

THE POSITIVE BERGMAN COMPLEX OF AN ORIENTED MATROID

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ABSTRACT. We study the positive Bergman complex $\mathcal{B}^+(M)$ of an oriented matroid M , which is a certain subcomplex of the Bergman complex $\mathcal{B}(\underline{M})$ of the underlying unoriented matroid \underline{M} . The positive Bergman complex is defined so that given a linear ideal I with associated oriented matroid M_I , the positive tropical variety associated to I is equal to the fan over $\mathcal{B}^+(M_I)$. Our main result is that a certain “fine” subdivision of $\mathcal{B}^+(M)$ is a geometric realization of the order complex of the proper part of the Las Vergnas face lattice of M . It follows that $\mathcal{B}^+(M)$ is homeomorphic to a sphere. For the oriented matroid of the complete graph K_n , we show that the face poset of the “coarse” subdivision of $\mathcal{B}^+(K_n)$ is dual to the face poset of the associahedron A_{n-2} , and we give a formula for the number of fine cells within a coarse cell.

1. INTRODUCTION

In [2], Bergman defined the *logarithmic limit-set* of an algebraic variety in order to study its exponential behavior at infinity. We follow [13] in calling this set the *Bergman complex* of the variety. Bergman complexes have recently received considerable attention in several areas, such as tropical algebraic geometry and dynamical systems. They are the *non-Archimedean amoebas* of [5] and the *tropical varieties* of [10, 13].

When the variety is a linear space, so that the defining ideal I is generated by linear forms, Sturmfels [13] showed that the Bergman complex can be described solely in terms of the matroid associated to the linear ideal. He used this description to define the Bergman complex $\mathcal{B}(M)$ of an arbitrary matroid M . Ardila and Klivans [1] showed that, appropriately subdivided, the Bergman complex of a matroid M is the order complex of the proper part of the lattice of flats L_M of the matroid. This result implies that the Bergman complex of an arbitrary matroid M is a finite, pure polyhedral complex, which is homotopy equivalent to a wedge of spheres.

Total positivity is another topic which has received a great deal of recent interest. Although the classical theory concerns matrices in which all minors are positive, in the past decade this theory has been extended by Lusztig [6, 7], who introduced the totally positive variety $G_{>0}$ in an arbitrary reductive group G and the totally positive part $B_{>0}$ of a real flag variety B . More recently, the positive part of the tropicalization of an affine variety (or positive tropical variety, for short) was introduced by Speyer and Williams [11].

Sturmfels [14] suggested the notion of a positive Bergman complex $\mathcal{B}^+(M)$ of an oriented matroid M and conjectured its relation to the Las Vergnas face lattice of M . We define the positive Bergman complex and positive Bergman fan so that given a linear ideal I with associated oriented matroid M_I , the positive tropical variety associated to I is equal to the positive Bergman fan of M_I .

We give a proof of Sturmfels’ conjecture: appropriately subdivided, $\mathcal{B}^+(M)$ is a geometric realization of the order complex of the proper part of the Las Vergnas face lattice of M . $\mathcal{B}^+(M)$ is homeomorphic to a sphere and naturally sits inside $\mathcal{B}(\underline{M})$, the Bergman complex of the underlying unoriented matroid of M . We conclude by showing that, for the oriented matroid of the complete graph K_n , the face poset of a certain “coarse” subdivision of $\mathcal{B}^+(K_n)$ is dual to the face poset of the associahedron A_{n-2} .

The paper is organized as follows. In Section 2 we introduce a certain oriented matroid M_ω which will play an important role in our work. In Section 3 we define the positive Bergman complex and prove our main theorem. In Section 4 we explain the relation between the positive Bergman complex of an oriented matroid and the positive tropical variety of a linear ideal. In Section 5 we describe the topology of the positive Bergman complex of an oriented matroid. Finally, in Sections 6 and 7 we describe in detail the positive Bergman complex of the oriented matroid of K_n : we relate it to the associahedron, and we give a formula for the number of full-dimensional fine cells within a full-dimensional coarse cell.

Throughout this paper we will abuse notation and use M to denote either a matroid or oriented matroid, depending on the context. Similarly, we will use the term “circuits” to describe either unsigned or signed circuits. When the distinction between matroids and oriented matroids is important, we will use \underline{M} to denote the underlying matroid of an oriented matroid M .

2. THE ORIENTED MATROID M_ω

Let M be an oriented matroid on the ground set $[n] = \{1, 2, \dots, n\}$ whose collection of signed circuits is \mathcal{C} . Let $\omega \in \mathbb{R}^n$ and regard ω as a weight function on $[n]$. For any circuit $C \in \mathcal{C}$ define $\text{in}_\omega(C)$ to be the ω -maximal subset of the circuit C – in other words, the collection of elements of C which have the largest weight. We will say that the circuit C *achieves its largest weight with respect to ω* at $\text{in}_\omega(C)$. Define $\text{in}_\omega(\mathcal{C})$ to be the collection of inclusion-minimal sets of the collection $\{\text{in}_\omega(C) \mid C \in \mathcal{C}\}$. We then define M_ω to be the oriented matroid on $[n]$ whose collection of circuits is $\text{in}_\omega(\mathcal{C})$.

It is not clear that M_ω is a well-defined oriented matroid; we will prove this shortly.

Given $\omega \in \mathbb{R}^n$, let $\mathcal{F}(\omega)$ denote the unique flag of subsets $\emptyset = F_0 \subset F_1 \subset \dots \subset F_k \subset F_{k+1} = [n]$ such that ω is constant on each set $F_i \setminus F_{i-1}$ and satisfies $\omega|_{F_i \setminus F_{i-1}} < \omega|_{F_{i+1} \setminus F_i}$ for all $1 \leq i \leq k$. We call $\mathcal{F}(\omega)$ the *flag* of ω , and we say that the *weight class* of ω or of the flag \mathcal{F} is the set of vectors ν such that $\mathcal{F}(\nu) = \mathcal{F}$.

It is clear that M_ω depends only on the flag $\mathcal{F} := \mathcal{F}(\omega)$ and so we also refer to this oriented matroid as $M_{\mathcal{F}}$.

Example 2.1. *Let M be the oriented matroid of the digraph D shown in Figure 1. Equivalently, let M be the oriented matroid of the point configuration shown in Figure 2.*

Note that D is an acyclic orientation of K_4 , the complete graph on 4 vertices. The signed circuits \mathcal{C} of M are $\{1\bar{2}4, 1\bar{3}5, 2\bar{3}6, 4\bar{5}6, 1\bar{2}5\bar{6}, 1\bar{3}4\bar{6}, 2\bar{3}4\bar{5}\}$, together with the negatives of every set of this collection. Choose ω such that $\omega_6 < \omega_1 = \omega_2 = \omega_3 = \omega_4 = \omega_5$, which corresponds to the flag $\emptyset \subset \{6\} \subset \{1, 2, 3, 4, 5, 6\}$.

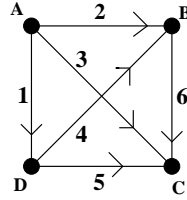


FIGURE 1. The digraph D

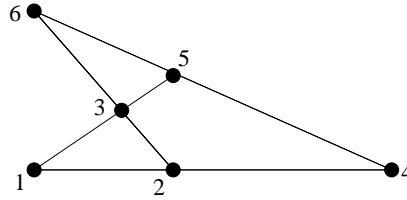


FIGURE 2. A point configuration

If we calculate $\text{in}_\omega(C)$ for each $C \in \mathcal{C}$ we get $\{\bar{1}\bar{2}4, \bar{1}\bar{3}5, \bar{2}\bar{3}, \bar{4}\bar{5}, \bar{1}\bar{2}5, \bar{1}\bar{3}4, \bar{2}\bar{3}\bar{4}\bar{5}\}$, together with the negatives of every set of this collection. However, $\bar{2}\bar{3}\bar{4}\bar{5}$ is not inclusion-minimal in this collection, as it contains $\bar{2}\bar{3}$ and $\bar{4}\bar{5}$. Thus $\text{in}_\omega(\mathcal{C})$ is equal to $\{\bar{1}\bar{2}4, \bar{1}\bar{3}5, \bar{2}\bar{3}, \bar{4}\bar{5}, \bar{1}\bar{2}5, \bar{1}\bar{3}4\}$, together with the negatives of every set, and M_ω is the oriented matroid whose collection of signed circuits is $\text{in}_\omega(\mathcal{C})$. Notice that in this case M_ω is the oriented matroid of the digraph D' in Figure 3.

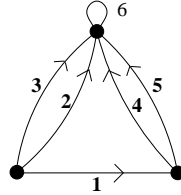


FIGURE 3. The digraph D' .

We must show that M_ω is well-defined. For convenience, we review here the *circuit axioms for oriented matroids* [4]:

- C1. \emptyset is not a signed circuit.
- C2. If X is a signed circuit, then so is $-X$.
- C3. No proper subset of a circuit is a circuit.
- C4. If X_0 and X_1 are circuits with $X_1 \neq -X_0$ and $e \in X_0^+ \cap X_1^-$, then there is a third circuit $X \in \mathcal{C}$ with $X^+ \subseteq (X_0^+ \cup X_1^+) \setminus \{e\}$ and $X^- \subseteq (X_0^- \cup X_1^-) \setminus \{e\}$.

We will also need the following stronger characterization of oriented matroids:

Theorem 2.2. [4, Theorem 3.2.5] *Let \mathcal{C} be a collection of signed subsets of a set E satisfying C1, C2, C3. Then C4 is equivalent to C4':*

- C4'. *for all $X_0, X_1 \in \mathcal{C}$, $e \in X_0^+ \cap X_1^-$ and $f \in (X_0^+ \setminus X_1^-) \cup (X_0^- \setminus X_1^+)$, there is a $Z \in \mathcal{C}$ such that $Z^+ \subseteq (X_0^+ \cup X_1^+) \setminus \{e\}$, $Z^- \subseteq (X_0^- \cup X_1^-) \setminus \{e\}$, and $f \in Z$.*

Proposition 2.3. *Let M be an oriented matroid on $[n]$ and $\omega \in \mathbb{R}^n$. Then M_ω is an oriented matroid.*

Proof. The strategy of our proof is to show that if C1, C2, C3 and C4' hold for M , then C1, C2, C3 and C4 hold for M_ω . First note that it is obvious that C1 and C2 hold for M_ω . C3 holds for M_ω because we defined $\text{in}_\omega(\mathcal{C})$ to consist of *inclusion-minimal* elements. It remains to show that C4 holds for M_ω . To do this, we start with two circuits in M_ω , lift them to circuits in M , and then use C4' for M to show that C4 holds for M_ω .

Take Y_0 and Y_1 in $\text{in}_\omega(\mathcal{C})$ such that $Y_1 \neq -Y_0$ and $e \in Y_0^+ \cap Y_1^-$. By definition, there exist circuits X_0 and X_1 of M such that $Y_0 = \text{in}_\omega(X_0)$ and $Y_1 = \text{in}_\omega(X_1)$. Notice that the presence of e in Y_0 and Y_1 guarantees that the maximum weights occurring in X_0 and in X_1 are both equal to ω_e .

Choose any $f \in (Y_0^+ \setminus Y_1^-) \cup (Y_0^- \setminus Y_1^+)$. Clearly such an f exists. Then $f \in (X_0^+ \setminus X_1^-) \cup (X_0^- \setminus X_1^+)$. By C4' for M , there exists a circuit X in \mathcal{C} such that

- (a₁) $X^+ \subseteq (X_0^+ \cup X_1^+) \setminus \{e\}$,
- (a₂) $X^- \subseteq (X_0^- \cup X_1^-) \setminus \{e\}$, and
- (a₃) $f \in X$.

Look at $\text{in}_\omega(X)$. We will prove that $\text{in}_\omega(X)$ contains the third circuit of M_ω which we are looking for. We want to show that

- (b₁) $\text{in}_\omega(X)^+ \subseteq (Y_0^+ \cup Y_1^+) \setminus \{e\}$
- (b₂) $\text{in}_\omega(X)^- \subseteq (Y_0^- \cup Y_1^-) \setminus \{e\}$

First, it is obvious that e is not in $\text{in}_\omega(X)^+$, since e was not in X^+ . Clearly $\text{in}_\omega(X)^+$ is a subset of $X_0^+ \cup X_1^+$. To show (b₁) and (b₂), we just need to show that the maximum weight which occurs in X is also equal to ω_e . By (a₁) and (a₂), this maximum weight is at most ω_e . By (a₃), equality is attained for $f \in X$, since $\omega_f = \omega_e$.

Note that if $\text{in}_\omega(X)$ is not inclusion-minimal in the set $\{\text{in}_\omega(C) \mid C \text{ a circuit of } M\}$, then it contains some inclusion-minimal $\text{in}_\omega(W)$ for another circuit W of M . And since $\text{in}_\omega(X)^+ \subseteq (Y_0^+ \cup Y_1^+) \setminus \{e\}$ and $\text{in}_\omega(X)^- \subseteq (Y_0^- \cup Y_1^-) \setminus \{e\}$, it is clear that we also have $\text{in}_\omega(W)^+ \subseteq (Y_0^+ \cup Y_1^+) \setminus \{e\}$ and $\text{in}_\omega(W)^- \subseteq (Y_0^- \cup Y_1^-) \setminus \{e\}$. □

In the following proposition, we describe the bases of M_ω and their orientations. Let us say that if $S \subset [n]$ and $\omega \in \mathbb{R}^n$, the ω -weight of S is the sum $\sum_{i \in S} \omega_i$.

Proposition 2.4. *The bases of M_ω are the bases of M which have minimal ω -weight. The basis orientations of M_ω are equal to their orientations in M .*

Proof. We know that if N is an oriented matroid on $[n]$ with signed circuits \mathcal{C} , then the bases of N are the maximal subsets of $[n]$ which contain no circuit. Thus, the bases \mathcal{B} of M_ω are the maximal subsets of $[n]$ which do not contain a set of the collection $\{\text{in}_\omega(C) \mid C \text{ a circuit of } M\}$. We want to show that \mathcal{B} is exactly the set of bases of M which have minimal ω -weight.

First let us choose a basis B of M which has minimal ω -weight. We claim that B is independent in M_ω . Suppose that B contains a subset of the form $\text{in}_\omega(C)$, where C is

a circuit of M . Write $B = \text{in}_\omega(C) \cup \{b_1, \dots, b_m\}$, and choose any $c \in \text{in}_\omega(C)$. We now construct a new basis B'' of M of smaller weight than B , as follows. Start with the set C . We know that $C \cup \{b_1, \dots, b_m\}$ is a spanning set of M . Let B_0 be a minimal subset (possibly empty) of $\{b_1, \dots, b_m\}$ such that $C \cup B_0$ is still a spanning set of M . Since C is minimally dependent in M , the set $B' := (C \setminus \{c\}) \cup B_0$ will be a basis of M . Now by the basis exchange axiom, for some $b \in B' \setminus B$, $B'' := (B \setminus \{c\}) \cup \{b\}$ is a basis of M . Since $b \in B' \setminus B$, it follows that $b \in C \setminus \text{in}_\omega(C)$. Thus, the weight of b is strictly less than the weight of c , which implies that the weight of B'' is less than the weight of B . This is a contradiction.

The previous argument shows that $r(M) \leq r(M_\omega)$. But every circuit of M_ω is contained in a circuit of M , so $r(M) \geq r(M_\omega)$. It follows that M and M_ω have the same rank.

Now let B be a basis of M_ω , *i.e.* B is a maximal subset of $[n]$ which does not contain an element of $\{\text{in}_\omega(C) \mid C \text{ a circuit of } M\}$. We claim that B is a basis of M with minimal ω -weight. First note that B is clearly independent in M : if it were dependent in M , it would contain some circuit C of M and hence would contain $\text{in}_\omega(C)$. Also, B has $r(M_\omega) = r(M)$ elements. Therefore it is a basis of M .

Finally, let us show that B has minimal ω -weight. Suppose not. Let c_1, \dots, c_r be the elements of B with highest weight. We claim that $\{c_1, \dots, c_r\} \supseteq \text{in}_\omega(C)$ for some circuit C of M , which will be a contradiction. Since B is a basis of M , adding to B any element of $[n] \setminus B$ creates a circuit. Since B is not a basis of minimal ω -weight, there must be an element $b \in [n] \setminus B$ such that the weight of b is strictly less than the weight of each of the elements c_1, \dots, c_r . Thus $B \cup b$ contains a circuit C , and $\text{in}_\omega(C) \subseteq \{c_1, \dots, c_r\}$, as claimed.

To prove the claim about orientations, start with a basis B of minimal weight of M . Recall that an oriented matroid has exactly two basis orientations, which are opposite to each other. Therefore we can assume without loss of generality that B has the same orientation in M and M_ω .

For any two ordered bases $B_1 = (e, x_2, \dots, x_r)$ and $B_2 = (f, x_2, \dots, x_r)$ of M_ω with $e \neq f$, we have

$$\chi_\omega(e, x_2, \dots, x_r) = -C_\omega(e)C_\omega(f)\chi_\omega(f, x_2, \dots, x_r),$$

where χ_ω is the chirotope of M_ω , and C_ω is one of the two opposite signed circuits of M_ω in $\{e, f, x_2, \dots, x_r\}$. Now B_1 and B_2 are also bases of M ; let C be one of the two opposite signed circuits of M in $\{e, f, x_2, \dots, x_r\}$. Then $\text{in}_\omega(C)$ contains a circuit of M_ω ; it must be either C_ω or $-C_\omega$. In any case, we have $C(e)C(f) = C_\omega(e)C_\omega(f)$, so $\chi_\omega(B_1)\chi_\omega(B_2) = \chi(B_1)\chi(B_2)$. It follows that if B_1 has the same orientation in M and M_ω , then so does B_2 .

Recall that one can obtain any basis of a matroid from any other by a sequence of simple basis exchanges of the type above. Since B has the same orientation in M and M_ω , so does any other basis of M_ω . \square

3. THE POSITIVE BERGMAN COMPLEX

Our goal in this section is to define the positive Bergman complex of an oriented matroid M and to relate it to the Las Vergnas face lattice of M , thus answering Sturmfels'

question [14]. We begin by giving some background on the Bergman complex and fan of a (unoriented) matroid.

The *Bergman fan* of a matroid M on the ground set $[n]$ is the set

$$\widetilde{\mathcal{B}}(M) := \{\omega \in \mathbb{R}^n : M_\omega \text{ has no loops}\}.$$

The *Bergman complex* of M is

$$\mathcal{B}(M) := \{\omega \in S^{n-2} : M_\omega \text{ has no loops}\},$$

where S^{n-2} is the sphere $\{\omega \in \mathbb{R}^n : \omega_1 + \cdots + \omega_n = 0, \omega_1^2 + \cdots + \omega_n^2 = 1\}$.

For simplicity, in this section we will concentrate on the Bergman complex of M , but similar arguments hold for the Bergman fan of M .

Since the matroid M_ω depends only on the weight class that ω is in, the Bergman complex of M is a disjoint union of the weight classes of flags \mathcal{F} such that $M_{\mathcal{F}}$ has no loops. We say that the weight class of a flag \mathcal{F} is *valid* for M if $M_{\mathcal{F}}$ has no loops.

There are two polyhedral subdivisions of $\mathcal{B}(M)$, one of which is clearly finer than the other. The *fine subdivision* of $\mathcal{B}(M)$ is the subdivision of $\mathcal{B}(M)$ into valid weight classes: two vectors u and v of $\mathcal{B}(M)$ are in the same class if and only if $\mathcal{F}(u) = \mathcal{F}(v)$. The *coarse subdivision* of $\mathcal{B}(M)$ is the subdivision of $\mathcal{B}(M)$ into M_ω -equivalence classes: two vectors u and v of $\mathcal{B}(M)$ are in the same class if and only if $M_u = M_v$.

The following results give alternative descriptions of $\mathcal{B}(M)$:

Theorem 3.1. [1] *Given an (unoriented) matroid M on the ground set $[n]$ and $\omega \in \mathbb{R}^n$ which corresponds to a flag $\mathcal{F} := \mathcal{F}(\omega)$, the following are equivalent:*

1. $M_{\mathcal{F}}$ has no loops.
2. For each circuit C of M , $\text{in}_\omega(C)$ contains at least two elements of C .
3. \mathcal{F} is a flag of flats of M .

Corollary 3.2. [1] *Let M be a (unoriented) matroid. Then the fine subdivision of the Bergman complex $\mathcal{B}(M)$ is a geometric realization of $\Delta(L_M - \{\hat{0}, \hat{1}\})$, the order complex of the proper part of the lattice of flats of M .*

We are now ready for the positive analogues of these concepts. The *positive Bergman fan* of an oriented matroid M on the ground set $[n]$ is

$$\widetilde{\mathcal{B}}^+(M) := \{\omega \in \mathbb{R}^n : M_\omega \text{ is acyclic}\}.$$

The *positive Bergman complex* of M is

$$\mathcal{B}^+(M) := \{\omega \in S^{n-2} : M_\omega \text{ is acyclic}\}.$$

Within each equivalence class of the coarse subdivision of $\mathcal{B}(M)$, the vectors ω give rise to the same unoriented M_ω . Since the orientation of M_ω is inherited from that of M , they also give rise to the same oriented matroid M_ω . Therefore each coarse cell of $\mathcal{B}(M)$ is either completely contained in or disjoint from $\mathcal{B}^+(M)$. Thus $\mathcal{B}^+(M)$ inherits the coarse and the fine subdivisions from $\mathcal{B}(M)$, and each subdivision of $\mathcal{B}^+(M)$ is a subcomplex of the corresponding subdivision of $\mathcal{B}(M)$.

Let M be an acyclic oriented matroid on the ground set $[n]$. We say that a covector $v \in \{+, -, 0\}^n$ of M is *positive* if each of its entries is $+$ or 0 . We say that a flat of M is

positive if it is the 0-set of a positive covector. Additionally, we consider the set $[n]$ to be a positive flat. For example, if M is the matroid of Example 2.1, then 16 is a positive flat which is the 0-set of the positive covector $(0 + + + + 0)$.

The *Las Vergnas face lattice* $\mathcal{F}_{lv}(M)$ is the lattice of positive flats of M , ordered by containment. Note that the lattice of positive flats of the oriented matroid M sits inside L_M , the lattice of flats of M .

Example 3.3. Let M be the oriented matroid from Example 2.1. The positive covectors of M are $\{0 + + + + 0, 00 + 0 + +, + + + 000, 0 + + + + +, + + + 0 + +, + + + + + 0, + + + + + +\}$ and the positive flats are $\{16, 124, 456, 1, 4, 6, \emptyset, 123456\}$. The lattice of positive flats of M is shown in Figure 4, alongside the lattice of flats of M .

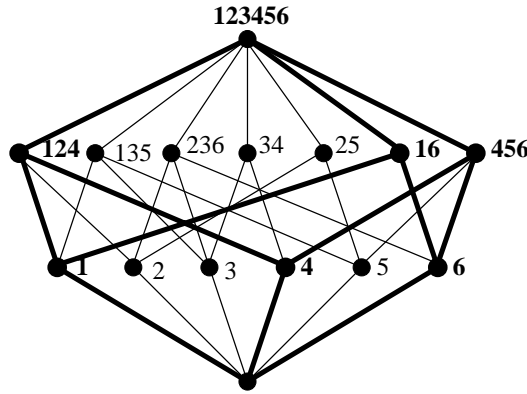


FIGURE 4. The lattice of positive flats and the lattice of flats.

We now give an analogue of Theorem 3.1.

Theorem 3.4. Given an oriented matroid M and $\omega \in \mathbb{R}^n$ which corresponds to a flag $\mathcal{F} := \mathcal{F}(\omega)$, the following are equivalent:

1. $M_{\mathcal{F}}$ is acyclic.
2. For each signed circuit C of M , $\text{in}_{\omega}(C)$ contains a positive element and a negative element of C .
3. \mathcal{F} is a flag of positive flats of M .

Proof. First we will show that 1 and 2 are equivalent. The statement that M_{ω} is acyclic means that M_{ω} has no all-positive circuit: in other words, each circuit of M_{ω} contains a positive and a negative term. Since M_{ω} is the matroid whose circuits are the inclusion-minimal elements of the set $\{\text{in}_{\omega}(C) \mid C \text{ a circuit of } M\}$, this means that for each circuit C of M , $\text{in}_{\omega} C$ contains a positive and a negative term. Finally, this is equivalent to the statement that for each circuit C of M , C achieves its maximum value with respect to ω on both C^+ and C^- .

Next we show that 3 implies 2. Assume we have an ω such that \mathcal{F} is a flag of positive flats. Let the flats of this flag be $F_1 \subset F_2 \subset \dots \subset F_k$. For each F_i , $([n] - F_i)$ is a positive covector. By orthogonality of circuits and covectors, we know that for any circuit C and

any covector Y , $(C^+ \cap Y)$ and $(C^- \cap Y)$ are either both empty or both non-empty. For any circuit C of M , consider the largest i such that $C \cap ([n] - F_i)$ is non-empty. Then clearly C will attain its maximum on $F_{i+1} - F_i$ and $\text{in}_\omega(C)$ contains a positive element and a negative element of C .

Finally, assume that 1 and 2 hold, but 3 does not. From 1 we know that $M_{\mathcal{F}}$ is acyclic; therefore the unoriented $M_{\mathcal{F}}$ has no loops, and \mathcal{F} is a flag of flats by Theorem 3.1. Let F_i be a flat which is not positive; by [4, Proposition 9.1.2] this is equivalent to saying that M/F_i is not acyclic. Let C be a positive circuit of M/F_i ; then we can find a circuit X of M such that $C = X - F_i$. Then X has positive elements of weight greater than ω_i , and no negative elements of weight greater than ω_i . It follows that $\text{in}_\omega(X)$ is positive, contradicting 2. \square

Corollary 3.5. *Let M be an oriented matroid. Then the fine subdivision of $\mathcal{B}^+(M)$ is a geometric realization of $\Delta(\mathcal{F}_{\text{lv}}(M) - \{\hat{0}, \hat{1}\})$, the order complex of the proper part of the Las Vergnas face lattice of M .*

4. CONNECTION WITH POSITIVE TROPICAL VARIETIES

In [11], the notion of the positive part of the tropicalization of an affine variety (or positive tropical variety, for short) was introduced, an object which has the structure of a polyhedral fan in \mathbb{R}^n . In order to describe this object, we must define an *initial ideal*.

Let $\mathcal{R} = \mathbb{R}[x_1, \dots, x_n]$ and $\omega \in \mathbb{R}^n$. If $f = \sum c_i \mathbf{x}^{\mathbf{a}_i} \in \mathcal{R}$, define the *initial form* $\text{in}_\omega(f) \in \mathcal{R}$ to be the sum of all terms $c_i \mathbf{x}^{\mathbf{a}_i}$ such that the inner product $\omega \cdot \mathbf{a}_i$ is maximal. For an ideal I of $\mathbb{R}[x_1, \dots, x_n]$, define the *initial ideal* $\text{in}_\omega(I)$ to be the ideal generated by $\text{in}_\omega(f)$ for all $f \in I$.

If I is an ideal in a polynomial ring with n variables, the positive tropical variety associated to I is denoted by $\text{Trop}^+ V(I)$ and can be characterized as follows:

$$\text{Trop}^+ V(I) = \{\omega \in \mathbb{R}^n \mid \text{in}_\omega(I) \text{ contains no nonzero polynomials in } \mathbb{R}^+[x_1, \dots, x_n]\}.$$

Now recall that if I is a linear ideal (an ideal generated by linear forms), we can associate to it an oriented matroid M_I as follows. Write each linear form $f \in I$ in the form $a_1 x_{i_1} + a_2 x_{i_2} + \dots + a_m x_{i_m} = b_1 x_{j_1} + b_2 x_{j_2} + \dots + b_n x_{j_n}$, where $a_i, b_i > 0$ for all i . We then define M_I to be the oriented matroid whose set of signed circuits consists of all minimal collections of the form $\{i_1 i_2 \dots i_m \bar{j}_1 \bar{j}_2 \dots \bar{j}_n\}$. We now prove the following easy statement.

Proposition 4.1. *If I is a linear ideal and M_I is the associated oriented matroid, then $\text{Trop}^+ V(I) = \tilde{\mathcal{B}}^+(M_I)$.*

Proof.

$$\begin{aligned} \tilde{\mathcal{B}}^+(M_I) &= \{\omega \in \mathbb{R}^n \mid (M_I)_\omega \text{ is acyclic}\} \\ &= \{\omega \in \mathbb{R}^n \mid (M_I)_\omega \text{ has no all-positive circuit}\} \\ &= \{\omega \in \mathbb{R}^n \mid M_{\text{in}_\omega(I)} \text{ has no all-positive circuit}\} \\ &= \{\omega \in \mathbb{R}^n \mid \text{in}_\omega(I) \text{ contains no nonzero polynomial in } \mathbb{R}^+[x_1, \dots, x_n]\} \\ &= \text{Trop}^+ V(I). \end{aligned}$$

□

5. TOPOLOGY OF THE POSITIVE BERGMAN COMPLEX

The topology of the positive Bergman complex of an oriented matroid is very simple: it is homeomorphic to a sphere. This follows from Corollary 3.5 together with results about the Las Vergnas face lattice, which we will review here.

Theorem 5.1. [4, Theorem 4.3.5] *Let M be an acyclic oriented matroid of rank r . Then the Las Vergnas lattice $\mathcal{F}_{lv}(M)$ is isomorphic to the face lattice of a PL regular cell decomposition of the $(r - 2)$ -sphere.*

Proposition 5.2. [4, Proposition 4.7.8] *Let Δ be a regular cell complex. Then its geometric realization is homeomorphic to the geometric realization of the order complex of its face poset.*

The previous two results imply that the geometric realization of the order complex of the Las Vergnas lattice is homeomorphic to a sphere.

Putting this together with Corollary 3.5, we get the following result.

Corollary 5.3. *The positive Bergman complex of an oriented matroid is homeomorphic to a sphere.*

6. THE POSITIVE BERGMAN COMPLEX OF THE COMPLETE GRAPH

In this section, we wish to describe the positive Bergman complex $\mathcal{B}^+(K_n)$ of the graphical oriented matroid $M(K_n)$ of an acyclic orientation of the complete graph K_n . We start by reviewing the description of the Bergman complex $\mathcal{B}(K_n)$ of the unoriented matroid $M(K_n)$, obtained in [1]. For the moment we need to consider K_n as an unoriented graph.

An *equidistant n -tree* T is a rooted tree with n leaves labeled $1, \dots, n$, and lengths assigned to each edge in such a way that the total distances from the root to each leaf are all equal. The internal edges are required to have positive lengths. Figure 5 shows an example of an equidistant 4-tree.

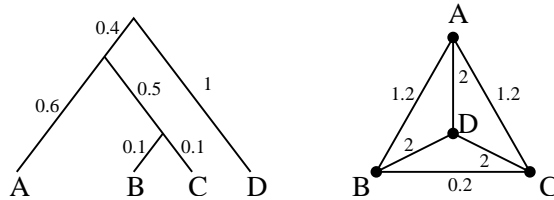


FIGURE 5. An equidistant tree and its corresponding distance vector.

To each equidistant n -tree T we assign a distance vector $d_T \in \mathbb{R}^{\binom{n}{2}}$: the distance d_{ij} is equal to the length of the path joining leaves i and j in T . Figure 5 also shows the distance vector of the tree, regarded as a weight function on the edges of K_4 .

The Bergman fan $\tilde{\mathcal{B}}(K_n)$ can be regarded as a space of equidistant n -trees, as the following theorem shows.

Theorem 6.1. [1, 9] *The distance vector of an equidistant n -tree, when regarded as a weight function on the edges of K_n , is in the Bergman fan $\tilde{\mathcal{B}}(K_n)$. Conversely, any point in $\tilde{\mathcal{B}}(K_n)$ is the distance vector of a unique equidistant n -tree.*

As mentioned earlier, the fine subdivision of $\tilde{\mathcal{B}}(M)$ is well understood for any matroid M . The following theorem shows that the coarse subdivision of $\tilde{\mathcal{B}}(K_n)$ also has a nice description: it is a geometric realization of the well-studied simplicial complex of trees T_n , sometimes called the Whitehouse complex [3, 8].

Theorem 6.2. [1] *Let $\omega, \omega' \in \tilde{\mathcal{B}}(K_n)$. Let T and T' be the corresponding equidistant n -trees. The following are equivalent:*

1. ω and ω' are in the same cell of the coarse subdivision.
2. T and T' have the same combinatorial type.

Now we return to the setting of oriented matroids. The positive Bergman complex $\mathcal{B}^+(K_n)$ is defined in terms of an acyclic orientation of K_n . This graph has $n!$ acyclic orientations, corresponding to the $n!$ permutations of $[n]$. The orientation corresponding to the permutation π is given by $\pi_i \rightarrow \pi_j$ for $i < j$. Clearly the $n!$ orientations of K_n will give rise to positive Bergman complexes which are equal up to relabeling. Therefore, throughout this section, the edges of K_n will be oriented $i \rightarrow j$ for $i < j$.

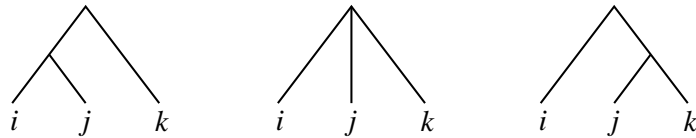
As we go around a cycle C of K_n , C^+ is the set of edges which are crossed in the forward direction, and C^- is the set of edges which are crossed in the backward direction.

Proposition 6.3. *Let ω be a weight vector on the edges of the oriented complete graph K_n . Let T be the corresponding equidistant tree. The following are equivalent:*

1. ω is in $\mathcal{B}^+(K_n)$.
2. T can be drawn in the plane without crossings in such a way that its leaves are numbered $1, 2, \dots, n$ from left to right.

Proof. We add three intermediate steps to the equivalence:

- (a) *In any cycle C , the ω -maximum is achieved in C^+ and C^- .*
- (b) *In any triangle C , the ω -maximum is achieved in C^+ and C^- .*
- (c) *For any three leaves $i < j < k$ in T , the leaf j does not branch off before i and k ; i.e., their branching order is one of the following:*



The equivalence $1 \Leftrightarrow (a)$ follows from Theorem 3.4, and the implication $(a) \Rightarrow (b)$ is trivial. Now we show that $(b) \Rightarrow (a)$. Proceed by contradiction. Consider a cycle $C = v_1 \dots v_k$, with k minimal, such that (a) is not satisfied. Consider the cycles $T = v_1 v_{k-1} v_k$ and $C' = v_1 v_2 \dots v_{k-1}$, which do satisfy (a) . Since C does not satisfy (a) , the edge $v_1 v_{k-1}$ must be ω -maximum in T , along with another edge e of the opposite orientation. Similarly, the edge $v_{k-1} v_1$ must be ω -maximum in C' , along with another edge f of the

opposite orientation. Therefore, in C , the edges e and f are ω -maximum and have opposite orientations. This is a contradiction.

Let us now show $(b) \Leftrightarrow (c)$. In triangle ijk (where we can assume $i < j < k$), (b) holds if and only if we have one of the following:

$$\omega_{ij} < \omega_{jk} = \omega_{ik}, \quad \text{or} \quad \omega_{ij} = \omega_{jk} = \omega_{ik}, \quad \text{or} \quad \omega_{jk} < \omega_{ij} = \omega_{ik}.$$

These three conditions correspond, in that order, to the three possible branching orders of i, j and k in T prescribed by condition (c) .

Finally we show $(c) \Leftrightarrow 2$. The backward implication is immediate. We prove the forward implication by induction on n . The case $n = 3$ is clear. Now let $n \geq 4$, and assume that condition (c) holds. Consider a lowest internal node v ; it is incident to several leaves, which must have consecutive labels $i, i + 1, \dots, j$ by (c) . Let T' be the tree obtained from T by removing leaf i . This smaller tree satisfies (c) , so it can be drawn in the plane with the leaves in order from left to right. Now we simply find node v in this drawing, and attach leaf i to it, putting it to the left of all the other leaves incident to v . This is a drawing of T satisfying 2. \square

The *associahedron* A_{n-2} is a well-known $(n - 2)$ -dimensional polytope whose vertices correspond to planar rooted trees [15]. There is a close relationship between $\mathcal{B}^+(K_n)$ and A_{n-2} .

Corollary 6.4. *The face poset of the coarse subdivision of $\mathcal{B}^+(K_n)$, with a $\hat{1}$ attached, is dual to the face poset of the associahedron A_{n-2} .*

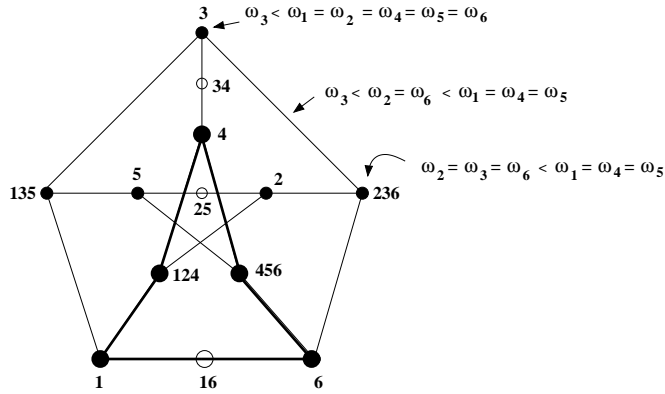
Proof. In the trees corresponding to the cells of $\mathcal{B}^+(K_n)$, the labeling of the leaves always increases from left to right. We can forget these labels and obtain the usual presentation of the dual to the associahedron, whose facets correspond to planar rooted trees. \square

Figure 6 shows the positive Bergman complex of K_4 (in bold) within the Bergman complex of K_4 . Vertices of the coarse subdivision are shown as black circles; vertices of the fine subdivision but not the coarse subdivision are shown as transparent circles. Observe that the coarse subdivision of $\mathcal{B}^+(K_4)$ is a pentagon, whose face poset is the face poset of the associahedron A_2 (which is self-dual).

Now, recall that different orientations of K_n give rise to different positive Bergman complexes. Let us make two comments about the way in which these positive Bergman complexes fit together.

Consider the $n!$ different acyclic orientations $o(\pi)$ of K_n , each corresponding to a permutation π of $[n]$. Each orientation $o(\pi)$ gives rise to a positive Bergman complex: it consists of those weight vectors such that the corresponding tree can be drawn with the leaves labeled π_1, \dots, π_n from left to right. Clearly, each permutation and its reverse give the same positive Bergman complex. The $\frac{n!}{2}$ possible positive Bergman complexes $\mathcal{B}^+(K_n)$ give a covering of $\mathcal{B}(K_n)$, and each one of them is dual to the associahedron A_{n-2} . This corresponds precisely to the known covering of the space of trees with $\frac{n!}{2}$ polytopes dual to the associahedron, as described in [3].

Also, recall from [1] that the Bergman complex $\mathcal{B}(K_n)$ is homotopic to a wedge of $(n - 1)!$ spheres. In fact, $\mathcal{B}(K_n)$ is covered by the $(n - 1)!$ dual associahedra corresponding to the

FIGURE 6. $\mathcal{B}^+(K_4) \subset \mathcal{B}(K_4)$

permutations π with $\pi_1 = 1$, because every tree can be drawn in the plane so that the leftmost leaf is labeled 1. This covering is optimal, since $\mathcal{B}(K_n)$ is homotopic to a wedge of $(n-1)!$ spheres.

7. THE NUMBER OF FINE CELLS IN $\mathcal{B}^+(K_n)$ AND $\mathcal{B}(K_n)$.

Since $\mathcal{B}^+(K_n)$ and $\mathcal{B}(K_n)$ are $(n-2)$ -dimensional, we will call the $(n-2)$ -dimensional cells inside them *full-dimensional*. In this section we will give a formula reminiscent of the “hook-length” formula for the number of full-dimensional fine cells within a full-dimensional coarse cell of $\mathcal{B}(K_n)$.

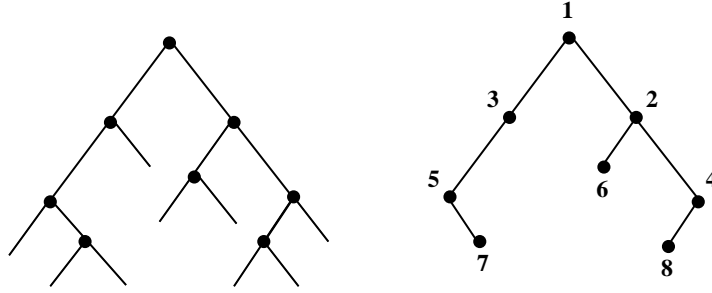
Proposition 7.1. *Let τ be a rooted binary tree with n labeled leaves. For each internal vertex v of τ , let $d(v)$ be the number of internal vertices of τ which are descendants of v , including v . Let $C(\tau)$ be the coarse cell of $\mathcal{B}(K_n)$ corresponding to tree τ . There are exactly*

$$\frac{(n-1)!}{\prod_v d(v)}$$

full-dimensional fine cells in $C(\tau)$.

Proof. The cell $C(\tau)$ consists of the distance vectors $d \in \mathbb{R}^{\binom{n}{2}}$ of all equidistant n -trees T of combinatorial type τ . Notice that $d_{ij} = 2h - 2h(v)$, where v is the lowest common ancestor of leaves i and j in T , $h(v)$ is the distance from v to the root of T , and h is the distance from the root of T to any of its leaves.

To specify a full-dimensional fine cell in $C(\tau)$, one needs to specify the relative order of the d_{ij} 's. Equivalently, in the tree T that d comes from, one needs to specify the relative order of the heights of the internal vertices, consistently with the combinatorial type of tree τ . Therefore, the fine cells in $C(\tau)$ correspond to the labellings of the $n-1$ internal vertices of τ with the numbers $1, 2, \dots, n-1$, such that the label of each vertex is smaller than the labels of its offspring. In the language of [12, Sec. 1.3], these are precisely the *increasing binary trees* of type τ' , where τ' is the result of removing the leaves of tree τ , and the edges incident to them. Figure 7 shows a tree type τ and one of the increasing binary trees of type τ' .

FIGURE 7. A tree type τ and an increasing binary tree of type τ' .

Suppose we choose one of the $(n-1)!$ labellings of τ' uniformly at random. Let $A_{\tau'}$ be the event that the chosen labelling L is increasing; it remains to show that $P(A_{\tau'}) = 1/\prod_v d(v)$.

Let τ'_1 and τ'_2 be the left and right subtrees of τ' . Let B_1 and B_2 be the events that τ'_1 and τ'_2 are labeled increasingly in L , and let B be the event that the root of τ is labeled 1. Then $A_{\tau'} = B \cap B_1 \cap B_2$. It is clear that B, B_1 and B_2 are independent events. It is also clear that $P(B_1) = P(A_{\tau'_1})$ and $P(B_2) = P(A_{\tau'_2})$. Therefore,

$$\begin{aligned} P(A_{\tau'}) &= P(B)P(B_1)P(B_2) \\ &= \frac{1}{n-1}P(A_{\tau'_1})P(A_{\tau'_2}). \end{aligned}$$

The result follows by induction. \square

It is also possible to obtain analogous formulas for the number of fine cells inside a lower-dimensional coarse cell, corresponding to a rooted tree which is not binary. We omit the details.

Notice that Proposition 7.1 is essentially equivalent to the formula for the number of linear extensions of a poset whose Hasse diagram is a tree [12, Supp. Ex. 3.1].

Corollary 7.2. *The positive Bergman complex $\mathcal{B}^+(K_n)$ contains exactly $(n-1)!$ full-dimensional fine cells. The Bergman complex $\mathcal{B}(K_n)$ contains exactly $n!(n-1)!/2^{n-1}$ full-dimensional fine cells.*

Proof. We recall the known bijection between increasing binary trees with vertices labeled $a_1 < \dots < a_k$, and permutations of $\{a_1, \dots, a_k\}$ [12, Sec. 1.3]. It is defined recursively: the permutation $\pi(T)$ corresponding to the increasing binary tree T is $\pi(T) = \pi(T_1) a_1 \pi(T_2)$, where T_1 and T_2 are the left and right subtrees of T . For example, the tree of Figure 7 corresponds to the permutation 57316284. It is not difficult to see how T can be recovered uniquely from $\pi(T)$.

Since the full-dimensional fine cells of $\mathcal{B}^+(K_n)$ are in correspondence with the increasing binary trees with labels $1, \dots, n-1$, the first result follows.

To show the second result, recall that the Bergman complex $\mathcal{B}(K_n)$ is covered by $n!$ positive Bergman complexes. Each permutation π of $[n]$ gives rise to a positive Bergman complex $\mathcal{B}^+(K_n)$; this complex parameterizes those trees which can be drawn in the plane so that its leaves are in the order prescribed by π . With $(n-1)!$ fine cells in each positive

Bergman complex, we get a covering of $\mathcal{B}(K_n)$ with $n!(n-1)!$ fine cells. Each fine cell appears several times in this covering, since it sits inside several positive Bergman complexes.

More precisely, each binary tree with n labeled leaves can be drawn in the plane in exactly 2^{n-1} ways: at each internal vertex, we may or may not switch the left and right subtrees. Therefore, each fine cell of the Bergman complex $\mathcal{B}(K_n)$ is inside 2^{n-1} different positive Bergman complexes. The desired result follows. \square

Recall that the maximum-dimensional fine cells of $\mathcal{B}(K_n)$ correspond to the maximal chains in the lattice of flats of K_n ; *i.e.*, the partition lattice Π_n . Thus we have given an alternative proof of the fact that there are $n!(n-1)!/2^{n-1}$ maximal chains in Π_n [12, Supp. Ex. 3.3].

As an illustration of Corollary 7.2, notice that, in Figure 6, the positive Bergman complex $\mathcal{B}^+(K_4)$ consists of $3! = 6$ fine cells, while the Bergman complex $\mathcal{B}(K_4)$ consists of $4!3!/2^3 = 18$ fine cells.

ACKNOWLEDGMENTS We are grateful to Bernd Sturmfels for suggesting this project to us, and to Lou Billera for helpful conversations.

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