Confluence in Labeled Chip-Firing

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The Labeled Chip-Firing Problem

Place *n* chips, labeled from 1 to *n*, at the origin in a 1D grid.

4
3
2
_ _ 1 _ _

Perform firing moves by choosing 2 chips at the same site, sending the smaller one left and the larger one right.

2
_ 3 1 4 _

1 2
_ 3 _ 4 _

1
_ 3 2 _ 4

3
1 _ 2 _ 4

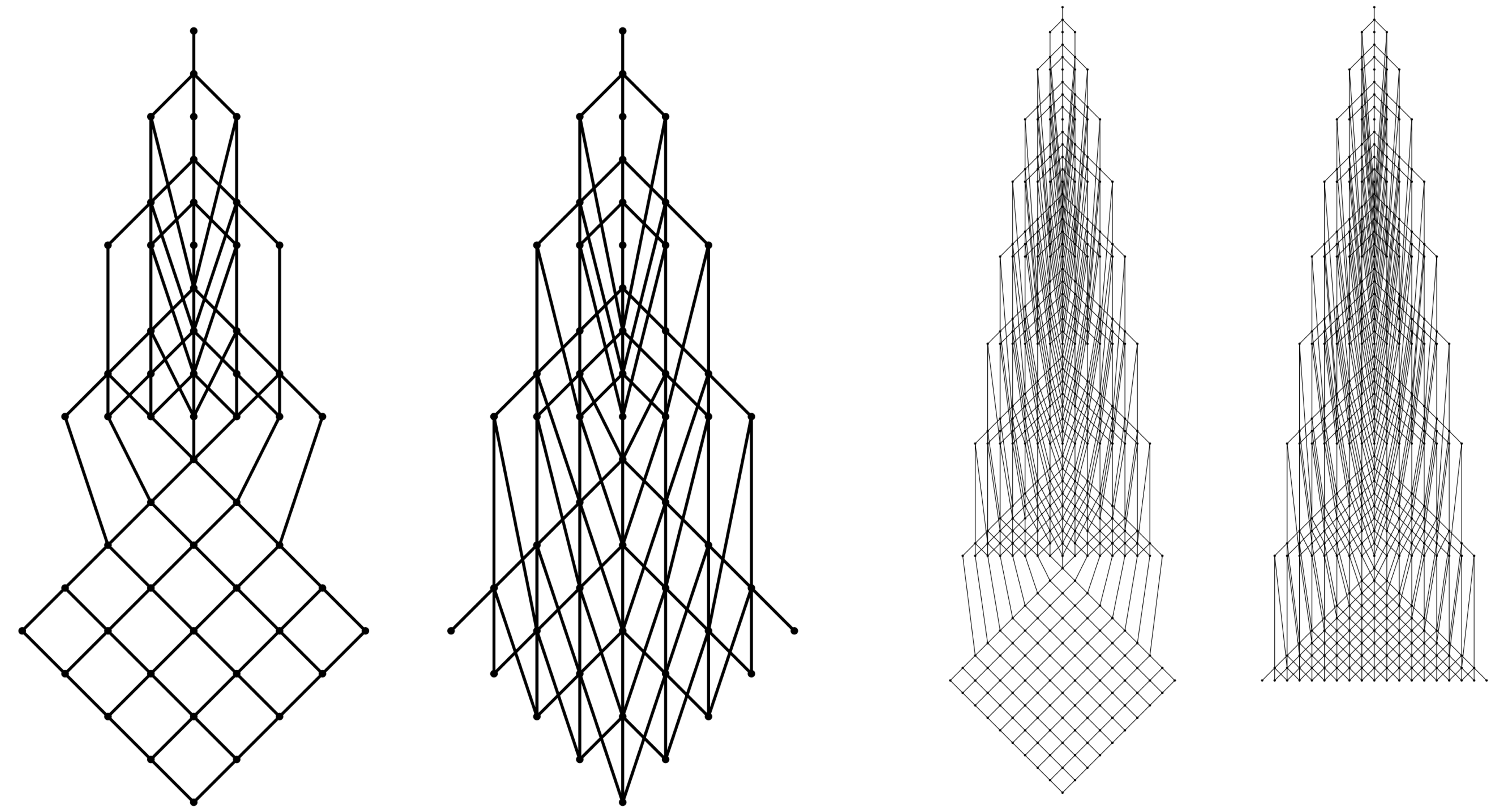
1 2 _ 3 4

Continue until no firing moves are available.

Theorem (Hopkins, McConville, Propp 2016): If *n* is even, then the chips end in sorted order, regardless of the choices of moves.

Novel Proof of Sorting Theorem

The firing moves in chip-firing on a line exhibit a nice poset structure:



From left to right: Hasse diagrams for firing move posets for *n*=10, 11, 20, and 21.

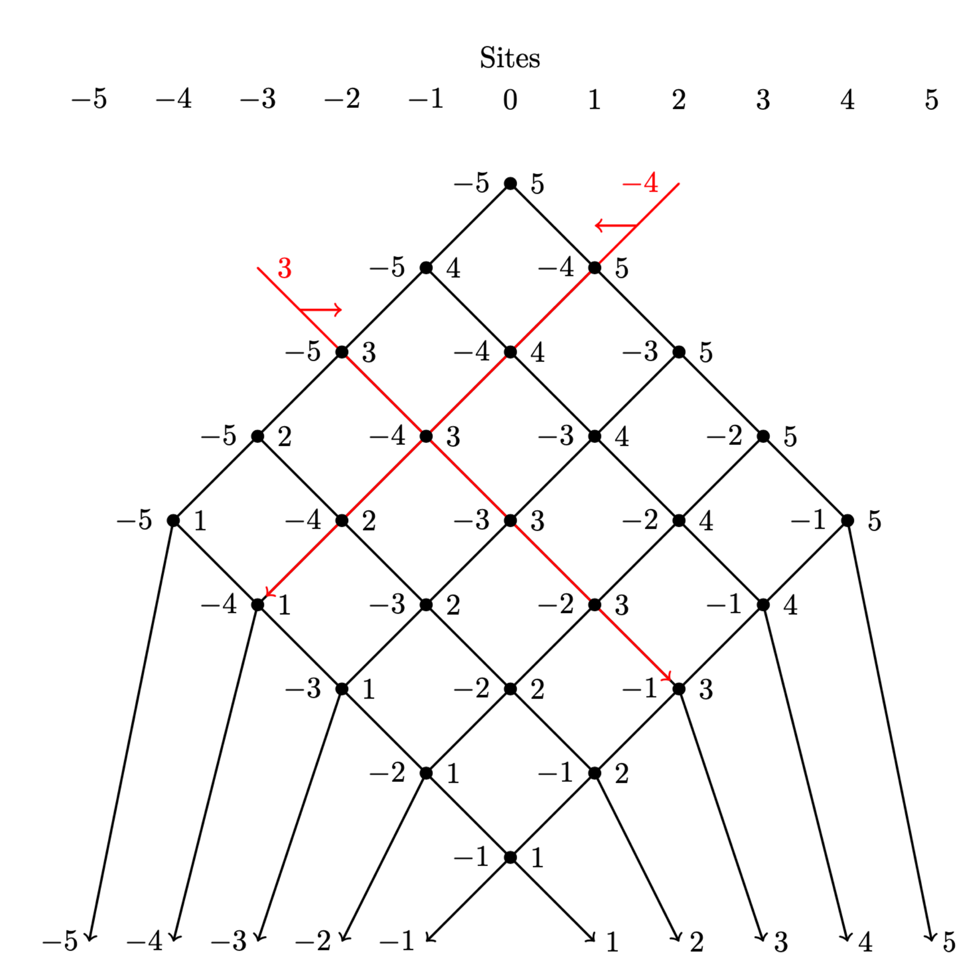
Firing Poset: $k_j \geq k'_{j'}$ if move j at site k must occur at or before move j' at site k'

Step 1: Prove the existence of the grid at the bottom of the Hasse diagram for *n* even.

Step 2: Prove straightforward bounds on the maximum or minimum position of each chip throughout the process.

Step 3: Prove that chips meeting these bounds will be sent to the correct final position by the moves in the grid.

Right: we renumber the chips to go from –*n* to 1 and 1 to n. Chip 3 must remain at or to the right of the red line labeled 3, while chip -4 must remain at or to the left of the red line labeled -4. These bound the final position of every chip in a way that can only be satisfied when the chips are sorted.

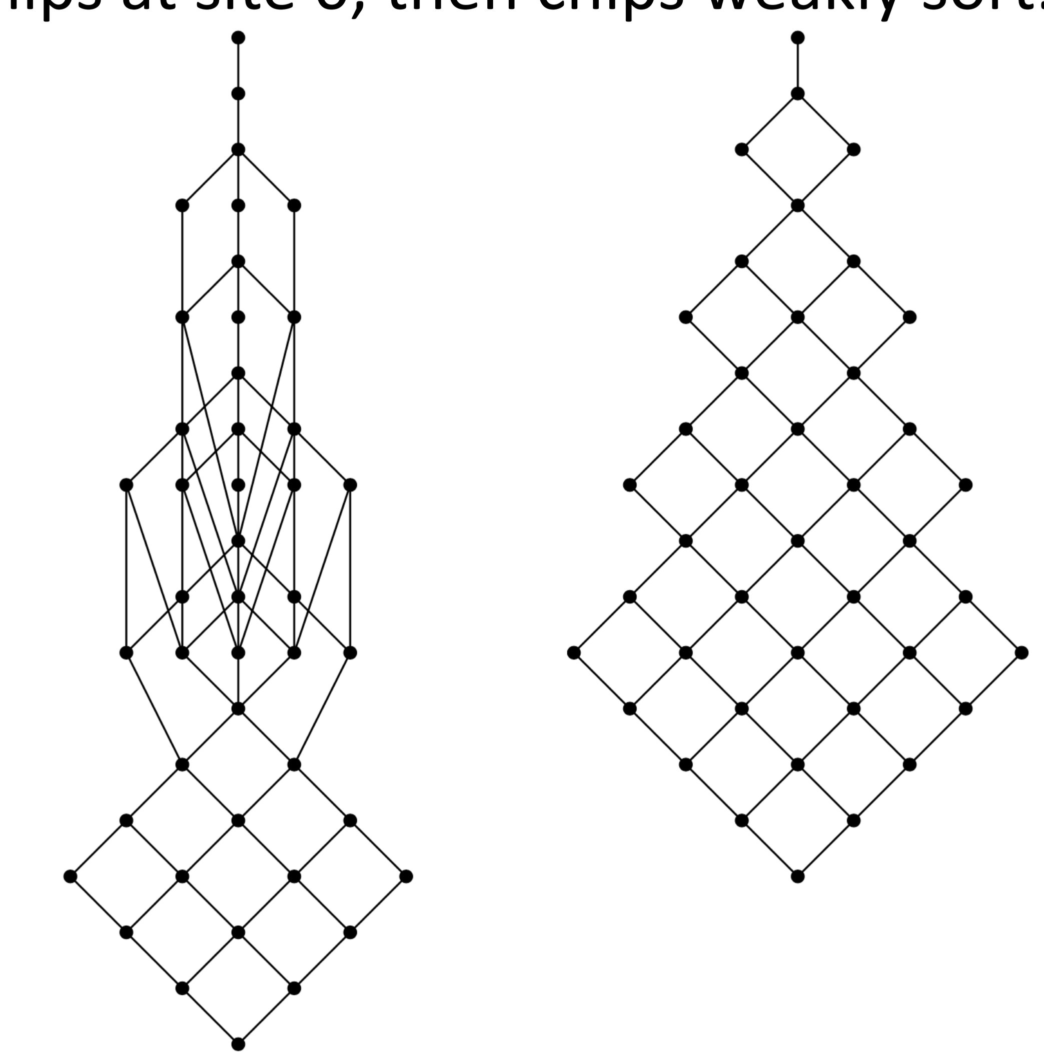


New Results

Theorem (conj. by Galashin et al 2017): if the initial configuration has *n* chips at site -1 and *n*+1 chips at site 0, then chips sort.

Theorem (conj. by Hopkins et al 2016): If each edge has *r* copies, and 2r divides *n*, then chips weakly sort.

Theorem (conj, by K., L.): For positive integer *t*, and *k* from 0 to *t*, place $2^{t-|k|}$ edges from site *k* to *k*+1, and from -*k* to -*k*-1. If the initial configuration has 2^{t+2} chips at site 0, then chips weakly sort.



Self-loop, *n*=15. Exponential, *t*=3, *n*=32.

Theorem (conj. by Hopkins et al 2016): if every site has a self-loop and an edge in each direction, and *n* is 1 mod 4, then chips weakly sort.

Theorem (conj. by Hopkins et al 2016): if every site has *r* self-loops, and *r* edges to the left and right, and *n* is *r* mod 4*r*, then chips weakly sort.