s_K; \theta_\nu, \theta_{\nu+1}) &= D_p{}^p(K^{(\nu)}, P_n^{(\nu)}). \end{aligned} \tag{5.4}$$

Apart from their interpretations as distances between convex sets, the terms

$e(s_K; \cdot, \cdot)$ may be viewed as distances between an arbitrary function s_K in $\mathscr{S}^2$ and a function s_{P_n} defined by (5.2) and (5.3); s_{P_n} is in the null space of L (5.3) and interpolates the values of s_K at θ_ν.

We adopt this latter view in carrying through the asymptotic analysis of $e(s_K; \theta_\nu, \theta_{\nu+1})$ as $\theta_{\nu+1} - \theta_\nu$ goes to zero. This local asymptotic analysis is directed at assumptions A2 and B2 of Section 4. The other assumptions there follow at once from the interpretation of $e(s_K; \cdot, \cdot)$ through (5.4).

A powerful tool in the evaluation of $e(s_K; \theta_\nu, \theta_{\nu+1})$ is Pólya's mean value theorem [5]. This analog of the familiar mean value theorem for the derivative says simply that

$$s_{P_n}(\theta) - s_K(\theta) = [Ls_K(\xi)]\, U(\theta), \qquad \theta_\nu \leqslant \theta \leqslant \theta_{\nu+1} \tag{5.5}$$

where

$$LU \equiv -1 \qquad \text{on} \qquad [\theta_\nu, \theta_{\nu+1}], \tag{5.6}$$

$$U(\theta_\nu) = U(\theta_{\nu+1}) = 0, \tag{5.7}$$

and ξ is an intermediate point $\theta_\nu \leqslant \xi \leqslant \theta_{\nu+1}$; the conditions for (5.5) are $s_K \in \mathscr{S}^2$, Eqs. (5.2) and (5.3), and $\theta_{\nu+1} - \theta_\nu < \pi$.

Equation (5.5) reduces the asymptotic analysis of $e(s_K; \theta_\nu, \theta_{\nu+1})$ to the simpler analysis of the function U prescribed by (5.6) and (5.7). Indeed, from (5.5), the continuity of $r_K = Ls_K$, and the intermediate value theorem,

$$e_\infty(s_K; \theta_\nu, \theta_{\nu+1}) = r_K(\xi) \max_{\theta_\nu \leqslant \theta \leqslant \theta_{\nu+1}} U(\theta), \tag{5.8}$$

$$e_\ell(s_K; \theta_\nu, \theta_{\nu+1}) = r_K(\xi) \int_{\theta_\nu}^{\theta_{\nu+1}} U(\theta)\, d\theta, \tag{5.9}$$

and

$$e_p(s_K; \theta_\nu, \theta_{\nu+1}) = [r_K(\xi)]^p \int_{\theta_\nu}^{\theta_{\nu+1}} [U(\theta)]^p\, d\theta. \tag{5.10}$$

The corresponding expression for the local area deviation requires one initial simplification. For $D_A(K^{(\nu)}, P_n^{(\nu)})$, the limits on the integral (2.6) become θ_ν and $\theta_{\nu+1}$. By integrating the portion $[\dot{s}_{P_n}^2 - \dot{s}_K{}^2]$ by parts and using (5.2) and (5.3), this integral reduces to

$$e_A(s_K; \theta_\nu, \theta_{\nu+1}) = \tfrac{1}{2} \int_{\theta_\nu}^{\theta_{\nu+1}} (s_{P_n}(\theta) - s_K(\theta))\, r_K(\theta)\, d\theta.$$

From the remainder expression (5.5), this becomes

$$e_A(s_K; \theta_\nu, \theta_{\nu+1}) = \tfrac{1}{2}[r_K(\xi)]^2 \int_{\theta_\nu}^{\theta_{\nu+1}} U(\theta)\, d\theta. \tag{5.11}$$

Note that ξ is an intermediate point, $\theta_\nu \leqslant \xi \leqslant \theta_{\nu+1}$, so that continuity of r_K implies

$$r_K(\xi) = r_K(\theta_\nu) + o(1).$$

It only remains to give the asymptotic form of the L_p-norms of U on $[\theta_\nu, \theta_{\nu+1}]$. For convenience, assume $\theta_\nu = 0$ and $\theta_{\nu+1} = h > 0$. From (5.6) and (5.7)

$$U(\theta) = -1 + \cos\theta + \frac{1 - \cos h}{\sin h}\sin\theta, \qquad 0 \leqslant \theta \leqslant h.$$

Change variables to write $U(\theta) = u_h(\tau)$, where $\theta = h\tau$ and

$$u_h(\tau) = -1 + \cos(h\tau) + \frac{1 - \cos h}{\sin h}\sin(h\tau), \qquad 0 \leqslant \tau \leqslant 1.$$

It is easily shown that $u_h(\tau)/h^2$ converges uniformly to $\frac{1}{2}\tau(1 - \tau)$ on $[0, 1]$ as h goes to zero. From observing this convergence, we obtain

$$h^{-2} \max_{[\theta_\nu, \theta_{\nu+1}]} U(\theta) = h^{-2} \max_{[0,1]} u_h(\tau) \to \tfrac{1}{8}$$

and

$$h^{-2p-1} \int_{\theta_\nu}^{\theta_{\nu+1}} [U(\theta)]^p \, d\theta = \int_0^1 [u_h(\tau)/h^2]^p \, d\tau \to 2^{-p} \int_0^1 [\tau(1 - \tau)]^p \, d\tau$$

as $h \to 0$.

These limits and Eqs. (5.8)–(5.11) yield

$$e_\infty(s_K; \theta_\nu, \theta_{\nu+1}) = \tfrac{1}{8} r_K(\theta_\nu)\, h^2 + o(h^2), \tag{5.12}$$

$$e_\ell(s_K; \theta_\nu, \theta_{\nu+1}) = \tfrac{1}{12}\, r_K(\theta_\nu)\, h^3 + o(h^3), \tag{5.13}$$

$$e_p(s_K; \theta_\nu, \theta_{\nu+1}) = 2^{-p}[r_K(\theta_\nu)]^2 \int_0^1 [\tau(1 - \tau)]^p \, d\tau \, h^{2p+1} + o(h^{2p+1}), \tag{5.14}$$

and

$$e_A(s_K; \theta_\nu, \theta_{\nu+1}) = \tfrac{1}{24}\, [r_K(\theta_\nu)]^2\, h^3 + o(h^3), \tag{5.15}$$

where $h = \theta_{\nu+1} - \theta_\nu$. The precise asymptotic forms supposed in assumptions A2 and B2 of Section 4 are given by these last four equations. Theorems 1–4 follow at once from the general results of Section 4 and the sets of equations (5.12)–(5.15), (5.4), and (4.1)–(4.4).

Error bounds for an approximation are easily deduced from Eqs. (5.8)–(5.11).

6. Inscribed Polygons

The local error analysis for inscribed polygons is slightly more complicated than the corresponding analysis for circumscribed polygons in Section 5. This occurs because the parameters $(\theta_1, \ldots, \theta_n)$ that describe the inscribed figure do not coincide with the knots of its spline support function.

Let K be a fixed convex body with support function s_K in $\mathscr{S}^2$. P_n will refer to a member of $\mathscr{P}_n$ that inscribes K. Recall the parametric representation $(\theta_1, \ldots, \theta_n)$ of P_n described after Theorem 5 in Section 3 and the decompositions of K and P_n into components $K^{(\nu)}$ and $P_n^{(\nu)}$ described after Theorem 7.

The parameters θ_ν of P_n are associated with vertices. The knots η_ν of s_{P_n}, however, identify with faces of P_n. This means that the knots η_ν and parameters θ_ν are related by inequalities

$$\theta_\nu < \eta_\nu < \theta_{\nu+1}. \tag{6.1}$$

Since $P_n \subseteq K$,

$$s_{P_n}(\theta) \leqslant s_K(\theta), \qquad 0 \leqslant \theta < 2\pi \tag{6.2}$$

and, since P_n inscribes K,

$$s_{P_n}(\theta_\nu) = s_K(\theta_\nu), \qquad \nu = 1, \ldots, n. \tag{6.3}$$

Equations (6.2) and (6.3) and the continuity of $\dot{s}_{P_n}$ on intervals $(\eta_{\nu-1}, \eta_\nu)$ imply, in addition,

$$\dot{s}_{P_n}(\theta_\nu) = \dot{s}_K(\theta_\nu). \tag{6.4}$$

The last three equations, together with

$$Ls_{P_n}(\theta) = 0, \qquad \theta \neq \eta_\nu \tag{6.5}$$

completely characterize s_{P_n} in terms of s_K. The double interpolation conditions (6.3) and (6.4) will allow us to invoke Pólya's theorem, as in Section 5.

Refer to Eqs. (4.1)–(4.4) and the discussion following them. Define

$$\begin{aligned} e_\infty(s_K; \theta_\nu, \theta_{\nu+1}) &= D_\infty(P_n^{(\nu)}, K^{(\nu)}), \\ e_A(s_K; \theta_\nu, \theta_{\nu+1}) &= D_A(P_n^{(\nu)}, K^{(\nu)}), \\ e_\ell(s_K; \theta_\nu, \theta_{\nu+1}) &= D_\ell(P_n^{(\nu)}, K^{(\nu)}), \\ \text{and} \quad e_p(s_K; \theta_\nu, \theta_{\nu+1}) &= D_p{}^p(P_n^{(\nu)}, K^{(\nu)}). \end{aligned} \tag{6.6}$$

We follow the pattern set in Section 5 and regard the terms $e(s_K; \cdot, \cdot)$ as distances between an arbitrary function s_K in $\mathscr{S}^2$ and a function s_{P_n} defined by (6.3)–(6.5). The analysis is directed at the verification of assumptions A2 and B2 of Section 4. The other assumptions in that section follow readily from the interpretation of $e(s_K; \cdot, \cdot)$ through (6.6).

Now s_{P_n} is twice continuously differentiable on $[\theta_\nu, \eta_\nu)$ and on $(\eta_\nu, \theta_{\nu+1}]$. From Pólya's mean value theorem [5] we obtain two remainder expressions,

$$s_K(\theta) - s_{P_n}(\theta) = [Ls_K(\xi_1)]\, U_1(\theta), \qquad \theta_\nu \leqslant \theta \leqslant \eta_\nu$$

and

$$s_K(\theta) - s_{P_n}(\theta) = [Ls_K(\xi_2)]\, U_2(\theta), \qquad \eta_\nu \leqslant \theta \leqslant \theta_{\nu+1} \tag{6.7}$$

where

$$\begin{aligned} LU_1 &\equiv 1 \qquad \text{on} \qquad [\theta_\nu, \eta_\nu], \\ LU_2 &\equiv 1 \qquad \text{on} \qquad [\eta_\nu, \theta_{\nu+1}], \end{aligned} \tag{6.8}$$

$$\begin{aligned} U_1(\theta_\nu) &= \dot{U}_1(\theta_\nu) = 0, \\ U_2(\theta_{\nu+1}) &= \dot{U}_2(\theta_{\nu+1}) = 0, \end{aligned} \tag{6.9}$$

and ξ_1 and ξ_2 are intermediate points, $\theta_\nu \leqslant \xi_1 \leqslant \eta_\nu \leqslant \xi_2 \leqslant \theta_{\nu+1}$. The conditions for (6.7) are $s_K \in \mathscr{S}^2$, Eqs. (6.3)–(6.5), and $\theta_{\nu+1} - \theta_\nu < \pi$.

From the remainder expressions (6.7), the continuity of $r_K = Ls_K$, and the intermediate value theorem, we obtain

$$e_\infty(s_K; \theta_\nu, \theta_{\nu+1}) = r_K(\xi) \max\{\max_{[\theta_\nu,\eta_\nu]} U_1(\theta), \max_{[\eta_\nu,\theta_{\nu+1}]} U_2(\theta)], \tag{6.10}$$

$$e_\ell(s_K; \theta_\nu, \theta_{\nu+1}) = r_K(\xi) \left(\int_{\theta_\nu}^{\eta_\nu} U_1(\theta)\, d\theta + \int_{\eta_\nu}^{\theta_{\nu+1}} U_2(\theta)\, d\theta\right), \tag{6.11}$$

$$e_p(s_K; \theta_\nu, \theta_{\nu+1}) = [r_K(\xi)]^p \left(\int_{\theta_\nu}^{\eta_\nu} [U_1(\theta)]^p\, d\theta + \int_{\eta_\nu}^{\theta_{\nu+1}} [U_2(\theta)]^p\, d\theta\right); \tag{6.12}$$

ξ is an intermediate point, $\theta_\nu \leqslant \xi \leqslant \theta_{\nu+1}$. The local area deviation is simplified by performing the integration by parts indicated in Section 5. Since η_ν is a point of discontinuity of $\dot{s}_{P_n}$, we obtain

$$e_A(s_K; \theta_\nu, \theta_{\nu+1}) = \tfrac{1}{2} \int_{\theta_\nu}^{\theta_{\nu+1}} (s_K(\theta) - s_{P_n}(\theta))\, r_K(\theta)\, d\theta + \tfrac{1}{2}[s_K(\eta_\nu) - s_{P_n}(\eta_\nu)]\, \Delta \dot{s}_{P_n}(\eta_\nu).$$

This reduces to

$$e_A(s_K; \theta_\nu, \theta_{\nu+1}) = \tfrac{1}{2}[r_K(\xi)]^2 \left(\int_{\theta_\nu}^{\eta_\nu} U_1(\theta)\, d\theta + \int_{\eta_\nu}^{\theta_{\nu+1}} U_2(\theta)\, d\theta\right) + \tfrac{1}{2}[s_K(\eta_\nu) - s_{P_n}(\eta_\nu)]\, \Delta \dot{s}_{P_n}(\eta_\nu); \tag{6.13}$$

$\Delta \dot{s}_{P_n}(\eta_\nu)$ is the jump in the derivative $\dot{s}_{P_n}$ at η_ν.

The asymptotic forms of the L_p-norms of U_1 and U_2 on $[\theta_\nu, \eta_\nu]$ and $[\eta_\nu, \theta_{\nu+1}]$ are obtained from explicit representations of these functions. For convenience, set $\theta_\nu = 0$, $\eta_\nu = h_1$, and $\theta_{\nu+1} - \eta_\nu = h_2$. Let

$$h = h_1 + h_2 = \theta_{\nu+1}.$$

From (6.8) and (6.9)

$$U_1(\theta) = 1 - \cos\theta, \qquad 0 \leqslant \theta \leqslant h_1$$

and a similar expression holds for U_2. Change variables to write $U_1(\theta) = u_{h_1}(\tau)$, where $\theta = h_1\tau$ and

$$u_{h_1}(\tau) = 1 - \cos(h_1\tau), \qquad 0 \leqslant \tau \leqslant 1.$$

The quotient $u_{h_1}(\tau)/h_1^2$ converges uniformly to $\tau^2/2$ on $[0, 1]$ as h_1 goes to zero. From this uniform convergence and the parallel development for U_2, we obtain

$$h_1^{-2} \max_{[\theta_\nu,\eta_\nu]} U_1(\theta) = h_1^{-2} \max_{[0,1]} u_{h_1}(\tau) \to \tfrac{1}{2},$$

$$h_2^{-2} \max_{[\eta_\nu,\theta_{\nu+1}]} U_2(\theta) = h_2^{-2} \max_{[0,1]} u_{h_2}(\tau) \to \tfrac{1}{2},$$

$$h_1^{-2} U_1(\eta_\nu) = h_1^{-2} u_{h_1}(1) \to \tfrac{1}{2},$$

$$h_1^{-2p-1} \int_{\theta_\nu}^{\eta_\nu} [U_1(\theta)]^p \, d\theta = \int_0^1 [u_{h_1}(\tau)/h_1^2]^p \, d\tau \to [(2p+1)\, 2^p]^{-},$$

and

$$h_2^{-2p-1} \int_{\eta_\nu}^{\theta_{\nu+1}} [U_2(\theta)]^p \, d\theta = \int_0^1 [u_{h_2}(\tau)/h_2^2]^p \, d\tau \to [(2p+1)\, 2^p]^{-1}$$

as $h_1 \to 0$ and $h_2 \to 0$.

In addition to these expressions, we also need asymptotic estimates of h_1/h and h_2/h in order to determine the exact order in h of (6.10)–(6.12) and we need an estimate of $\Delta\dot{s}_{P_n}(\eta_\nu)$ for (6.13). These estimates are deduced from the conditions (6.3)–(6.5) that define s_{P_n} and from the fact that s_{P_n} is at least continuous at η_ν. Explicitly,

$$s_{P_n}(\theta) = s_K(\theta_\nu)\cos(\theta - \theta_\nu) + \dot{s}_K(\theta_\nu)\sin(\theta - \theta_\nu), \qquad \eta_{\nu-1} \leqslant \theta \leqslant \eta_\nu.$$

The continuity of s_{P_n} at η_ν gives us the equation

$$s_K(\theta_\nu)\cos h_1 + \dot{s}_K(\theta_\nu)\sin h_1 = s_K(\theta_{\nu+1})\cos h_2 - \dot{s}_K(\theta_{\nu+1})\sin h_2.$$

Express h_1 and h_2 as $h_1 = \alpha h$ and $h_2 = (1 - \alpha) h$, for some α, $0 \leqslant \alpha \leqslant 1$, and rewrite the last equation as

$$\dot{s}_K(\theta_{\nu+1}) \sin(1 - \alpha) h + \dot{s}_K(\theta_\nu) \sin(\alpha h) = s_K(\theta_{\nu+1}) \cos(1 - \alpha) h - s_K(\theta_\nu) \cos(\alpha h).$$

By equating terms of order h^2 in this equation, we obtain

$$0 = \tfrac{1}{2} r_K(\theta_\nu) [2\alpha - 1] h^2 + o(h^2),$$

which implies $\alpha = \frac{1}{2} + o(1)$ or

$$\lim_{h\to 0}(h_1/h) = \lim_{h\to 0}(h_2/h) = \tfrac{1}{2}, \tag{6.14}$$

when $r_K(\theta_\nu) > 0$. Similar analysis based on the explicit expression for s_{P_n} yields

$$\Delta \dot{s}_{P_n}(\eta_\nu) = r_K(\theta_\nu) h + o(h). \tag{6.15}$$

By using (6.14), (6.15), and the asymptotic expressions for the L_p-norms of U_1 and U_2 in Eqs. (6.10)–(6.13), we arrive at the following estimates:

$$e_\infty(s_K; \theta_\nu, \theta_{\nu+1}) = \tfrac{1}{8} r_K(\theta_\nu) h^2 + o(h^2), \tag{6.16}$$

$$e_\ell(s_K; \theta_\nu, \theta_{\nu+1}) = \tfrac{1}{24} r_K(\theta_\nu) h^3 + o(h^3), \tag{6.17}$$

$$e_p(s_K; \theta_\nu, \theta_{\nu+1}) = \frac{[r_K(\theta_\nu)]^p}{8^p(2p + 1)} h^{2p+1} + o(h^{2p+1}), \tag{6.18}$$

and

$$e_A(s_K; \theta_\nu, \theta_{\nu+1}) = \tfrac{1}{12} [r_K(\theta_\nu)]^2 h^3 + o(h^3), \tag{6.19}$$

where $h = \theta_{\nu+1} - \theta_\nu$. Here we have also invoked the continuity of r_K to write $r_K(\xi) = r_K(\theta_\nu) + o(1)$; recall $\theta_\nu \leqslant \xi \leqslant \theta_\nu + h$.

The precise asymptotic forms supposed in assumptions A2 and B2 of Section 4 are given by these last four equations. Theorems 5–8 follow by the results of Section 4 and the sets of equations (6.16)–(6.19), (6.6), and (4.1)–(4.4).

Error bounds for an inscribed polygonal approximation are deduced from Eqs. (6.10)–(6.13).

Note added in proof. The limit given in Theorem 5 for the Hausdorff metric is equivalent to a result of Fejes Tóth, Approximation by polygons and polyhedra, *Bull. Amer. Math. Soc.* **54** (1948), 431–438. It is derived there by different methods and under slightly different assumptions. Fejes Tóth pursues other interesting problems in his paper.

References

1. S. Carlsson and U. Grenander, Statistical approximation of plane convex sets, *Skand. Aktuarietidskr.* **3/4** (1967), 113–127.
2. C. H. Dowker, On minimum circumscribed polygons, *Bull. Amer. Math. Soc.* **50** (1944), 120–122.
3. H. G. Eggleston, "Problems in Euclidean Space: Application of Convexity," Pergamon Press, London, 1957.
4. D. E. McClure, Nonlinear segmented function approximation and analysis of line patterns, Center for Computer and Information Sciences and Division of Applied Mathematics Report, Brown University, 1973; *Quart. Appl. Math.* **33** (1975), 1–37.
5. G. Pólya, On the mean-value theorem corresponding to a given linear homogeneous differential equation. *Trans. Amer. Math. Soc.* **24** (1922), 312–324.
6. H. Poritsky, Convex spaces associated with a family of linear inequalities, *in* "Proceedings of Symposia on Pure Mathematics," Vol. VII, American Mathematical Society, Providence, RI 1963.
7. F. A. Valentine, "Convex Sets," McGraw-Hill, New York, 1964.
8. R. A. Vitale, A representation theorem for compact convex sets in the plane, Division of Applied Mathematics Report, Brown University, 1974.

Printed by the St Catherine Press Ltd., Tempelhof 37, Bruges, Belgium.