Combinatorial Properties of Shifted Complexes

by

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Abstract

In this thesis we study the class of shifted simplicial complexes. A simplicial complex on n nodes is shifted if there exists a labelling of the nodes by 1 through n such that for any face, replacing any node of the face with a node of smaller label results in a collection which is also a face.

A primary motivation for considering shifted complexes is a procedure called shifting. Shifting associates a shifted complex to any simplicial complex in a way which preserves meaningful information, while simplifying the structure of the complex. For example, shifting preserves the f-vector of a complex but always reduces the topology to a wedge of spheres. Shifting has proved to be a successful tool for answering questions regarding f-vectors.

While most of the previous results on shifted complexes are algebraic or topological in nature, we explore the combinatorics of shifted complexes. We give intrinsic characterization theorems for shifted complexes and shifted matroid complexes. In addition, we show results on the enumeration of shifted complexes and connections to various combinatorial structures.

Thesis Supervisor: Richard P. Stanley
Title: Norman Levinson Professor of Applied Mathematics

Acknowledgments

First I would like to thank my advisor, Richard Stanley, who was a wonderful advisor in so many ways. Always generous with his time and ideas, I can only hope to be such a good advisor someday.

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Contents

Chapter 1

Introduction

In this thesis we study the combinatorics of the class of shifted simplicial complexes. A simplicial complex on n nodes is shifted if there exists a labelling of the nodes by 1 through n such that for any face $\{v_1, v_2, \ldots, v_k\}$, replacing any v_i by a node with a smaller label results in a collection which is also a face. An equivalent definition of shifted complexes is in terms of order ideals. Consider the poset P_s on strings of increasing integers where $(x_1 < x_2 < \cdots < x_k)$ is less than $(y_1 < y_2 < \cdots < y_k)$ if $x_i \leq y_i$ for all i. Shifted complexes are exactly order ideals of P_s.

A primary motivation for the study of shifted complexes is the fact that any simplicial complex can be transformed into a shifted complex in a way which preserves meaningful information. There are many such shifting procedures. The original form was initially developed by Erdös, Ko, and Rado [5]. More recently, Kalai introduced the notion of algebraic shifting [7]. These shifting operations preserve certain properties of a complex while simplifying others. For example, shifting preserves the f-vector of a complex, but the topology is always reduced to a wedge of spheres. The f-vector of a complex is the vector $\{f_0, f_1, \ldots, f_d\}$, where f_i is the number of i dimensional faces of the complex. In fact, studying the f-vectors of simplicial complexes is the main application of algebraic shifting. Shifting is analogous to the compression procedure. While shifting maps a simplicial complex to an order ideal of the shifted partial ordering, compression maps a simplicial complex to an initial segment of the lexicographical ordering. Compression preserves the f-vector of a complex but loses essentially all other information. This procedure was used to prove the Kruskal-Katona theorem which characterizes the f-vectors of all simplicial complexes. Similarly, shifting has proved to be a successful tool for answering questions regarding f-vectors. For example, it has been used to characterize the f-vectors of simplicial complexes with prescribed Betti numbers [3]. A nice exposition of results in shifting and motivation for the study of shifted complexes is Vic Reiner's talk [12].

While much work has concentrated on studying shifting procedures, surprisingly little attention had been given to the class of shifted complexes itself except in the one dimensional case. One dimensional shifted complexes are the same as threshold graphs. Threshold graphs have been extensively studied [8] and this connection provides many insights into the structure of shifted complexes. We first extend a characterization of threshold graphs in terms of the vicinal preorder by defining a generalized vicinal preorder for simplicial complexes of any dimension. Our result shows that a complex is shifted if and only if the generalized preorder is total; see Theorem 3.1.1. Building on this, we arrive at a second characterization for shifted complexes in terms of obstructions. Threshold graphs are known to be characterized by a finite list of forbidden induced subgraphs. We give the range of the number of nodes on which there exist obstructions to shiftedness. In particular we show that there exists a finite number of obstructions to shiftedness in each dimension; see Theorem 3.2.2. Chapter 2 reviews many of the concepts of threshold graphs, and Chapter 3 contains the characterization theorems.

In Chapter 4 we consider the enumeration of shifted complexes. It is not difficult to show that there are 2^{n-1} threshold graphs on n nodes. We provide the number of two dimensional shifted complexes by giving a surprising but very simple bijection between these complexes and totally symmetric plane partitions. The question of enumerating totally symmetric plane partitions has a long history [14], and the result was eventually proved by Stembridge [16]. Beyond dimensions one and two, we can say very little about the total number of shifted complexes. The question has been looked at from the perspective of plane partitions, but there is not even a conjecture for dimension three. We do give the first few entries of the number of pure shifted complexes; see Figure 4-5.

In [1] a construction is given which forms a shifted complex from any subpartition of a fixed partition. If the subpartition is taken to be the first row, the construction yields a complex which is not only shifted but even the complex of independent sets of a matroid. This led to considering shifted matroid complexes. These shifted matroids have previously been considered as nested transversal matroids [11]. In Chapter 5 we provide two characterizations of this class of matroids. The first is a construction involving two basic operations, adding a disjoint vertex and starring a vertex; see Theorem 5.2.3. For the second result, recall that shifted complexes are order ideals in P_s. Our characterization theorem shows that shifted matroids are exactly the principal order ideals of P_s; see Theorem 5.4.1. We also show that this class of matroids is closed under duality and minors. In addition, we give results on the Tutte polynomials and broken circuit complexes of shifted matroids.

Chapter 6 contains results on other subclasses of shifted complexes. The original construction which led to shifted matroids can be generalized to form a class of linear extension complexes [1]. These complexes are shifted but not always matroids. We show partial results on determining the cases which do yield shifted matroids. Next, we consider the class of constructible shifted complexes. This is the class of all complexes formed from the two operations mentioned above - adding a vertex and starring a vertex. We give results on the form of the corresponding order ideals; see Theorem 6.2.1. In the next section we consider shifted independence complexes of graphs, also called flag complexes. We show that a graph is threshold if and only if its independence complex is shifted. In addition, we give a constructive characterization of these complexes. Finally, we show that shiftedness is preserved under a generalized independence complex construction.

Chapter 2

Definitions and Preliminaries

2.1 Shifted Complexes

Definition 2.1.1. *A simplicial complex on n nodes is shifted if there exists a labelling of the nodes by one through n such that for any face $\{v_1, v_2, \ldots, v_k\}$, replacing any v_i by a node with a smaller label results in a collection which is also a face.*

Example:

A simplicial complex with the faces $\{1, 2, 3\}$ and $\{2, 4\}$ must also have the face $\{1, 4\}$ to be shifted.

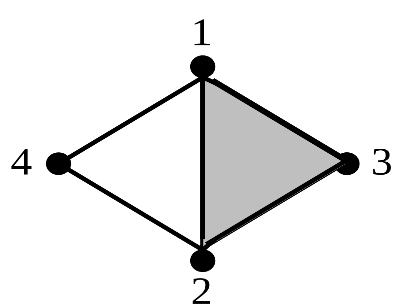


Figure 2-1: A two dimensional shifted complex.

Definition 2.1.2. *An order ideal I of a poset P is a subset of P such that if x is in I and y is less than x in the partial order then y is in I.*

An equivalent definition of shifted complexes is in terms of order ideals. Consider the partial ordering on strings of increasing integers where $(x_1 < x_2 < \cdots < x_k)$ is taken to be less than $(y_1 < y_2 < \cdots < y_k)$ if $x_i \leq y_i$ for all i. Shifted complexes are exactly order ideals in this poset.

Example:

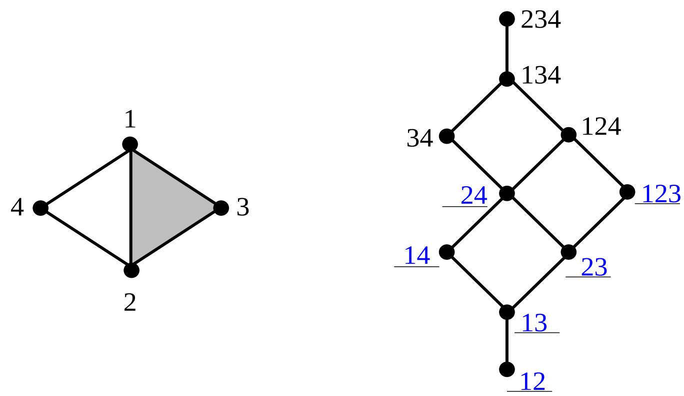


Figure 2-2: Shifted complexes as order ideals.

Let us consider some complexes that are not shifted.

Example:

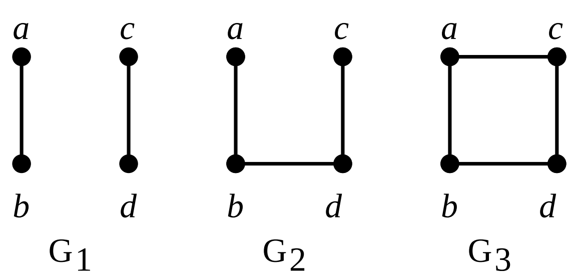


Figure 2-3: Non-shifted complexes.

We can easily see that the graphs of Figure 2-3 are not shifted. For example, in G_1 let node a have label 1. Then the label of c must be larger than the label of a. So we should be able to replace c with a in the collection $\{c, d\}$ and have a face of the complex. But $\{a, d\}$ is not a face. Therefore G_1 is not shifted, and we could similarly prove G_2 and G_3 are not shifted. These three graphs are actually all of the obstructions to shiftedness in dimension one. This result is a consequence of the connection between shifted complexes and threshold graphs which we state in the next section.

2.2 Threshold Graphs

Definition 2.2.1. *An independent set of a graph is a collection of nodes no two of which are connected by an edge.*

Definition 2.2.2. *A graph is threshold if for all $v \in V$ there exists weights $w(v)$, and $t \in \mathbb{R}$ such that the following condition holds: $w(U) \leq t$ if and only if U is an independent set, where $w(U) = \sum_{v \in U} w(v)$.*

Example:

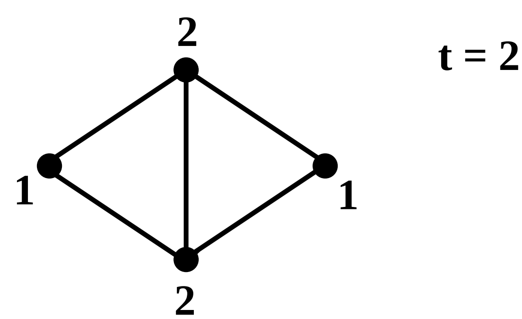


Figure 2-4: A threshold graph with threshold 2.

Threshold graphs are an extensively studied class of graphs. There are many equivalent characterizations of these graphs. For a thorough look at threshold graphs, we refer the interested reader to [8]. Here we will look at some of these characterizations and their generalizations because of the following property:

Proposition 2.2.1. *One dimensional shifted complexes are exactly the threshold graphs.*

This is not hard to prove directly, but we leave it for now since it will be implied in the next chapter.

2.2.1 Obstructions

Above we claimed that G_1, G_2, and G_3 are all the obstructions to shiftedness in dimension one. This statement is simply a translate of the following result.

Theorem 2.2.1. *[8] A graph is threshold if and only if it does not contain G_1, G_2, or G_3 as an induced subgraph.*

In the next chapter we generalize this characterization, in the sense that we can characterize shifted complexes in terms of obstructions.

2.2.2 Polytope of Degree Sequences

Another characterization of threshold graphs is in terms of the *polytope of degree sequences*. To form this polytope, fix a number n and consider all simple graphs on n nodes. For each graph G, form an n-vector $d(G) = (d_1, d_2, \ldots, d_n)$ where d_i is the degree of each node i. The polytope P_d of degree sequences is the convex hull of these points in $\mathbb{R}^n$. Threshold graphs are extremal with respect to this structure.

Theorem 2.2.2. *[8] A graph is threshold if and only if its degree sequence is a vertex of P_d.*

Definition 2.2.3. *A Hypergraph is a pair (V, E) where V is a finite set and E is a collection of subsets of V. If every element of E has the same size then we call the hypergraph regular. For a regular hypergraph with each element of E having size d, we call the hypergraph d-regular.*

Notice that a 2-regular hypergraph is the same as a simple graph.
Golumbic considered various hypergraph analogs of threshold graphs [6]. In particular he asked whether the following three generalizations are equivalent for d-regular hypergraphs:

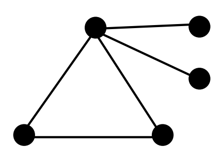
- T_1 - There exists a labelling of V by positive integers and a threshold t such that for all subsets X of V, X does not contain an edge if and only if $\sum_{x \in X} w(x) \leq t$.
- T_2 - There exists a labelling of V such that for all subsets $X \subset V$ of size $d+1$, $X \in G$ if and only if $\sum_{x \in X} w(x) > t$.
- T_3 - For vertices x and y define $x \succeq y$ if x can replace y in any hyperedge. Then, for all x and y either $x \succeq y$ or $y \succeq x$.

First we notice that condition T_3 is the same as shiftedness for a pure complex. A hypergraph is called threshold if it satisfies condition T_2. This second condition is the direct analogue of the condition in Theorem 3. For a d-regular hypergraph, we may generalize the notion of a degree sequence to reflect the number of hyperedges each vertex is adjacent to. As above we may form a polytope of these more general degree sequences. The threshold complexes will be the vertices of this polytope. And while $T_1 \Rightarrow T_2 \Rightarrow T_3$, we have, $T_3 \not\Rightarrow T_2 \not\Rightarrow T_1$ so shifted complexes are not the vertices of this polytope [13].

2.2.3 Construction

Threshold graphs can be described constructively in terms of two basic operations. Let D stand for adding a disjoint node. Let S stand for starring a node, namely placing a new node adjacent to all previous nodes of the graph.

Theorem 2.2.3. *Threshold graphs are exactly those graphs formed from the empty graph by successive applications of the operations D and S.*



DSDDS

Figure 2-5: A threshold graph with its construction.

It is easy to see that these two operations preserve shiftedness. Starring in higher dimensions also preserves shiftedness. By starring a node v in dimension d, we simply mean forming all faces $\{v \cup f \mid f \text{ is a face of dimension } \leq (d-1)\}$. This is different from coning in that we do not necessarily increase the dimension of the complex and hence we can star multiple times in the same dimension. All complexes formed by the operations of adding a disjoint node and starring in the general sense are shifted. These do not form all shifted complexes however. We refer to these complexes as *constructible*. See Figure 2-6 for examples of constructible and non-constructible shifted complexes, where a vertical bar denotes an increase in dimension.

Figure 2-6: Examples of constructible and non constructible shifted complexes.

2.2.4 Vicinal Preorder

The last characterization we look at here is in terms of the vicinal preorder. First, let us recall the definition of the vicinal preorder for graphs, which we call the 1-vicinal preorder for convenience later on.

For any simple graph G and any node $v \in G$, let

$$N_1(v) = \{w \in G \mid wv \in G\} \text{ and}$$
$$N_1[v] = N_1(v) \cup \{v\}.$$

Note that $N_1(v)$ is just the usual neighborhood of a vertex of a graph.

Definition 2.2.4. (1-*Vicinal Preorder* $\precsim_1$)
$x \succsim_1 y$ if and only if $N_1[x]$ contains $N_1(y)$.
If $x \succsim_1 y$ and $y \succsim_1 x$ then we write $x \sim_1 y$.

Theorem 2.2.4. [8] *G is a threshold graph if and only if the vicinal preorder of G is total.*

Example:

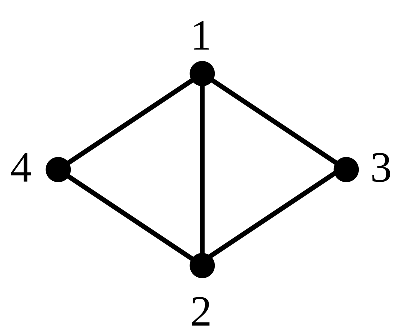


Figure 2-7: A threshold graph with a shifted labelling.

$$1 \sim_1 2 \succ_1 3 \sim_1 4$$

In this example, the graph has been labeled with a shifted labelling. Notice that the shifted labelling is exactly opposite the vicinal order. This is true in general, so once we determine the vicinal ordering of a graph, we may obtain a shifted labelling.

Chapter 3

Characterizations

3.1 Generalized Vicinal Preorders

In the last chapter we noted that one dimensional shifted complexes are the same as threshold graphs. First we extend the characterization in terms of the vicinal preorder. We have for any one dimensional simplicial complex K,

K is shifted if and only if the 1-vicinal preorder of K is total.

In order to generalize this result, we will need the concept of the vicinal preorder for general simplicial complexes, not just graphs. The *star* of a vertex v of a simplicial complex is the set of faces of the complex which contain v. The *link* of a vertex v of a simplicial complex K is the set of faces of the star(v) which do not contain v. Namely, the link of a vertex v is equal to the set of faces $\{f \in K \mid f \cup v \in K \text{ and } v \notin f\}$. Next we define the generalized preorder. For a d-dimensional simplicial complex K, and v a node in K, let

$$N_d(v) = \{(d-1)\text{-dimensional faces of the link}(v)\} \text{ and}$$
$$N_d[v] = \{(d-1)\text{-dimensional faces of the star}(v)\}.$$

Note that for $d = 1$ $N_d(v)$ and $N_d[v]$ are the same as in the graphical case.

Definition 3.1.1. *(d-vicinal preorder $\succsim_d$)*
$x \succsim_d y$ if and only if $N_d[x]$ contains $N_d(y)$.

We need to check that this is a preorder, namely that it is reflexive and transitive.

Reflexivity requires that $N_d[x] \supseteq N_d(x)$, which is true by definition. Transitivity requires that $N_d[x] \supseteq N_d(y)$ and $N_d[y] \supseteq N_d(z)$ imply $N_d[x] \supseteq N_d(z)$. Suppose $yz \notin K$. Then $N_d(z) \subseteq N_d(y) \subseteq N_d[x]$. If $yz \in K$ then for some face f, $yf \in N_d(z)$. We must show $yf \in N_d[x]$. We have

$$\begin{aligned} yf &\in N_d(z) \\ \Rightarrow zf &\in N_d(y) \\ \Rightarrow zf &\in N_d(x) \\ \Rightarrow xf &\in N_d(z) \\ \Rightarrow xf &\in N_d(y) \\ \Rightarrow yf &\in N_d(x). \end{aligned}$$

Definition 3.1.2. *A simplicial complex is pure if all maximal faces under containment are the same size.*

Theorem 3.1.1. *For a pure d-dimensional simplicial complex K, K is shifted if and only if the d-vicinal preorder is total.*

Proof. ($\Rightarrow$) Suppose K is shifted and the d-vicinal preorder is not total.
This implies there exists nodes $x, y \in K$ such that x and y are incomparable.
Then we have $N_d[x] \not\supseteq N_d(y)$ and $N_d[y] \not\supseteq N_d(x)$.
Hence there exists faces f_1 and f_2 such that $xf_1 \in K$, $yf_1 \notin K$ and $yf_2 \in K$, $xf_2 \notin K$.
Let l be a shifted labelling for K. Without loss of generality, we may assume $l(x) < l(y)$. Then $yf_2 \in K$ implies $xf_2 \in K$, a contradiction.
($\Leftarrow$) Suppose the d-vicinal preorder is total. Label the nodes of K in non-increasing order with respect to the vicinal preorder. We claim this is a shifted labelling. Consider any face $(x_1, x_2, \ldots, x_{d+1}) \in K$ and any node w such that $l(w) < l(x_i)$ for some i. Since the labelling is non-increasing, $N_d[w] \supseteq N_d(x_i)$ implies $(x_1, x_2, \ldots, \hat{x}_i, \ldots, x_{d+1}) \in N_d[w]$ which implies $(w, x_1, x_2, \ldots, \hat{x}_i, \ldots, x_{d+1}) \in K$.

$\square$

This is our first intrinsic look at shifted complexes of dimension greater than one.

However, it is restricted to pure complexes. It is tempting to think that a complex is shifted if each skeleton is shifted, where the ith skeleton is the collection of all i dimensional faces. It is important to point out that having all the vicinal preorders total does **not** imply shiftedness. For example, consider the simplicial complex with maximal faces $= \{abc, ad, ae, cd, ce, de\}$ (see Figure 3-1). Both the 1 and 2 vicinal preorders are total,

$$a \sim_1 c \succ_1 e \sim_1 d \succ_1 b$$
$$a \sim_2 b \sim_2 c \succ_2 e \sim_2 d$$

but the complex is not shifted.

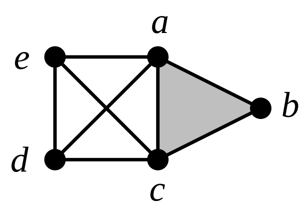


Figure 3-1: A non-shifted complex with both preorders total.

On the other hand, for a complex to be shifted, we do need that the preorders are total. What still may be missing is a single labelling which is a shifted labelling in all dimensions. Any shifted labelling must label the nodes in non-increasing order with respect to all preorders. Otherwise, we would have two nodes, x and y, such that $l(x) < l(y)$ but $y \succ_i x$ for some i. This means $N_i[y] \supset N_i(x)$ with strict containment. Therefore we would have $yf \in K$ but $xf \notin K$ for some face f, showing K is not shifted. Thus we see that for a complex to be shifted it must have all preorders total *and* a labelling which is non-increasing with respect to them all. This means that two nodes may be equivalent in one preorder and have one larger than the other in another preorder. But, we can not have one node larger than the other in one preorder and then smaller in another preorder.

In Figure 3-2 we have $a \sim_1 b \succ_1 c \sim_1 d$ and $a \sim_2 b \sim_2 c \succ_2 d$.

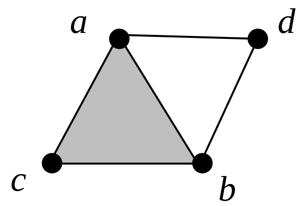


Figure 3-2: A shifted complex with different but compatible preorders.

3.2 Obstructions

In the last chapter we saw that threshold graphs may also be characterized in terms of forbidden induced subgraphs.

Theorem 3.2.1. *[8] G is a threshold graph if and only if it does not contain any of the following as induced subgraphs:* $\{ab, cd\}, \{ab, bc, cd\}$ *or* $\{ab, bc, cd, ad\}$ *(see Figure 3-3).*

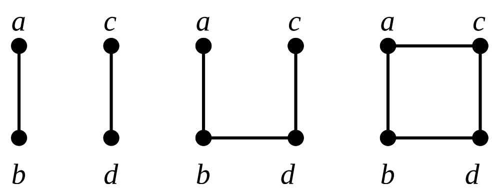


Figure 3-3: Forbidden induced subgraphs for threshold graphs.

Our goal is to characterize shifted complexes in terms of forbidden induced subcomplexes. We consider an obstruction to shiftedness to be a non-shifted simplicial complex all of whose induced subcomplexes are shifted. We will give the range of nodes on which there are obstructions to shiftedness. For a complex to be shifted, it must be shifted in all dimensions. By Theorem 6, we may easily check if the 1-skeleton of a complex is shifted. Therefore in general, when we consider the d-dimensional case, we allow ourselves to assume the $(d-1)$-skeleton is shifted.

Theorem 3.2.2. *In d dimensions all obstructions to shiftedness with shifted $(d-1)$-skeleton are on $(d+3) \leq n \leq (2d+2)$ nodes, and there exist obstructions on each of these values.*

Proof. Let K be a d-dimensional obstruction with shifted $(d-1)$-skeleton on $n > (2d+2)$ nodes.

Case 1: The d skeleton of K is not shifted.

Then the d-vicinal preorder is not total. This is the case if and only if there exists $x, y \in K$ such that x and y are incomparable. Again this is equivalent to $N_d[x] \not\supseteq N_d(y)$ and $N_d[y] \not\supseteq N_d(x)$. Hence we have $(d-1)$ faces e_1 and e_2 such that $xe_1 \in K$, $ye_1 \notin K$ and $ye_2 \in K$, $xe_2 \notin K$.

The number of nodes in $\{x, y, e_1, e_2\}$ is at least $(d+3)$ and at most $(2d+2)$. Since $n > (2d+2)$, there must exist a node a, not equal to x or y and not in e_1 or e_2. Removing a from K clearly cannot affect xe_1, ye_1, ye_2, or xe_2. This implies the d-vicinal preorder is not total on $K \setminus a$. Therefore $K \setminus a$ can not be shifted, which contradicts that K is an obstruction.

Case 2: The d skeleton of K is shifted.

Then we know all preorders are total. Since K is not shifted, there is no labelling of the nodes which is non-increasing with respect to all of them. We have assumed the $(d-1)$ skeleton is shifted, therefore it is the d-vicinal order which does not agree with the first $(d-1)$ orders. This happens if and only if there exists x and y such that $x \succ_i y$ and $y \succ_d x$ for some $i \leq d-1$. Hence we have a $(d-2)$-face w and a $(d-1)$-face f such that $xw, yf \in K$ and $yw, xf \notin K$.

The total number of nodes in $\{x, y, w, f\}$ is at least $(d+3)$ and at most $(2d+1)$. Since $n > (2d+2)$, there must exist a node a, not equal to x or y and not in f or w. Removing a from K cannot affect xw, yw, xf, or yf. This implies there cannot exist a shifted labelling on $K \setminus a$, which contradicts that K is an obstruction.

To finish the proof, we first note that if $n < (d+3)$ then we cannot have any of the obstructing structures above. Next, we show a family of obstructions on $(d+3) \leq n \leq (2d+2)$ nodes. For each of the following complexes, let the $(d-1)$-skeleton be complete. Take two d-faces, $(x, w_1, w_2, \ldots, w_d)$ and $(y, v_1, v_2, \ldots, v_d)$. Consider the amount of overlap between the v_is and w_is. They may overlap on 0 to at most $(d-1)$ nodes. In each case, removing any node leaves at most one d-face on a complete $(d-1)$-skeleton, which is shifted (see Figure 3-4 and Figure 3-5).

□

One of the most important consequences of this theorem is that there are finitely

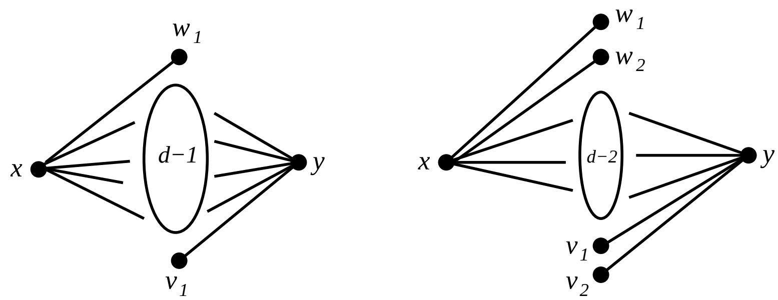


Figure 3-4: A family of obstructions.

Figure 3-5: The 2-skeletons of the 2-dim members of the family of obstructions.

many obstructions to shiftedness in each dimension. This means that we can check for shiftedness in a fixed dimension.

Chapter 4

Enumeration

4.1 Threshold Graphs

In one dimension, the number of shifted complexes is the number of threshold graphs.

Theorem 4.1.1. *There are 2^{n-1} non-isomorphic unlabeled threshold graphs on n nodes.*

Proof. Consider the constructive characterization of threshold graphs. Recall that all threshold graphs can be formed by successively performing two operations. This would give us 2^n strings, except that at the first step, starring a node and adding a disjoint node are equivalent. No two of these graphs are isomorphic to each other. One way to see this is to note that each string gives a unique degree sequence. □

4.2 Totally Symmetric Plane Partitions

Theorem 4.2.1. *The number of two dimensional shifted complexes on $(n+1)$ nodes is given by:*

$$\prod_{1 \leq i \leq j \leq k \leq n} \frac{i+j+k-1}{i+j+k-2}$$

.

The first few numbers of this series are:

$$1, 2, 5, 16, 66, 352, 2431, \ldots$$

This is a result of the connection between shifted complexes and totally symmetric plane partitions. We show that two dimensional shifted complexes are in bijection with totally symmetric plane partitions.

Definition 4.2.1. *A plane partition $\pi = (\pi_{ij})_{i,j\geq 1}$ is an array of nonnegative integers with non-increasing rows and columns.*

Example:

3	3	2
2	2	1
1	1	0

Figure 4-1: A plane partition.

Definition 4.2.2. *A plane partition is totally symmetric if $\pi_{ij} = \pi_{ji}$ and each row, when considered as an ordinary partition, is self-conjugate.*

Example:

3	3	2
3	2	1
2	1	0

Figure 4-2: A totally symmetric plane partition.

Plane partitions can also be thought of as collections of blocks in $\mathbb{R}^3$ where entry ij gives us the height of the blocks at that location. Then we can look at different symmetry classes of this structure. Totally symmetric plane partitions correspond to plane partitions which are invariant under the action of S_3. In this setting, it is not hard to see that TSPPs are order ideals in the poset of Figure 4-3.

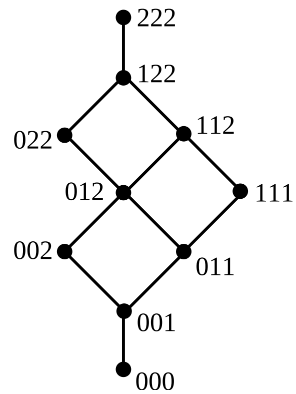


Figure 4-3: TSPPs as order ideals.

Where a string (abc) represents all permutations of the elements $\{a, b, c\}$. Then, given an order ideal π in $\mathbb{N}^3$, we recover the plane partition as follows: $\pi_{ij} = |\{k : (i, j, k) \in \pi\}|$. We can move between this poset and the shifted poset simply by adding (012) to each entry.

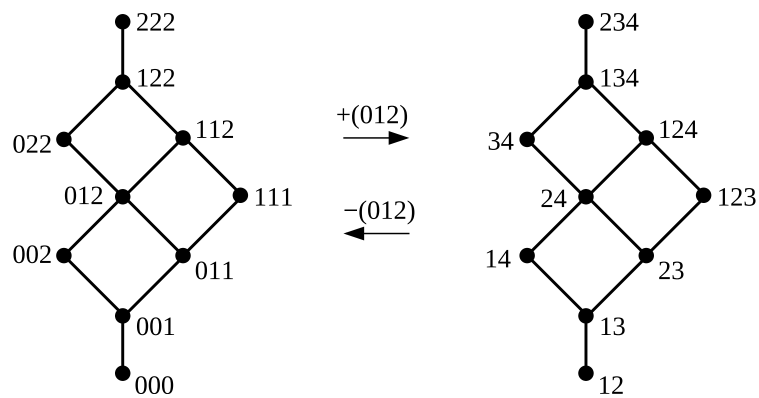


Figure 4-4: Bijection between TSPPs and 2-dimensional shifted complexes.

Now we can see that *two dimensional shifted complexes are the same as totally symmetric plane partitions* as they are order ideals in the same poset.

Theorem 4.2.1 is simply a restatement of the following:

Theorem 4.2.2. *[16] The number of TSPPs which fit in an $(n \times n \times n)$ box is:*

$$\prod_{1 \le i \le j \le k \le n} \frac{i + j + k - 1}{i + j + k - 2}$$

.

These questions can be generalized by looking at stacks of cubes in higher dimension which are invariant under certain symmetries. The d-dimensional shifted complexes exactly correspond to the d-dimensional analogue of a totally symmetric plane partition. Essentially nothing is known in this direction. There is not even a conjecture of the number of such structures for dimension four and higher. We refer the interested reader to [15] for much more on plane partitions.

4.3 Pure complexes

Here we give the first entries in the number of pure shifted complexes (see Figure 4-5). In zero dimensions, the only pure shifted complexes are the graphs consisting of n disjoint nodes. In one dimension, exactly half of all shifted complexes are pure. These are the threshold graphs which have the last node starred. In two dimensions, we use the total number of two dimensional shifted complexes on n nodes T_n to count those that are pure. The number of pure two dimensional shifted complexes on n nodes is $T_{n-1} - T_{n-2}$. To see this, consider the shifted poset of only the strings of length three on n nodes. The pure complexes will correspond to order ideals in this poset with at least one element containing n. The total number of order ideals in this poset is T_{n-1}. The number of order ideals which do not have an element with the node n is T_{n-2}.

The main diagonal in Figure 4-5 reflects the d-simplex. The diagonal just below the main diagonal counts order ideals with top elements of length $(n-1)$ which contain the node n. Clearly there are $(n-1)$ of these. The next diagonal consists of the Eulerian numbers, $2^k - k - 1$, with $k = n - 1$. Along this diagonal, $n - d = 3$. Consider a pure shifted d-dimensional complex on n nodes with $n - d = 3$. Taking the complement of each d-face forms a one dimensional shifted complex (with a reverse shifted labelling). Hence these complexes are in bijection with certain threshold graphs. A pure complex will correspond to an order ideal whose top elements are all the same size and at least one of which contain the node n. The complement, will not have n in all of its top faces. There are $(n-1)$ faces with n as an element in one dimension, and all of these faces are comparable. This gives $(n-1)$ order ideals which do not yield a pure complex. The final step is to notice that the shifted poset of only one dimensional faces on n nodes is the same as the entire shifted poset on $(n-1)$ nodes. Hence the number of order ideals which give us pure shifted complexes is $2^{n-1} - (n-1) - 1 = 2^{n-1} - n$, where the last 1 is to account for the empty set.

n \ d	Total	0	1	2	3	4	5
1	**1**	**1**					
2	**2**	**1**	**1**				
3	**4**	**1**	**2**	**1**			
4	**9**	**1**	**4**	**3**	**1**		
5	**25**	**1**	**8**	**11**	**4**	**1**	
6	**99**	**1**	**16**	**50**	**26**	**5**	**1**

Figure 4-5: The number of pure shifted complexes in d dimensions on n nodes.

Chapter 5

Shifted Matroids

5.1 Matroids

In this chapter we look at the connections between shifted complexes and matroids. We essentially follow the terminology of [10]. We start with a number of definitions and motivation.

Definition 5.1.1. *A matroid is a collection of subsets $\mathcal{I}$ of a base set E such that*
1) $\emptyset \in \mathcal{I}$.
2) If $I \in \mathcal{I}$ and $J \subseteq I$ then $J \in \mathcal{I}$.
3) If I and J are in $\mathcal{I}$ and $|J| < |I|$ then there exists $e \in I \backslash J$ such that $J \cup e \in \mathcal{I}$.

The elements of $\mathcal{I}$ are referred to as the *independent sets* of the matroid. Many concepts for a matroid follow naturally from the idea of the elements of $\mathcal{I}$ being independent. For example, a *basis* of M is a maximally independent set and a *circuit* of M is a minimally dependent set. Also, there exists a well-defined rank function associated to any matroid in terms of these concepts.

Definition 5.1.2. *The rank function ρ of a matroid M is given by:*
The rank of any subset X of the base set is the size of the largest independent set contained in X. The rank of a matroid M is the size of any basis of M.

Let us give the motivating example for this terminology. Consider a finite collection of vectors in a vector space. If we let these vectors be our base set, then the collections of vectors which are linearly independent form the independent sets of a matroid. This is easy to check from the definition. A matroid which corresponds to some collection of vectors in a vector space is called *representable*. Matroids are a more general concept however and include many structures which do not correspond to any such collection of vectors.

Matroids can be formed from many other combinatorial structures. For example, if we take the edges of a graph as our base set, then the collection of forests of the graph form the independent sets of a matroid. The connections between matroids and graph theory motivate other terminology.

Definition 5.1.3. *An element $e \in E$ is called a loop if $\{e\}$ is a circuit.*

Definition 5.1.4. *Two elements e and f are called parallel if $\{e, f\}$ is a circuit. A parallel class of M is a maximal collection of parallel elements.*

5.1.1 Affine Representation

One convenient way to work with matroids is by their affine representations. This is a representation by surfaces in $\mathbb{R}^n$ where independent sets are represented by certain incidences of the surfaces. The general form of these depictions is quite complicated and so we refer the interested reader to [9]. In general, we only use the fact that such a representation exists.

For rank 3 matroids, the form is simple and extremely useful. For a matroid of rank 3 on a base set E, place the points of E in the plane and put a line through all closed sets of size greater than 3 but whose rank is only 2. Then the bases of the matroid are the subsets of points of size three which are not collinear. In the other direction, any collection of points and lines in the plane such that any two lines meet in at most one point is a representation of a rank 3 matroid whose bases are the subsets of size 3 which are not collinear.

Example:

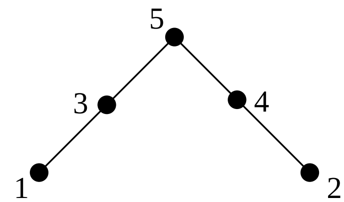


Figure 5-1: An affine representation of a matroid.

Figure 5-1 corresponds to a matroid with bases $= \{123, 124, 134, 145, 234, 235, 345\}$.

5.2 Shifted Matroid Complexes

Notice that the second condition in the definition of a matroid simply states that the independent sets form a simplicial complex. We want to consider those matroids whose complexes of independent sets are shifted. All matroid complexes are pure and so we can characterize shiftedness by a total vicinal preorder. Recall that the basic obstruction to the vicinal preorder being total is faces f_1 and f_2 and nodes x and y such that $f_1x, f_2y \in K$ and $f_1y, f_2x \notin K$.

We first note that any matroid with a shifted matroid complex can have at most one parallel class of size greater than one. Suppose we had two parallel classes $\{a, b\}$ and $\{c, d\}$ in a matroid M. Let K_M be the corresponding matroid complex. Then we would have $ac, bd \in K_M$ and $ab, cd \notin K_M$. Hence the vicinal preorder is not total and the complex is not shifted.

5.2.1 Rank 2

In this section, we consider all matroids in terms of their affine representations. There are only three cases of rank two matroids; a single point, two points, or a line. To be shifted we know we can have at most one multiplicity. It is easy to see that *all* of these cases give shifted matroid complexes. In fact, we can see how to construct these complexes. The only non-adjacencies in this graph are between the members of the parallel class of size greater than one. (If we have no multiplicities, then we have a complete graph.) We can form any such complex by starting with a collection of disjoint nodes and then starring on all other nodes.

Let D stand for adding a disjoint node and S for starring a node.

Theorem 5.2.1. *Rank two shifted matroid complexes are exactly those complexes of the form* $DD \cdots DSS \cdots S$.

In particular we can understand the structure of these graphs as a clique with nodes starred on.

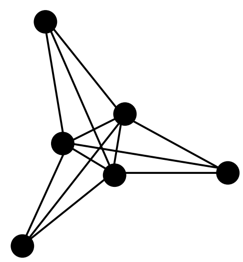


Figure 5-2: A one dimensional shifted matroid.

5.2.2 Rank 3

Rank 3 matroids are represented by collections of lines which obey certain incidence rules. However, we quickly see that we cannot have more than one line for a shifted matroid complex. Let $\{a_1, b_1, c_1\}$ and $\{a_2, b_2, c_2\}$ be two not necessarily disjoint lines (for example c_1 and c_2 could be the same point) of a matroid M with complex K_M. This would give us $(a_1b_1c_1), (a_2b_2c_2) \notin K_M$ and $(a_1b_2c_1), (a_2b_1c_2) \in K_M$ showing K_M is not shifted. So a shifted rank 3 matroid can have at most one line and possibly disjoint points. The line corresponds to the rank 2 case. Therefore rank 3 shifted matroids have the structure of the rank 2 case with additional disjoint points. Again, it is easy to see that all matroids of this form do have shifted matroid complexes. The extra disjoint points are also starred points of the complex but this time in one higher dimension. Therefore we started with a shifted one dimensional complex and starred nodes in the second dimension, which clearly leaves us with a shifted complex.

Let a vertical bar denote an increase in dimension.

Theorem 5.2.2. *Rank 3 shifted matroid complexes are exactly those complexes of the form $DD \cdots DSS \cdots S|SS \cdots S$.*

5.2.3 General Rank

Rank n matroids are represented by collections of $n-1$ dimensional euclidean surfaces which obey certain incidence rules [9]. But, by the same argument above, we can not have more than one such surface. Therefore our only option again becomes the rank $n-1$ case with disjoint points added. As before this results in a shifted matroid complex since the disjoint nodes simply contribute by starring in the top dimension.

Theorem 5.2.3. *Rank n shifted matroid complexes are exactly those complexes of the form $DD \cdots DSS \cdots S|SS \cdots S| \cdots |SS \cdots S$ (with exactly $n-2$ vertical bars).*

5.3 Partition Matroids

In this section we show specific examples of shifted matroids and find their explicit constructive representations. Recently, Ardila introduced a family of shifted matroids [1]. For ease of exposition, we start by considering a special case, the Catalan matroid.

5.3.1 Catalan Matroids

The Catalan matroid is formed from the collection of Dyck paths of a fixed length. A Dyck path is a lattice path in $\mathbb{Z}^2$ which starts at the origin, ends on the x-axis, and takes steps of the form $(x, y) \to (x+1, y+1)$ and $(x, y) \to (x+1, y-1)$. For Dyck paths of length $2n$, the base set of our matroid is $[1, 2, \ldots, 2n]$. The bases are the collections of 'up' or 'northeast' steps of each Dyck path. It is easy to see that this matroid is shifted. We show that it takes on a particularly nice form. Consider all Dyck paths of length $2n$. We concentrate on the non-faces of the corresponding matroid complex. Immediately we see that $\{2n\}$ is not a face since the last step of any Dyck path must be a down step. All other elements of the base set are independent. Next consider faces of size two. The pair $\{2n-2, 2n-1\}$ is also not a face of the matroid complex since there is only one step left in the Dyck path but two down steps are now needed to reach the x-axis. Similarly, $\{2n-2, 2n-3, 2n-4\}$, $\{2n-3, 2n-4, 2n-5, 2n-6\} \cdots \{n, n-1, \dots 2\}$ can not be faces of the matroid complex. We know that the complex is shifted so no subset greater than these can be a face either. It is easy to see that all subsets less than these are in fact independent. Now, we need to understand the form of this complex. By our first observation we see that the corresponding matroid complex is on only $2n-1$ nodes. We can construct the complex as disjoint nodes and stars starting with the highest labeled nodes. The pair $\{2n-2, 2n-1\}$ is the only non-edge. We must add these as disjoint nodes to not have an edge between them. Therefore we first add on $2n-2$ and $2n-1$ as disjoint nodes. All other nodes are starred on later so they have edges between themselves and $2n-2$ and $2n-1$. Next we need to avoid $\{2n-2, 2n-3, 2n-4\}$. In order to not achieve this 2-face, we star on the nodes $2n-3$ and $2n-4$ in dimension one. Notice that this also avoids $\{2n-3, 2n-4, 2n-1\}$ as is necessary. All smaller faces appear since the nodes not added yet have to be starred in a higher dimension. We continue in this fashion by starring on the necessary nodes down one dimension. And, in each dimension there are exactly two new nodes that appear in the smallest non-face. This gives us all but the maximal faces of the complex (notice we have not yet dealt with node 1). All Dyck paths begin with an initial up step. This puts 1 in every basis. So, our final step is to star by 1 in the top dimension. This brings us to the following result:

Theorem 5.3.1. *The Catalan matroid has the form $DDSS|SS|SS|\cdots|SS|S$.*

5.3.2 Partition Matroids

As mentioned above, the Catalan matroid is just a special case of a larger collection of shifted matroids introduced in [1]. Given a partition $\lambda \vdash n$ we can form a matroid by looking at all standard Young tableaux of shape λ and taking the collection of first rows as bases.

Example:

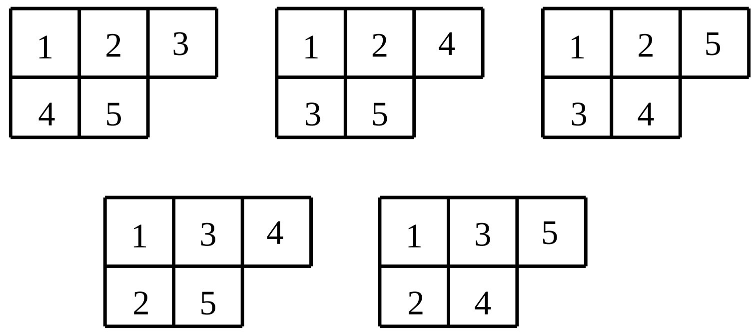


Figure 5-3: All standard Young tableaux of shape (3,2).

The matroid formed by the partition $\lambda = (3, 2)$ has bases $\{123, 124, 125, 134, 135\}$.

The Catalan matroid corresponds to the special case of a $2 \times n$ partition. All partition matroids are shifted. And, we are able to construct them in the same way as the Catalan matroid. The key observation is that these matroids have a unique highest basis in terms of the shifted ordering. This top face determines all other faces for us. Let $(x_1, x_2, \ldots, x_n)$ be the top face. All elements larger than x_n are not independent. Now consider all elements between x_{n-1} and x_n. These elements are never in faces with each other or with x_n. As before, to have no adjacencies we must add these nodes as disjoint nodes before all others. Next we consider the elements between x_{n-2} and x_{n-1}. Again to take care of these non-faces, we must star on the elements at the next stage. At each step we are adding the 'length of each column' number of nodes. Namely, the conjugate partition tells us how many nodes we add in each dimension. These partition matroids account for all shifted matroids which have a non-decreasing number of nodes added in each dimension except for the top dimension which has just one starred node.

For example, if we take a partition of shape $4 \times n$, we get:
$DDDDSSSS|SSSS|\cdots|SSSS|S$.

5.4 Principal Order Ideals

The ease of determining the partition matroids really stems from the fact that they provided a unique top face. In terms of the shifted ordering, we had a principal order ideal. In fact, Ardila's class of shifted complexes is of this generality. He defined a family, SM, which are exactly these principal order ideals and showed they are matroids. We show that this is the right concept for *all* shifted matroid complexes.

Theorem 5.4.1. *Shifted matroid complexes are exactly the principal order ideals under the shifted partial ordering.*

Proof. Let $(x_1, x_2, \ldots, x_n)$ be any element of the shifted partial ordering. We can apply the same procedure used above in section 5.3.2 to form a shifted matroid complex from this element. When we formed a matroid from the first row of a partition, we did not use properties of partitions. We only needed that the first row provided a unique maximal element in the shifted partial ordering.

Now suppose we had an order ideal with at least two top elements, $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$. Let us say we have an incomparability of the form $x_i < y_i$ and $x_j > y_j$ with $i < j$. Then we have the face (y_i, y_j) but (y_i, x_j) and (y_j, x_j) are not faces. We have no way to place y_i, y_j, and x_j under our construction of shifted matroids to form these adjacencies. The two non-faces would require that all three be added as disjoint nodes, but then we do not have y_i and y_j adjacent. Therefore we can not have a shifted matroid complex.

□

Corollary 5.4.1. *There are $\binom{n}{k}$ shifted matroids of rank k on n nodes.*

5.5 Matroid Operations

5.5.1 Minors

First we define two operations for a matroid, deletion and contraction. Both operations result in new matroids. We define the operations by giving the independent sets of the resulting matroids.

Definition 5.5.1. *Let M be a matroid on the base set E, and let $T \subseteq E$. Define the deletion $M \backslash T$ to be the matroid on $E - T$ whose independent sets are given by:*

$$\mathcal{I}(M \backslash T) = \{I \subseteq E - T : I \in \mathcal{I}(M)\}.$$

Definition 5.5.2. *Let M be a matroid on the base set E, and let $T \subseteq E$. Define the contraction M/T to be the matroid on $E - T$ whose independent sets are given by:*

$$\mathcal{I}(M/T) = \{I \subseteq E - T : I \cup B \in \mathcal{I}(M)\}, \text{ where } B \text{ is a basis for } M \backslash (E - T).$$

Definition 5.5.3. *A minor of a matroid is any matroid formed by a sequence of contractions and deletions.*

Theorem 5.5.1. *The class of shifted matroids is closed under taking minors.*

Proof. Let M be a shifted matroid on the base set E and e be any element of E. First we consider $M \backslash e$, whose independent sets are the independent sets of M which do not involve e. In terms of the complex of independent sets, this is equivalent to the geometric deletion of e from the complex. Since the class of shifted complexes is closed under deletion, we have that $M \backslash e$ is a shifted matroid.
Next we look at M/e. First we note that $M \backslash (E - e)$ is the matroid on just one element, e. Therefore, $\{I \subseteq E - T : I \cup B \in \mathcal{I}(M)\}$ are those subsets which form an independent set with e. Again, in terms of the matroid complex, this is simply the link of e and the link of any node in a shifted complex is shifted. Therefore M/e is a shifted matroid. ☐

5.5.2 Duality

There is also a natural duality operation for matroids. Let $\mathcal{B}(M)$ be the collection of bases of a matroid M on base set E. Let $\mathcal{B}^*(M) = \{E - B : B \in \mathcal{B}(M)\}$. The collection $\mathcal{B}^*(M)$ is the collection of bases of a matroid on E. We denote this matroid by M^* and call it the *dual matroid* of M. This is a well-defined duality operation and in particular, $(M^*)^* = M$.

Theorem 5.5.2. *The class of shifted matroids is closed under duality.*

Proof. Consider any shifted matroid as a principal order ideal in the shifted partial ordering with unique top element $(x_1, x_2, \ldots x_k)$. The bases of the matroid are $\{(y_1, y_2, \ldots, y_k) \,|\, y_i \leq x_i \text{ and } y_i \neq 0 \,\forall i\}$. Clearly, if we have two bases B_1 and B_2 such that B_1 is less than B_2 in the shifted partial ordering, then $E - B_1$ is greater than $E - B_2$ in the shifted partial ordering. Taking the complement of all bases gives a principal filter in the shifted partial ordering. This is equivalent to a principal order ideal since we may simply reverse the labelling. Therefore the dual of a shifted matroid is a shifted matroid. □

5.5.3 Shifting

Shifting a complex which is a matroid does not necessarily result in a matroid. Consider the simplicial complex which is the boundary complex of the octahedron. It is not hard to check that this is a matroid complex. For both symmetric and exterior shifting, the result is the shifted complex with top faces 136 and 234 and hence not a matroid [7]. Also, we can combinatorial shift the boundary complex of an octahedron to the shifted complex generated by 145 and 136. We can observe that matroids are not preserved under shifting in a more general sense. Recall that shifting preserves the f-vector of a complex. A shifted matroid is always a principal order ideal in the shifted partial ordering, and so we are limited by the size of such ideals. For the octahedron, we have eight two dimensional faces on six nodes. But there are no principal order ideals in the shifted partial ordering on six nodes with eight two dimensional elements. Hence no shifting procedure could associate a shifted matroid to the boundary complex of a octahedron.

5.6 Broken Circuit Complex

Next we investigate shifted broken circuit complexes. Again we begin with some definitions and examples. The results used here may all be found in [4].

Definition 5.6.1. *Given a matroid M on the base set E and a linear ordering of the base set, a broken circuit of M is a subset $C - \{x_i\}$ where C is a circuit and x_i is the smallest element of C with respect to the linear ordering.*

Definition 5.6.2. *The broken circuit complex $BC(M)$ of a matroid M on a base set E is defined by:*

$$BC(M) = \{S \subseteq M : S \text{ contains no broken circuit}\}.$$

Definition 5.6.3. *The reduced broken circuit complex $BC_r(M)$ of a matroid M with broken circuit complex $BC(M)$ is defined by:*

$$BC_r(M) = \{S \in BC(M) : x_m \notin S\}, \text{ where } x_m \text{ is the node of smallest label.}$$

First we give some examples. Let M be the matroid with two three point lines which intersect in one point (see Figure 5-4).

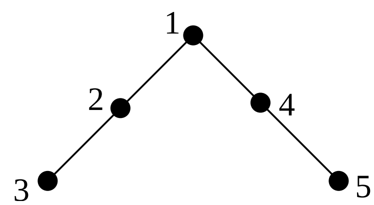


Figure 5-4: A labeled matroid whose broken circuit complex is not shifted.

The circuits of this matroids are $C = \{123, 145, 2345\}$, and the broken circuits are $BC = \{23, 45, 345\}$. $BC(M)$ is generated by maximal faces $\{124, 125, 134, 135\}$.

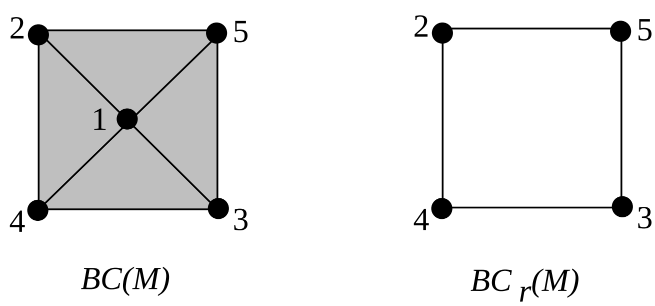


Figure 5-5: The broken circuit complex of the matroid in Figure 5-4.

The construction of the broken circuit complex uses the labelling of the matroid. This construction is **not** independent of the labelling. Let us consider the same matroid as in the last example but with a different labelling (See Figure 5-6).

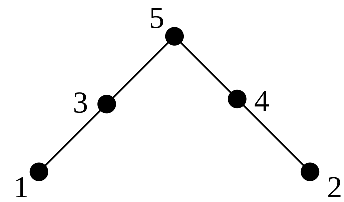


Figure 5-6: The matroid of Figure 5-4 with a different labelling.

With this labelling we have $C = \{135, 245, 1234\}$, $BC = \{35, 45, 234\}$, and $BC(M)$ generated by $\{123, 124, 125, 134\}$.

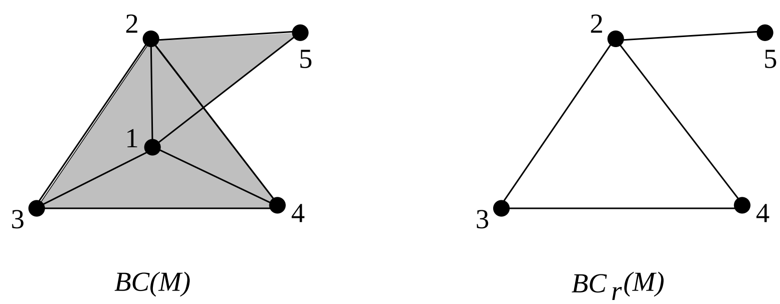


Figure 5-7: The broken circuit complex of the matroid in Figure 5-6.

The broken circuit complex has clearly changed under this new labelling. We note two results on broken circuit complexes before moving to our result on shifted broken circuit complexes.

Proposition 5.6.1. *$BC(M)$ is pure of dimension $\rho(M) - 1$.*

Proposition 5.6.2. *$BC(M) = Cone(BC_r(M))$. $BC(M) = BC_r(M) * x_m$ where x_m is the node of smallest label.*

Now let us consider the case where we have a rank n matroid M with a shifted matroid complex and a shifted labelling of the nodes.

Theorem 5.6.1. *Broken circuit complexes of shifted matroids are shifted and inherit a shifted labelling.*

Proof. $BC(M)$ is a pure n-dimensional cone with cone point 1, and a subcomplex of the matroid complex. Purity allows us to only check shiftedness in the top dimension. If $BC(M)$ had all n-faces that included 1 from the matroid complex, then it would be shifted since the matroid complex is shifted. But some such face may contain a broken circuit. Say we have a circuit $(x_1, x_2, \ldots, x_{d+1})$, broken circuit $(x_2, \ldots, x_{d+1})$, and $(1, x_2, \ldots, x_{d+1})$ is a face of the matroid complex. Then $(1, x_2, \ldots, x_{d+1})$ is not a face of $BC(M)$. So we need any greater face, say $(1, y_2, \ldots, y_{d+1})$, to also not be a face of $BC(M)$. If $(y_2, \ldots, y_{d+1})$ is not a face of the matroid complex then we are done. Otherwise we need it to be a broken circuit. Now since $(x_1, x_2, \ldots, x_{d+1})$ is a circuit, it had to be a non-face of the matroid complex. But then $(x_1, y_2, \ldots y_{d+1})$ must also be a non-face or the matroid complex would not be shifted. Hence $(y_2, \ldots y_{d+1})$ is also a broken circuit, and $BC(M)$ is shifted under the initial shifted labelling of M. $\square$

We list here some other results on shifted broken circuit complexes.

Proposition 5.6.3. *The same matroid with two different labellings can result in one shifted broken circuit complex and one not shifted.*

Proof. The two examples given above demonstrate this. The first labelling gives a non-shifted broken circuit complex. An easy way to see this is to notice that $BC_r(M)$ is a square, one of the obstructions to shiftedness. The second labelling gives a shifted complex and induces a shifted labelling. $\square$

Proposition 5.6.4. *All shifted matroid complexes with a single S (see section 5.2) in the top dimension appear as broken circuit complexes.*

This is a direct consequence of the following result:

Theorem 5.6.2. *For any matroid M there exists a matroid N such that $BC_r(N) = \mathcal{I}(M)$.*

Proposition 5.6.5. *All one dimensional broken circuit complexes are shifted.*

Proof. All one dimensional broken circuit complexes must be cones over zero dimensional complexes. The only zero dimensional complexes are collections of disjoint nodes. Hence all one dimensional broken circuit complexes are star graphs which are shifted. In fact they are matroids, and this structure falls under the situation of the previous proposition. $\square$

Proposition 5.6.6. *All matroids of rank less than four and on less than six nodes give a shifted broken circuit complex with an induced shifted labelling.*

Proposition 5.6.7. *The matroid with two 3-point lines can not induce a shifted labelling on a broken circuit complex.*

Proof. Consider this matroid with the labels as given in Figure 5-8. Suppose with out loss of generality a has the smallest label among $\{a, b, c\}$, d has the smallest label among $\{d, e, f\}$, and $l(a) < l(d)$. Then $C = \{abc, def, abde, abdf, abef, acde, acdf, acef, bcde, bcdf, bcef\}$. BC contains $\{bc, ef, bde, bdf, bef, cde, cdf, cef\}$. The last three circuits contribute differently depending on the labelling, but none affect $BC(M)$. Hence we have $BC(M) = \{abd, abf, ace, abe, acd, acf, ade, adf\}$. Consider $BC(M) - \{a, d\}$, which equals $\{ce, eb, bf, fc\}$, a square. Hence $BC(M)$ can not be shifted. □

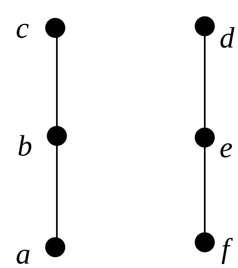


Figure 5-8: The matroid with two 3-point lines.

5.7 The Tutte polynomial

Next we give some preliminary results on Tutte polynomials of shifted matroids. For a general introduction to Tutte polynomials see [17]. For a matroid M on a base set E with rank function ρ the Tutte polynomial is defined as follows:

Definition 5.7.1. $T(M; x, y) = \sum_{A \subseteq E} (x-1)^{\rho(E)-\rho(A)}(y-1)^{|A|-\rho(A)}$.

The Tutte polynomial carries a great deal of information about a matroid. Different evaluations provide different combinatorial invariants. For example:

$T(M;1,1) =$ number of bases of M.

$T(M;2,1) =$ number of independent sets of M.

$T(M;1,2) =$ number of spanning sets of M.

$T(M;2,2) = 2^{|E|}$.

We utilize the following recurrences satisfied by the Tutte polynomial:

- $T(M) = T(M-e) + T(M/e)$, if e is not equal to a loop or isthmus.
- $T(M) = T(M(e))T(M-e)$, if e is equal to a loop or isthmus.

For a shifted matroid, let d be the number of initial disjoint nodes and s_i the number of starred nodes in dimension i. Also, we always use n for the size of our base set.

5.7.1 Rank 1

In rank 1 the only shifted matroid is the uniform matroid $U_{1,n}$, and

$$T(U_{1,n}) = x + y + y^2 + \cdots + y^{n-1}.$$

5.7.2 Rank 2

In rank 2, we have matroids of the form $M = DD\cdots DSS\cdots S$.

If $s_1 = 1$, then the unique starred node is a coloop since it is in all maximal faces. In this case we get $T(M) = x(x + y + y^2 + \cdots + y^{n-1})$.

If $s_1 > 1$, we notice that contracting along any starred node gives us $U_{1,(n-1)}$. This is because contracting gives us the link of the node, and a starred node forms a face with all other nodes. Using the contraction/deletion recurrences we get

$$T(M) = T(DD\cdots D\underbrace{SS\cdots S}_{s_1-1}) + T(\underbrace{DD\cdots D}_{n-1}).$$

To simplify notation, for a matroid M with non-zero parameters $\{d, s_1, s_2, \ldots s_k\}$, write $T(M)$ as $T(d, s_1, s_2, \ldots, s_k)$. Then the above statement is simply

$$T(d, s_1) = T(d, s_1 - 1) + T(n-1).$$

The second term is simply the rank 1 case. Using this recurrence, we get a general form for all rank 2 matroids.

Theorem 5.7.1. *We have*

$$T(d,s_1) = x(x+y+\cdots+y^{d-1}) + (s_1-1)(x+y+\cdots+y^d) + (s_1-2)y^{d+1} \\ +(s_1-3)y^{d+2}+\cdots+y^{n-2}.$$

From this we see that

$$T(d+1,s) = T(d,s) + xy^d + y^{d+1} + y^{d+2} + \cdots + y^{d+s-1}$$
$$T(d,s+1) = T(d,s) + x + y + y^2 + y^3 + \cdots + y^{d+s}.$$

5.7.3 General Rank

Now we consider $T(d,s_1,s_2,\ldots,s_k)$ for a shifted matroid of rank $k+1$. As before, if $s_k = 1$ then the unique node starred in dimension k is a coloop and we have $T(M) = xT(d,s_1,s_2,\ldots,s_{k-1})$. If $s_k > 1$ we consider contraction along a starred node. Again we get the link of the node which is all $(k-1)$-faces. This gives us a complex with parameters $\{d,s_1,s_2,\ldots,s'_{k-1}\}$ where $s'_{k-1} = s_{k-1}+s_k-1$. In the previous notation of Ds and Ss, we are removing the last vertical bar and one S. Here the recurrence is

$$T(d,s_1,s_2,\ldots,s_k) = T(d,s_1,s_2,\ldots,s_k-1) + T(d,s_1,s_2,\ldots,s_{k-1}+s_k-1).$$

In [1] the Tutte polynomial for Catalan matroids is given in terms of simple properties of Dyck paths.

Open Problem: Find the explicit form of the Tutte polynomial for an arbitrary shifted matroid.

5.8 Nested Transversal Matroids

Let S be a finite set and $A = (A_1,A_2,\ldots,A_k)$ be a family of subset of S. A *transversal* of A is a subset $\{s_1,s_2,\ldots,s_k\}$ of S such that $s_j \in A_j$ for all j. A subset $X \subset S$ is a *partial transversal* of A if X is a transversal of some subset of A.

Theorem 5.8.1. *[10] Let A be a family of subsets of a finite set S. Let $\mathcal{I}$ be the set of partial transversals of A. Then $\mathcal{I}$ is the collection of independent sets of a matroid on S.*

A transversal A is called *nested* if $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_k$. In [11] a class of matroids $\mathcal{M}$ was introduced which are formed from the domains of integer valued functions. These matroids are exactly the transversal matroids with nested presentations. This class is also the same as the class of shifted matroids.

Given a function f from a finite base set E to the integers, form a function f' from 2^E to the integers by:

$$f'(A) = \begin{cases} \max\{f(a) | \, a \in A\} & \text{if A} \neq \emptyset \\ \min\{f(a) | \, a \in E\} & \text{otherwise.} \end{cases}$$

Definition 5.8.1. *M_f is the matroid with $\mathcal{I} = \{I \subseteq E$ such that $max\{f(a) | \, a \in J\} \geq |J|$ for all $J \subseteq I$, $J \neq \emptyset$.*

$\mathcal{M}$ is the collection of matroids obtained from integer valued functions in this way.

In [11] it is shown that $\mathcal{M}$ is closed under many matroid operations, and is exactly the class of nested transversal matroids. Also, they give a characterization of this class in terms of excluded minors which we include below.

For any $k \geq 2$, form a base set E which is the disjoint union of two k element sets E_1 and E_2. Form a matroid N^k on E by letting the circuits be given by

$$\mathcal{C}(N^k) = \{E_1, E_2\} \cup \{C \, | C \not\supseteq E_1, \, C \not\supseteq E_2, \, C \subset E, \, |C| = k+1\}.$$

Theorem 5.8.2. *[11] $\mathcal{M}$ is the class of matroids having no minor isomorphic to N^k for $k \geq 2$.*

In terms of the independent set complex, this class of matroids has the form:

- $(k-1)$ dimensional,
- all faces of dimension less than or equal to $(k-2)$,
- all but two disjoint $(k-1)$ faces.

For example, for $k = 2$, N^k is the matroid with complex a square.

Chapter 6

Connections

In this chapter we investigate many of the places where shifted complexes appear in connection to other common combinatorial structures. We saw some examples already. The enumeration of shifted complexes made a connection with plane partitions. Also, Catalan paths and first rows of tableaux induce shifted matroids.

6.1 Linear Extensions

In the previous chapter, we saw that shifted matroids can be formed by the collection of first rows of all standard young tableaux of a fixed partition. This result can be partially extended. Namely, the collection of entries of any fixed subpartition of all standard Young tableaux of a partition forms a shifted complex. This construction does not however always yield a matroid.

For example, consider $\lambda = (5, 1, 1)$ and $\mu = (3, 1)$. This pair gives a complex formed by the two top faces (1245) and (1236) in the shifted partial ordering.

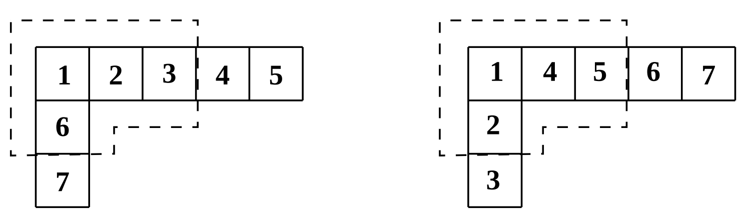


Figure 6-1: A partition complex that is not a matroid.

In Figure 6-1, we show the realization of these two faces. Due to the large number of standard Young tableaux of this shape, we do not show all possibilities, but it is not hard to check these are the maximal cases. We can see the complex is not a matroid since it does not have a unique top face.

These partition complexes are actually all special cases of the following even more general construction. A *linear extension* of a poset P on n elements is a bijection f from the elements of P to $[1, 2, \ldots, n]$ such that if x is less than y in P then $f(x)$ is less than $f(y)$. Consider any poset, P and a fixed order ideal $I \subseteq P$. Take a linear extension f of P and consider the set of values which are assigned to the elements of the order ideal, $\{f(x_i) \mid x_i \in I\}$. The collection of these sets over all linear extensions of P forms a shifted complex. In [2] Ardila posed the question of studying the pairs (I, P) which form a matroid. Here we show some partial results on this question. First we demonstrate some of the complications of this question. Consider the following examples:

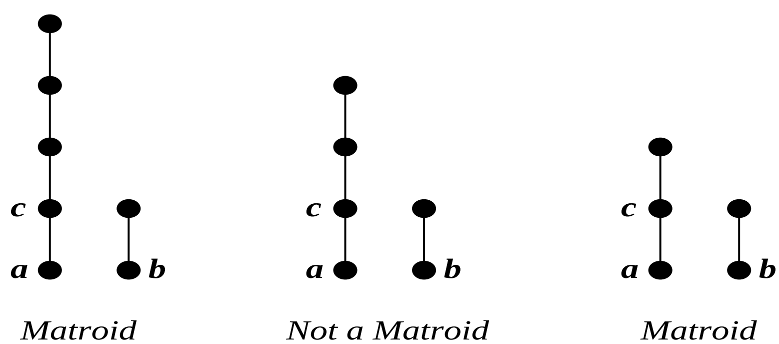


Figure 6-2: Posets with order ideals = $\{abc\}$.

The non-matroid example above is minimal in the following sense:

All linear extension complexes formed from posets of size at most 5 are matroids.

All linear extension complexes formed from order ideals of size at most 2 are matroids.

The second result is actually a consequence of a slightly more general result.

Proposition 6.1.1. *For any linear extension complex, the first entry in all top faces is the same.*

Proof. Suppose we have two faces $(x_1, x_2, \ldots, x_k)$ and $(y_1, y_2, \ldots, y_k)$ with $x_1 < y_1$ and $x_i > y_i$ for some $i \neq 1$. We will show that $(y_1, x_2, \ldots, x_k)$ is also a face of the complex. First we note that we can take $x_1 + 1 = y_1$. Since the complex is shifted, the face $(y_1, y_2, \ldots, y_k)$ implies $(x_1 + 1, y_2, \ldots, y_k)$ is also a face and also incomparable to $(x_1, x_2, \ldots, x_k)$. Now consider the linear extension which admits the face $(x_1, x_2, \ldots, x_k)$. Reassign the point given the value x_1 with y_1. Reassign the point given the value y_1 with $y_1 + 1$, $y_1 + 1$ with $y_1 + 2$ and so on. Continue until we are at the first incomparable element to the original element with value x_1. This element is the smallest incomparable element that had a value larger than y_1. We may give this element the value x_1. Now we have moved y_1 into x_1 position showing that $(y_1, x_2, \ldots, x_k)$ can be realized.

□

Hence an order ideal with only 2 elements can not give incomparable top faces since the first entry will be the same in both.

Now we give the first positive result on realizing matroids via this construction.

Proposition 6.1.2. *Every shifted matroid can be realized as a linear extension complex.*

Proof. Suppose we have a shifted matroid with top element $\{x_1, x_2, \ldots, x_k\}$. If x_1 is not equal to 1 then we first form a chain of length $x_1 - 1$. Next we form a second chain with "branches". Namely, form a poset with a primary chain of length k and chains of length $x_i - x_{i-1} - 1$ at height i for all $i \neq 1$. To obtain our matroid, we take the primary chain of length k as our order ideal. If x_1 is equal to 1, we simply skip the first step of forming an extra chain.

For example, consider the matroid with top face $\{3, 7, 9\}$. We form a chain of length 2 and then another primary chain of length 3 with two branches (See Figure 6-3).

It is easy to see that this will result in the desired shifted matroid.

□

Now we see that there are pairs of posets and order ideals which will give us any shifted matroid. The representation from the proof above is not unique. Many different posets and order ideals can give the same complex. Although the construction above

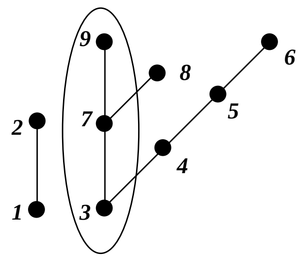


Figure 6-3: A poset and order ideal corresponding to M generated by (379).

forms a principal order ideal, in general restricting to principal order ideals does not necessarily give a matroid.

Example:

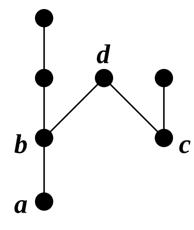


Figure 6-4: A principal order ideal complex that does not yield a matroid.

In Figure 6-4 the linear extension complex corresponding to the order ideal $\{a, b, c, d\}$ is not a matroid. This complex has two top faces, $\{1, 2, 5, 7\}$ and $\{1, 3, 4, 7\}$. Therefore although shifted matroids correspond to principal order ideals in the shifted partial ordering, principal order ideals do not necessarily give us matroids in this setting.

6.2 General Order Ideals

We have seen that shifted matroids are exactly the principal order ideals under the shifted partial ordering. Now that we understand the principal order ideals, we are naturally led to consider order ideals with any fixed number of top elements. We can provide results on these for the class of constructible shifted complexes.

In general, the D and S operations preserve shiftedness. Matroid complexes are those in which all the disjoint added nodes come first. We have seen that these will correspond to principal order ideals. Thus we ask if having multiple switches from adding disjoint nodes to starring nodes might lead to having multiple top elements in the order ideal. We now answer this in the affirmative.

Theorem 6.2.1. *Given a shifted complex formed by the two operations D and S, the number of switches from D to S is exactly the number of top elements in the corresponding order ideal. (In dimension one, all shifted complexes have this form.)*

Proof. First, given a complex formed by the D and S operations, we show how to obtain a shifted labelling. Consider the complex written out as a string with the nodes added left to right. Label the Ss starting with 1 and increasing from right to left (one each time) until the last S. Say we have k S operations in all. Then label the Ds starting with $k+1$ and increasing left to right (one each time). For example:

$$\frac{D}{7}\,\frac{D}{8}\,\frac{S}{6}\,|\,\frac{S}{5}\,\frac{D}{9}\,\frac{S}{4}\,\frac{S}{3}\,|\,\frac{S}{2}\,\frac{S}{1}$$

To see that this is a shifted labelling, first consider the final node added. If it is disjoint, clearly we may give it the highest labelling. If it is starred on, then we know it has the most possible adjacencies and we may give it the smallest labelling. Removing the last node leaves an induced subcomplex on $n-1$ nodes. This subcomplex is shifted and we may repeat our argument with the remaining labels.

Now suppose we have a complex formed by Ds and Ss and we have labelled it as above. Consider the switches from D to S. Let us think of them as pairs: from left to right (D_1, S_1), (D_2, S_2), $\ldots$,(D_m, S_m). Notice by our labelling, $D_1 < D_2 < \cdots < D_m$ and $S_1 > S_2 > \cdots > S_m$ so faces with different switch pairs will be incomparable. Also, each switch pair appears in a *top* face since for a pair (D_i, S_i), S_i is the largest node adjacent to D_i. Finally, two switch pairs cannot appear in the same face since the D with higher label will not be adjacent to the nodes in the other pair. Therefore, we now have that the number of top elements in the order ideal is at least the number of switches from D to S. Now we just need to show that these are all the top elements. Suppose we had another top element not arising from a switch pair; then all nodes of this face must be added by starring. Otherwise, we would have a D with an S to the right of it and at least one other element in between. If the node in between is a

D, then it has higher label than the original D, while if it is an S, it has higher label than the original S. Either way, we can replace it with one of the original nodes to get a higher face. So a top face with no switch pair must have all Ss. But a face with all Ss cannot be a top face because there is always at least one D (the first node) to the left of any S. So we could replace any element with the initial D and have a face with a higher labelling.

□

6.3 Independence Complex

Recall that an independent set of a graph is a collection of nodes no two of which are connected by an edge. Let $I(G)$ denote the independence complex of a graph G. This complex is formed by taking the collection of independent sets of G. Clearly removing a node from an independent set results in an independent set so this collection is a simplicial complex.

Proposition 6.3.1. *G is a threshold graph if and only if $I(G)$ is shifted.*

Proof. Let G be a threshold graph. Then we know G is shifted. Let l be a shifted labelling of the nodes of G. Consider any face $F = \{v_1, v_2 \dots v_k\}$ of $I(G)$ and a node w such that $l(w) > l(v_i)$ for some i. We need to show that $F' = \{v_1, v_2, \dots, \hat{v_i}, w, \dots, v_k\}$ is a face of $I(G)$. If not, then w must be connected to some v_j $(j \neq i)$ in G. Then w has a larger label than v_i so if $\{wv_j\} \in E(G)$ then $\{v_iv_j\}$ must be in G for G to be shifted, which contradicts F being a face of $I(G)$.

Now let $I(G)$ be shifted and l a shifted labelling. Consider any edge, $\{v_1v_2\}$ of G and a node w such that $l(w) > l(v_2)$. We need to show that $\{v_1w\}$ is an edge of G. If not, then $\{v_1w\}$ is an independent set of G and hence a face of $I(G)$. $I(G)$ is shifted and v_2 had a smaller label than w which means $\{v_2w\}$ must be a face of $I(G)$ and not an edge of G, again a contradiction.

□

6.3.1 Flag complexes

Definition 6.3.1. *A flag complex is a complex such that every minimal non-face has exactly two elements.*

Every flag complex can be formed as the independence complex of a graph. And, the independence complex of any graph is a flag complex. So we now see that all shifted flag complexes are formed from threshold graphs. This allows us to determine the form of these complexes.

Theorem 6.3.1. *Shifted flag complexes are the constructible complexes with exactly one S in each dimension.*

Proof. Every shifted flag complex arises as the independence complex of a threshold graph. Every threshold graph can be represented as a string of Ds and Ss standing for adding a disjoint node and starring a node. Consider mapping this string under the following rules: $D \to |S$ and $S \to D$. Namely, switch every S to a D and switch every D to an S but also increase dimension with every such switch.
Example: $DDSDSDSSD \to S|SD|SD|SDD|S$
Notice that usually we always start the strings with a D but it is equivalent for the first operation to be a D or an S.
First, we want to determine the independent sets of a threshold graph from its string of Ds and Ss. They are the set of all Ds and all collections which consist of an S and all Ds that come after it.
Next, given the image of the string, we want to determine its facets. They are the set of all Ss and all collections which consist of a D and all Ss that come after it. In particular they are exactly the independent sets of G.
This procedure is invertible showing that all strings of Ds and Ss with exactly one S in each dimension are flag complexes.

□

We note here for completeness that shifting does not preserve flag complexes. As an example consider the graph $K_{3,3}$. It is easy to check that this a flag complex.

Symmetric shifting yields the complex generated by top face $\{2,6\}$ and exterior shifting yields the complex generated by top faces $\{2,5\}$ and $\{3,4\}$. In the first graph, the collection $\{1,2,3\}$ is a minimal non-face showing it is not a flag complex. In the second graph, $\{1,2,3\}$ is also a minimal non-face. As with shifting matroids, we can see that shifting could not preserve flag complexes from a broader consideration. Recall that shifting preserves the f-vector of a complex. The graph $K_{3,3}$ has 6 nodes, 9 edges, and no faces of dimension 2 or greater. But any order ideal in the shifted partial order ordering on 6 nodes with 9 one dimensional faces will include the edges $\{1,2\}$, $\{1,3\}$, and $\{2,3\}$. Since we can not add any two dimensional faces, this will generate a minimal non-face with three elements.

6.3.2 Generalized Independence Complex

Forming the independence complex of a graph can be generalized to arbitrary simplicial complexes. For a simplicial complex Δ, we form $I(\Delta)$ by declaring the facets of Δ to be the minimal non-faces of $I(\Delta)$. Ehrenborg suggested considering this construction with respect to shifted complexes. The statement that Δ is shifted if and only if $I(\Delta)$ is shifted is false in both directions. Consider Δ generated by $\{123, 14, 24, 15\}$. Then $I(\Delta)$ is generated by $\{235, 345, 12, 13\}$. We can easily see that this complex is not shifted by looking at the induced subcomplex on $\{1,2,4,5\}$. This subcomplex is a path of length three, an obstruction to shiftedness in dimension one.

We can continue applying the procedure to disprove the other direction. $I(I(\Delta))$ is generated by $\{245, 234, 145, 35\}$ which is also not shifted. But, $I(I(I(\Delta)))$ is generated by $\{123, 124, 125, 134, 45\}$ which is shifted (mapping 3 to 4 gives a shifted labelling). Note that the counterexamples to this result are non-pure complexes. We could ask the question again for the case of pure complexes. This is actually a more natural generalization of the independence complex of a graph. The generalized procedure only restricts to the same procedure on graphs if the graph is connected (i.e. pure). Namely, if we have a graph with disjoint nodes, under the generalized procedure they would be minimal non-faces of $I(\Delta)$. On the other hand, a disjoint node is in all maximal faces of the independence complex of the graph. In the pure case, we come to the following result:

Proposition 6.3.2. *For Δ pure, Δ is shifted if and only if $I(\Delta)$ is shifted.*

Proof. ($\Rightarrow$) Suppose Δ is shifted but $I(\Delta)$ is not shifted. Then there exists x, y, f_1, f_2 such that $xf_1, yf_2 \in I(\Delta)$ and $yf_1, xf_2 \notin I(\Delta)$. Since yf_1 and xf_2 are not in $I(\Delta)$, they must be facets or contain facets of Δ. First we note that the facets involved here must not be strictly contained in f_1 and f_2 or xf_2 and yf_1 could not be in $I(\Delta)$. Suppose yf_1 and xf_2 are facets of Δ. Let l be a shifted labelling for Δ and with out loss of generality, let $l(x) < l(y)$. Since Δ is shifted, we have that $xf_1 \in \Delta$. But, $|xf_1|$ $= |yf_1|$ which implies xf_1 is a facet of Δ and can not be in $I(\Delta)$ a contradiction. Suppose at least one of yf_1 and xf_2 is not a facet of Δ. Then they must contain a facet. Let $g_1 \subseteq f_1$, $g_2 \subseteq f_2$, and xg_2, yg_1 be facets of Δ. They will not be in $I(\Delta)$, but $yg_2 \subseteq yf_2 \in I(\Delta)$ and $xg_1 \subseteq xf_1 \in I(\Delta)$ so we are back in the first case.

($\Leftarrow$) Now suppose $I(\Delta)$ is shifted but Δ is not shifted. Then there exists x, y, f_1, f_2 such that $xf_1, yf_2 \in \Delta$ and $yf_1, xf_2 \notin \Delta$. First notice that we may take xf_1 and yf_2 to be maximal faces since Δ is pure, and in particular this gives that $|xf_1| = |yf_2|$. Since xf_1 and yf_2 are facets of Δ, they are not in $I(\Delta)$. Next consider xf_2 and yf_1; we must show they are also not in $I(\Delta)$. For these faces not to be in $I(\Delta)$, they must contain facets. However, $|xf_2| = |yf_2| = |xf_1| = |yf_1|$ so if they contained a facet it would be of smaller size, and this can not be because Δ is pure. Hence xf_2 and yf_1 are in $I(\Delta)$, which contradicts $I(\Delta)$ being shifted.

$\square$

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