CHIP FIRING ON DYNKIN DIAGRAMS AND MCKAY QUIVERS

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ABSTRACT. Two classes of avalanche-finite matrices and their critical groups (integer cokernels) are studied from the viewpoint of chip-firing/sandpile dynamics, namely, the Cartan matrices of finite root systems and the McKay-Cartan matrices for finite subgroups G of general linear groups. In the root system case, the recurrent and superstable configurations are identified explicitly and are related to minuscule dominant weights. In the McKay-Cartan case for finite subgroups of the special linear group, the cokernel is related to the abelianization of the subgroup G. In the special case of the classical McKay correspondence, the critical group and the abelianization are shown to be isomorphic.

1. INTRODUCTION

The chip-firing model is a discrete dynamical system classically modeling the distribution of a discrete commodity on a graphical network by a Laplacian matrix. Much early work was done in the context of the *abelian sandpile model*, studying the avalanching dynamics of granular flow on a grid. The long-term stabilizing configurations exhibit a phenomenon deemed *self-organized criticality* [10, 13]. Chip-firing dynamics and long-term behavior of the model have been related to areas such as economic models [3], energy minimization [2], and face numbers of matroids [28].

Recent work has established that much of the good behavior of abelian sandpiles or chip-firing models of graphs and directed graphs (digraphs) generalizes naturally to certain integer matrices that have been called *avalanche-finite matrices* or *nonsingular M-matrices* (see, e.g., Guzmán and Klivans [22] or the paper [31] by Postnikov and Shapiro, where they are called toppling matrices). Assume C is such a matrix in $\mathbb{Z}^{\ell \times \ell}$, and C^t is its transpose. The *critical group* of C is

$$\mathsf{K}(C) := \mathsf{coker}(C^t : \mathbb{Z}^\ell \to \mathbb{Z}^\ell) = \mathbb{Z}^\ell / \mathsf{im}(C^t).$$

The stable configurations alluded to above are known as the *critical* or *recurrent* configurations, and they form a system of distinguished coset representatives for $\mathsf{im}(C^t)$ in $\mathbb{Z}^\ell$. Closely related is an alternative system of coset representatives called the *superstable* configurations. Basic facts on avalanche-finite matrices, recurrent and superstable configurations are reviewed in Section 2.

With this in mind, the present paper considers two kinds of matrices previously unidentified as avalanche-finite matrices. The first is the Cartan matrix C for a finite, crystallographic, irreducible root system, studied in Section 3. Here the critical group $\mathsf{K}(C)$ has an auxiliary interpretation as the *fundamental group* of the root system, that is, the weight lattice modulo the root lattice. Our first main result, proven in Section 4, identifies the superstable and recurrent configurations of $\mathsf{K}(C)$ in terms of the *Weyl vector* ϱ (the half-sum of all positive roots) and the *minuscule dominant weights* λ.

Theorem 1.1. *For the Cartan matrix C of a finite, crystallographic, irreducible root system,*

   (i) *the superstable configurations are the zero vector $\mathbf{0}$ and the minuscule dominant weights λ;*
   (ii) *the recurrent configurations are ϱ and $\varrho - \lambda$ for all minuscule dominant weights λ.*

The second kind of avalanche-finite matrix is what we refer to as the *McKay-Cartan matrix C* associated to an n-dimensional faithful representation $\gamma : G \hookrightarrow \mathsf{GL}_n(\mathbb{C})$ of a finite group G. Assume $\{\mathbf{1}_G = \chi_0, \chi_1, \ldots, \chi_\ell\}$ is the set of irreducible complex characters of G, and $\chi_\gamma \cdot \chi_i = \sum_{j=0}^{\ell} m_{ij} \chi_j$. Then C is the $\ell \times \ell$ matrix with (i,j)-entry given by $c_{ij} := n\delta_{ij} - m_{ij}$ for $1 \le i, j \le \ell$, where δ_{ij} is the Kronecker delta. Our second main result is established in Section 5.

Theorem 1.2. *The McKay-Cartan matrix C of a faithful representation $\gamma : G \hookrightarrow \mathsf{GL}_n(\mathbb{C})$ of a finite group G is an avalanche-finite matrix.*

As we discuss in Section 5.2, the abelian group $\mathsf{K}(C)$ coming from the McKay-Cartan matrix C has a *multiplicative* structure as a *rng* (=ring without unit). This results from viewing $\mathsf{K}(C)$ as an *ideal* inside the quotient ring which is the *(virtual) representation ring $R(G)$* of G with the principal ideal generated by $n \cdot 1 - \chi_\gamma$ factored out. Here n is the degree γ, and χ_γ is its character.

Section 6 proves the following result for faithful representations $\gamma : G \hookrightarrow \mathsf{SL}_n(\mathbb{C})$ into special linear groups, relating $\mathsf{K}(C)$ to the *abelianization $G^{\mathsf{ab}} = G/[G,G]$* and its *Pontrjagin dual* or *character group $\widehat{G^{\mathsf{ab}}}$*.

Theorem 1.3. *For a faithful representation $\gamma : G \hookrightarrow \mathsf{SL}_n(\mathbb{C})$ of a finite group G, there is a surjection $\mathsf{K}(C) \twoheadrightarrow \widehat{G^{\mathsf{ab}}}$.*

Section 6 discusses examples, including the motivating case where all our results apply: McKay's original correspondence [27] for finite subgroups G of $\mathsf{SL}_2(\mathbb{C})$. McKay observed that the extended matrix $\tilde{C} = (c_{ij})$, with $c_{ij} = n\delta_{ij} - m_{ij}$ for $0 \le i, j \le \ell$, for the natural 2-dimensional representation $\gamma : G \hookrightarrow \mathsf{SL}_2(\mathbb{C})$ coming from the action of G on $\mathbb{C}^2$ by matrix multiplication coincides with an affine Cartan matrix for a simply-laced finite root system. Our last result is the following.

Theorem 1.4. *For a faithful representation $\gamma : G \hookrightarrow \mathsf{SL}_2(\mathbb{C})$ of a finite group G, there is an isomorphism $\mathsf{K}(C) \cong \widehat{G^{\mathsf{ab}}}$.*

Thus, the fundamental group of a simply-laced finite root system is (noncanonically) isomorphic to G^{ab} for its McKay subgroup G. This turns out to be equivalent to a result of Steinberg; see Remark 6.14 below.

2. Avalanche-finite matrices and chip firing

In this section, we review some basic notions (Z-matrix, M-matrix, avalanche-finite matrix, critical group, recurrent and superstable configurations) that can be found, for example, in Gabrielov [17], Guzmán and Klivans [22], and Postnikov and Shapiro [31, §13]. We show variants of certain concepts, (e.g. burning configurations), that originated in the context of abelian sandpile models

and chip firing on graphs can be adapted to the matrix case (see Definition 2.15). In Theorem 2.16 we establish a useful relation between burning configurations and recurrent configurations for an avalanche-finite matrix.

A matrix $C = (c_{ij})$ in $\mathbb{Z}^{\ell \times \ell}$ with $c_{ij} \leq 0$ for all $i \neq j$ is called a *Z-matrix*. Assume $\mathbb{N} = \{0, 1, 2, \ldots\}$, and say that the elements $v = [v_1, \ldots, v_\ell]^t \in \mathbb{N}^\ell$ are *(nonnegative) chip configurations*, viewed as assigning v_i chips to state i for each $1, 2, \ldots, \ell$. For a fixed Z-matrix C, call a nonnegative chip configuration v *stable* if $(0 \leq) v_i < c_{ii}$ for $i = 1, 2, \ldots, \ell$. If v is unstable, then $v_i \geq c_{ii}$ for some i, and a new nonnegative chip configuration v' can be created by subtracting the ith row of C from v, that is, $v' := [v'_1, \ldots, v'_\ell]^t \in \mathbb{N}^\ell$, where $v'_j := v_j - c_{ij}$ for $j = 1, 2, \ldots, \ell$. The result v' is referred to as the *(C-)firing* or *(C-)toppling* of v at state i.

Definition 2.1. A Z-matrix is called an *avalanche-finite* matrix if every nonnegative chip configuration can be brought to a stable one by a sequence of such topplings.

Denote the zero and all-ones vectors in $\mathbb{R}^\ell$ by

$$\mathbf{0} := [0, 0, \ldots, 0]^t,$$
$$\mathbf{1} := [1, 1, \ldots, 1]^t.$$

For u, v in $\mathbb{R}^\ell$, let $u \geq v$ (resp. $u > v$) mean that $u_i \geq v_i$ (resp. $u_i > v_i$) for $i = 1, 2, \ldots, \ell$. Call a diagonal matrix D *positive* if $[D_{11}, D_{22}, \ldots, D_{\ell\ell}]^t > \mathbf{0}$.

Proposition 2.2. *For a Z-matrix C in $\mathbb{Z}^{\ell \times \ell}$, the following conditions are equivalent:*
 (i) *C is an avalanche-finite matrix.*
 (ii) *C^t is an avalanche-finite matrix.*
 (iii) *There exists a positive diagonal matrix D with $DC + (DC)^t$ positive definite (that is, all its eigenvalues are positive).*
 (iv) *The eigenvalues of C all have positive real part.*
 (v) *C^{-1} exists and has all nonnegative entries.*
 (vi) *There exists r in $\mathbb{R}^\ell$ with $r > \mathbf{0}$ and $Cr > \mathbf{0}$.*

Proof. The equivalence of (iii),(iv),(v),(vi) can be found, for example, in Plemmons [30, Thm. 1]. The equivalence of (i) and (v) is due to Gabrielov [17]. The equivalence of (i) and (ii) then follows. $\square$

The matrices of Proposition 2.2 are commonly known as *(nonsingular) M-matrices* and arise in a broad range of mathematical disciplines. The paper [30] by Plemmons contains 40 equivalent conditions for a Z-matrix to be an M-matrix.

Definition 2.3. The *critical group* $\mathsf{K}(C)$ of an avalanche-finite matrix C is the cokernel,
$$\mathsf{K}(C) := \mathsf{coker}(C^t : \mathbb{Z}^\ell \to \mathbb{Z}^\ell) := \mathbb{Z}^\ell / \mathsf{im}(C^t).$$

Remark 2.4. In defining $\mathsf{K}(C) = \mathsf{coker}(C^t)$, there is little danger in replacing C by its transpose C^t when convenient. This is because Proposition 2.2 shows C^t is an avalanche-finite matrix if and only if C is, and nonsingular integer matrices C have a (non-canonical) abelian group isomorphism $\mathsf{coker}(C^t) \cong \mathsf{coker}(C)$, or a canonical isomorphism
$$\mathsf{coker}(C^t) \cong \widehat{\mathsf{coker}(C)}.$$

Here $\widehat{A}$ for a finite abelian group A is its group of characters or *Pontrjagin dual* [14, Exer. 5.2.14],
$$\widehat{A} := \mathsf{Hom}(A, \mathbb{C}^\times) \cong \mathsf{Hom}(A, \mathbb{Q}/\mathbb{Z}),$$
which satisfies $\widehat{A} \cong A$, but not canonically.

The terminology "critical group" for $\mathsf{K}(C)$ comes from certain coset representatives, called the critical or recurrent configurations, which are distinguished by their toppling dynamics, as we explain next. The details about their existence and uniqueness can be found in [31, Lemma 13.2].

Definition 2.5. For an avalanche-finite matrix C and a nonnegative configuration v, there is a unique stable configuration, denoted $\mathsf{stab}_C(v)$ and called the *stabilization of* v, that is reachable by a sequence of valid C-topplings from v; moreover, $\mathsf{stab}_C(v)$ is independent of the topplings used to reach stability.

Note that this implies

$$(2.6) \qquad \mathsf{stab}_C(v + p) = \mathsf{stab}_C(\mathsf{stab}_C(v) + p) \text{ for } p \in \mathbb{N}^\ell,$$

since any sequence of topplings that stabilize $v \longmapsto \mathsf{stab}_C(v)$ gives a sequence of valid topplings $v + p \longmapsto \mathsf{stab}_C(v) + p$, which can be performed first when computing $\mathsf{stab}_C(v + p)$.

Definition 2.7. For each $i = 1, 2, \ldots, \ell$, the ith *avalanche operator* X_i is the map on the set of stable configurations defined by $X_i(v) := \mathsf{stab}_C(v + e_i)$, where e_i is the ith standard unit basis vector of $\mathbb{Z}^\ell$.

It turns out (see [13] or [31, Lemma 13.3]) that the avalanche operators commute: $X_i X_j = X_j X_i$. In fact, (2.6) implies the more general result,

$$(2.8) \qquad X_1^{p_1} \cdots X_\ell^{p_\ell}(v) = \mathsf{stab}_C(v + p),$$

for any vector $p = (p_1, \ldots, p_\ell) \in \mathbb{N}^\ell$. The *abelian sandpile model* is a Markov chain whose state space is the set of stable configurations and whose transitions are given by randomly choosing a site $i \in \{1, \ldots, \ell\}$ and performing the avalanche operator X_i.
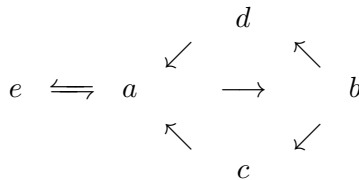
The next proposition gives various known equivalent definitions of the recurrent configurations of the model. For example, (d) is the definition used in [31, §13, p. 3138], while (c) is used in [22, Defn. 4.10], and (e) is related to Dhar's burning algorithm, see Theorem 2.16. Moreover, (e) is the definition that we will appeal to in the proof of Theorem 1.1.

To state the proposition-definition, we introduce the *support* $\mathsf{supp}(p) := \{i : p_i \neq 0\}$ of a vector $p = [p_1, \ldots, p_\ell]^t$ in $\mathbb{Z}^\ell$ and the *digraph* $D(C)$ associated to C, which has node set $\{1, 2, \ldots, \ell\}$, and directed arcs $i \to j$ whenever $c_{ij} < 0$, that is, whenever chip firing at node i adds at least one chip to node j.

Proposition 2.9. *Let C in $\mathbb{Z}^{\ell \times \ell}$ be an avalanche-finite matrix. The following are equivalent for v in $\mathbb{Z}^\ell$. Define v to be (C-)recurrent if one of them holds (hence all of them hold):*

(a) $v = \mathsf{stab}_C(v + p)$ *for some* $p \in \mathbb{N}^\ell$, $p > \mathbf{0}$.
(b) $v = \mathsf{stab}_C(v + Np)$ *for some* $p \in \mathbb{N}^\ell$, $p > \mathbf{0}$, *and every integer* $N \geq 1$.
(c) $v = \mathsf{stab}_C(u)$ *for some* $u \in \mathbb{N}^\ell$ *with* $u_i \geq c_{ii}$ *for all* i.
(d) $v = \mathsf{stab}_C(v + p_i e_i)(= X_i^{p_i}(v))$ *for* $i = 1, 2, \ldots, \ell$ *for some* $p = (p_1, \ldots, p_\ell) \in \mathbb{N}^\ell$, $p > \mathbf{0}$.
(e) $v = \mathsf{stab}_C(v + p)$ *for some* $p \in \mathbb{N}^\ell$, *with the property that every node* $j = 1, 2, \ldots, \ell$ *has at least one directed path* $i = i_0 \to i_1 \to \cdots \to i_{m-1} \to i_m = j$ *in the digraph* $D(C)$ *from a node* i *in* $\mathsf{supp}(p)$.

Proof. We will check these implications:

$$
\begin{array}{ccccc}
 & & d & & \\
 & \swarrow & & \nwarrow & \\
e \Longrightarrow a & & \longrightarrow & & b \\
 & \nwarrow & & \swarrow & \\
 & & c & &
\end{array}
$$

(a) implies (b):   Note $v = \mathsf{stab}_C(v + p)$ implies, by iterating (2.6), that

$$v = \mathsf{stab}_C(v + p) = \mathsf{stab}_C((v + p) + p) = \cdots = \mathsf{stab}_C(v + Np).$$

(b) implies (c):   If $p > \mathbf{0}$, then $u := v + Np$ has $u_i \geq c_{ii}$ for large N.

(c) implies (a): If $v = \mathsf{stab}_C(u)$, then v is stable, and hence $u_i \geq c_{ii} > v_i$, so one has $p := u - v > \mathbf{0}$ with $v = \mathsf{stab}(u) = \mathsf{stab}_C(v + p)$.

(b) implies (d): If $v = \mathsf{stab}_C(v + Np)$ with $p > \mathbf{0}$ and $N = 1, 2, ...$, choose $N > 0$ sufficiently large so that the self-map $Y := X_1^{p_1} \cdots X_{i-1}^{p_{i-1}} X_i^{p_i-1} X_{i+1}^{p_{i+1}} \cdots X_\ell^{p_\ell}$ acting on the (finite) set of all stable configurations has $Y^N = Y^{N+M} = Y^{N+2M} \cdots$ for some finite order $M > 0$. In particular, $Y^N(v) = Y^{N+M}(v)$, and hence

$$v = \mathsf{stab}_C(v + (N + M)p) = (X_iY)^{N+M}(v) = X_i^{N+M}Y^{N+M}(v) = X_i^{N+M}Y^N(v)$$
$$= X_i^M(X_iY)^N(v) = X_i^M\mathsf{stab}_C(v + Np) = X_i^M(v).$$

(d) implies (a):   Note $v = \mathsf{stab}_C(v + p_ie_i)$ for all i implies, by iterating (2.6), that

$$v = \mathsf{stab}_C(v + p_1e_1) = \mathsf{stab}_C((v + p_1e_1) + p_2e_2) = \cdots = \mathsf{stab}_C(v + p).$$

(a) implies (e):   Trivial, since $p > \mathbf{0}$ means $\mathsf{supp}(p) = \{1, 2, \ldots, \ell\}$.

(e) implies (a):   If $v = \mathsf{stab}_C(v + p)$ with p as in (e), let $M := \max_{i=1}^\ell\{c_{ii}\}$ and choose a tower of integers $1 =: N_\ell \ll N_{\ell-1} \ll \cdots N_2 \ll N_1 \ll N_0$ where $N_dM < N_{d-1}$. Then iterating (2.6) gives $v = \mathsf{stab}_C(v + N_0p)$. We claim $v = \mathsf{stab}_C(v + N_0p)$ can be computed by first "flooding the network with chips" as follows.

Let $S_0 := \mathsf{supp}(p)$, and let S_d for $d = 1, 2, \ldots, \ell - 1$ denote the nodes whose shortest directed path from $\mathsf{supp}(p)$ has d steps. One can first do N_1 topplings at each node in S_1, then N_2 topplings at each node in S_2, and so on, finishing with $N_{\ell-1}$ topplings at each node in $S_{\ell-1}$. At the dth stage, each node in S_d will have received at least N_{d-1} chips from nodes in S_{d-1}, and since $N_{d-1} > N_dM \geq N_dc_{ii}$, it will have the N_dc_{ii} chips that it needs to do N_d valid topplings.

After these "flooding" topplings, the result has the form $v + p'$ where $p_i' \geq N_\ell = 1$ for all i, which one can continue toppling until stability is achieved. Hence $v = \mathsf{stab}_C(v + N_0p) = \mathsf{stab}_C(v + p')$, so (a) is satisfied. $\qquad\square$

Theorem 2.10. [13], [31, Thm. 13.4] *For any avalanche-finite matrix C in $\mathbb{Z}^{\ell\times\ell}$, the recurrent configurations in $\mathbb{Z}^\ell$ form a system of coset representatives for $\mathsf{coker}(C^t) = \mathbb{Z}^\ell/\mathsf{im}(C^t)$.*

Closely related is the following notion.

Definition 2.11. A configuration $u \in \mathbb{N}^\ell$ is said to be *superstable for a Z-matrix* C if $z \in \mathbb{N}^\ell$ and $u - C^tz \in \mathbb{N}^\ell$ together imply that $z = \mathbf{0}$.

When C is an avalanche-finite matrix, the superstable configurations give another set of coset representatives for $\mathsf{K}(C) = \mathsf{coker}(C^t) = \mathbb{Z}^\ell/\mathsf{im}(C^t)$, which are distinguished as follows (see [22, Thm. 4.6]): u is superstable if and only if u *uniquely* minimizes the energy function

$$(2.12) \qquad\qquad E(u) := \|C^{-1}u\|^2 = (C^{-1}u, C^{-1}u)$$

among all nonnegative vectors within its coset $u + \mathsf{im}(C^t)$, where $(\cdot, \cdot)$ denotes the usual inner product on $\mathbb{R}^\ell$. There is also a simple relation between the superstable configurations and the recurrent configurations.

Theorem 2.13. [22, Thms. 4.14, 4.15] *For an avalanche-finite matrix* $C = (c_{ij})$, *the vector* v^C *defined by*

$$(2.14) \qquad\qquad v^C := [c_{11} - 1, \ldots, c_{\ell\ell} - 1]^t$$

has the property that u *in* $\mathbb{N}^\ell$ *is superstable if and only if* $v^C - u$ *is recurrent.*

For a given avalanche-finite matrix, in general it is hard to predict or parameterize its set of recurrent (or superstable) configurations. However, we will show in Section 4 that the Cartan matrix of a finite, crystallographic, irreducible *root system* is always an avalanche-finite matrix, and we will identify its recurrent and superstable configurations explicitly.

One way to test whether or not a configuration is recurrent is to use a *burning configuration*. The following conditions were stated by Dhar [13] for undirected graphs and by Speer [32] for directed graphs, (see also [29, Thm. 2.27]). We give the necessary variant for a general avalanche-finite matrix.

Definition 2.15. A vector b in $\mathbb{N}^\ell$ is a burning configuration for an avalanche-finite matrix C in $\mathbb{Z}^{\ell \times \ell}$ if

    (i) b is the image of some element of $\mathbb{Z}^\ell$ under C^t, and
    (ii) every node $j = 1, 2, \ldots, \ell$ has at least one directed path from i to j in the digraph $D(C)$ from a node i in $\mathsf{supp}(b)$ to j.

Note it follows from (ii) that a burning configuration $b \neq \mathbf{0}$.

Theorem 2.16. *Assume* b *is a burning configuration for the avalanche-finite matrix* C. *Then a configuration* v *is recurrent if and only if* $\mathsf{stab}_C(v + b) = v$.

Proof. ($\Longleftarrow$) If $\mathsf{stab}_C(v + b) = v$, then v is recurrent by Proposition 2.9(e).

($\Longrightarrow$) We suppose now that v is recurrent and $b = C^t z$ for $z \in \mathbb{Z}^\ell$, and first argue that the configuration $v + b$ is unstable. Since v is recurrent for C, the vector $u := v^C - v$ is superstable, where v^C is as in (2.14), and hence

$$v^C - (v + b) = u - b = u - C^t z$$

cannot lie in $\mathbb{N}^\ell$, that is, $v + b$ is not componentwise less than v^C, so $v + b$ is unstable. Now we show $\mathsf{stab}_C(v + b)$ is recurrent. By definition, it must be stable. To see recurrence, consider expanding b in terms of avalanche operators:

$$\mathsf{stab}_C(v + b) = X_1^{b_1} \cdots X_\ell^{b_\ell}(v).$$

If v is recurrent, and performing a sequence of topplings on v results in a stable configuration x, then x is also recurrent. Hence $\mathsf{stab}_C(v + b)$ is recurrent. Since b lies in $\mathsf{im}(C^t)$, v and $\mathsf{stab}_C(v + b)$ are in the same coset modulo $\mathsf{im}(C^t)$. But, recurrent configurations are unique per equivalence class, and so v must equal $\mathsf{stab}_C(v + b)$. $\qquad\square$

Remark 2.17. Given a burning configuration b for C, let $z = [z_1, \ldots, z_\ell]^t := (C^t)^{-1} b$. Then $z \in \mathbb{N}^\ell$, since C^{-1} has nonnegative entries by Proposition 2.2. Any stabilization process from $v + b$ to v for any recurrent v has exactly z_i firings of node i.

Remark 2.18. There is an extensive theory of critical groups for avalanche-finite matrices C that come from directed graphs (see [11, 5, 23, 32, 36]). In it, one starts with a directed graph D on node set $\{0, 1, 2, \ldots, \ell\}$ with m_{ij} arcs directed from node i to node j, and assumes the distinguished "source" node 0 has at least one directed path to every other node j. One then defines a *Laplacian matrix* $\tilde{L}$ in which $\tilde{L}_{ij} = \delta_{ij} d_i - m_{ij}$, where d_i is the outdegree of vertex i, and δ_{ij} is the Kronecker delta. From this one derives the *reduced Laplacian* L from $\tilde{L}$ by striking out the 0th row and column. This always gives a avalanche-finite matrix; see Postnikov and Shapiro [31, Prop. 13.1.2].

Its cokernel is the critical group $\mathsf{K}(D) := \mathsf{coker}(L) \cong \mathsf{coker}(L^t)$. Here the cardinality $|\mathsf{K}(D)|$ also counts *arborescences* in D: directed trees in which every vertex has a directed path toward vertex 0.

In fact, we will often obtain our avalanche-finite matrix $C \in \mathbb{Z}^{\ell \times \ell}$ by striking out a row and column in a singular matrix $\tilde{C}$ in $\mathbb{Z}^{(\ell+1) \times (\ell+1)}$. Here we collect for later use some equivalent descriptions of the cokernels in this context. For this purpose, we fix an ordered $\mathbb{Z}$-basis $\{e_0, e_1, \ldots, e_\ell\}$ for $\mathbb{Z}^{\ell+1}$.

Proposition 2.19. *Assume C in $\mathbb{Z}^{\ell \times \ell}$ is obtained from some $\tilde{C} = (c_{ij})_{i,j=0,1,\ldots,\ell}$ in $\mathbb{Z}^{(\ell+1) \times (\ell+1)}$ by removing the 0th row and 0th column. Let $\delta = [\delta_0, \delta_1, \ldots, \delta_\ell]^t$ be a primitive vector in $\mathbb{Z}^{\ell+1}$ in the nullspace of $\tilde{C}$, that is, $\tilde{C}\delta = \mathbf{0}$ with $\gcd(\delta) = 1$, so that there is an inclusion of sublattices*

$$(2.20) \qquad \mathsf{im}(C^t) \quad \subseteq \quad \delta^\perp := \{x \in \mathbb{Z}^{\ell+1} : x \cdot \delta = 0\} \quad \subseteq \quad \mathbb{Z}^{\ell+1}.$$

Then one has

$$(2.21) \qquad \mathsf{coker}(\tilde{C}^t) \cong \mathbb{Z} \oplus \left(\delta^\perp / \mathsf{im}(\tilde{C}^t)\right).$$

Assuming the stronger condition that $\delta_0 = 1$ gives

$$(2.22) \qquad \mathsf{coker}(C) \cong \mathbb{Z}^{\ell+1} / \left(\mathbb{Z}e_0 + \mathsf{im}(\tilde{C})\right).$$

Under the even stronger assumption that $\tilde{C}\delta = \mathbf{0} = \tilde{C}^t\gamma$ for some $\gamma, \delta \in \mathbb{Z}^{\ell+1}$ with $\gamma_0 = \delta_0 = 1$, then

$$(2.23) \qquad \mathsf{coker}(C^t) \cong \delta^\perp / \mathsf{im}(\tilde{C}^t),$$

$$(2.24) \qquad \mathsf{coker}(\tilde{C}^t) \cong \mathbb{Z} \oplus \mathsf{coker}(C^t).$$

Proof. To prove (2.21), note that primitivity of δ gives surjectivity at the end of this short exact sequence

$$(2.25) \qquad 0 \to \delta^\perp \longrightarrow \mathbb{Z}^{\ell+1} \xrightarrow{(-)\cdot\delta} \mathbb{Z} \to 0.$$

The inclusions in (2.20) combined with (2.25) give this short exact sequence

$$0 \to \delta^\perp / \mathsf{im}(\tilde{C}^t) \longrightarrow \mathbb{Z}^{\ell+1} / \mathsf{im}(\tilde{C}^t) \xrightarrow{(-)\cdot\delta} \mathbb{Z} \to 0.$$

This sequence must split, since $\mathbb{Z}$ is a free (hence projective) $\mathbb{Z}$-module, showing (2.21).

To see (2.22), observe that

$$\mathbb{Z}e_0 + \mathsf{im}(\tilde{C}) = \mathsf{im} \begin{bmatrix} 1 & c_{00} & c_{01} & \cdots & c_{0\ell} \\ 0 & c_{10} & c_{11} & \cdots & c_{1\ell} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & c_{\ell 0} & c_{\ell 1} & \cdots & c_{\ell\ell} \end{bmatrix} = \mathsf{im} \begin{bmatrix} 1 & c_{01} & \cdots & c_{0\ell} \\ 0 & c_{11} & \cdots & c_{1\ell} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & c_{\ell 1} & \cdots & c_{\ell\ell} \end{bmatrix} = \mathsf{im} \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & C \end{bmatrix},$$

where the second equality used $\tilde{C}\delta = \mathbf{0}$ and $\delta_0 = 1$. This proves (2.22), since

$$\mathbb{Z}^{\ell+1} / \left(\mathbb{Z}e_0 + \mathsf{im}(\tilde{C})\right) \cong \mathsf{coker} \begin{bmatrix} 1 & \mathbf{0} \\ \mathbf{0} & C \end{bmatrix} \cong \mathsf{coker}(C).$$

To verify (2.23) holds, note that the projection

$$\begin{array}{ccc} \mathbb{Z}^{\ell+1} & \xrightarrow{\pi} & \mathbb{Z}^\ell \\ [x_0, x_1, \ldots, x_\ell]^t & \longmapsto & [x_1, \ldots, x_\ell]^t \end{array}$$

restricts to a lattice isomorphism $\pi : \delta^\perp \to \mathbb{Z}^\ell$ since $\delta_0 = 1$ forces $x_0 = -\sum_{i=1}^\ell \delta_i x_i$ for x in $\delta^\perp$. We further claim that $\pi(\mathsf{im}(\tilde{C}^t)) = \mathsf{im}(C^t)$, since

$$
\mathsf{im}(\tilde{C}^t) = \mathsf{im} \begin{bmatrix} c_{00} & c_{10} & \cdots & c_{\ell,0} \\ c_{01} & c_{11} & \cdots & c_{\ell 1} \\ \vdots & \vdots & \ddots & \vdots \\ c_{0\ell} & c_{1\ell} & \cdots & c_{\ell\ell} \end{bmatrix} = \mathsf{im} \begin{bmatrix} c_{10} & \cdots & c_{\ell,0} \\ c_{11} & \cdots & c_{\ell 1} \\ \vdots & \ddots & \vdots \\ c_{1\ell} & \cdots & c_{\ell\ell} \end{bmatrix} = \pi^{-1}\mathsf{im}(C^t),
$$

where the second (resp. third) equality used $\tilde{C}^t\gamma = \mathbf{0}$ (resp. $\tilde{C}\delta = \mathbf{0}$). Hence (2.23) follows.

Lastly, (2.24) follows by combining (2.21) and (2.23).   □

Example 2.26. One can compute for the matrix $\tilde{C} = \begin{bmatrix} 30 & -15 \\ -20 & 10 \end{bmatrix}$ that

$$
\mathsf{coker}(\tilde{C}) = \mathsf{coker}(\tilde{C}^t) \cong \mathbb{Z} \oplus \mathbb{Z}/5\mathbb{Z}.
$$

One has $\tilde{C}\delta = 0$ for $\delta = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ having $\delta_0 = 1$, which agrees with (2.21), since

$$
\delta^\perp/\mathsf{im}\tilde{C}^t = \mathbb{Z}\begin{bmatrix} -2 \\ 1 \end{bmatrix} / \left(\mathbb{Z}\begin{bmatrix} 30 \\ -15 \end{bmatrix} + \mathbb{Z}\begin{bmatrix} -20 \\ 10 \end{bmatrix} \right) \cong \mathbb{Z}/(-15\mathbb{Z} + 10\mathbb{Z}) = \mathbb{Z}/5\mathbb{Z}.
$$

Removing the 0th row and column from $\tilde{C}$ gives the matrix $C = [10] \in \mathbb{Z}^{1\times 1}$, having

$$
\mathsf{coker}(C) \cong \mathsf{coker}(C^t) = \mathbb{Z}/10\mathbb{Z},
$$

which agrees with (2.22), since

$$
\mathbb{Z}^2/ \left(\mathbb{Z}e_0 + \mathsf{im}\tilde{C} \right) = \mathbb{Z}^2/\mathsf{im}\begin{bmatrix} 1 & 30 & -15 \\ 0 & 20 & 10 \end{bmatrix} = \mathbb{Z}^2/\mathsf{im}\begin{bmatrix} 1 & 0 \\ 0 & 10 \end{bmatrix}
$$

On the other hand, both (2.23), (2.24) fail here, because $\tilde{C}^t\gamma = \mathbf{0}$, where $\gamma = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\gamma_0 \neq 1$.

3. Review of root systems

Here we briefly review standard definitions and facts about root systems focusing on what is needed for Sections 4 and 6. Good references are Humphreys [24, §III.9,10], and Bourbaki [6, §IV.1].

3.1. **Basic definitions.** For $\alpha \neq \mathbf{0}$ in a real vector space $V = \mathbb{R}^\ell$, with a positive definite inner product $(\cdot, \cdot)$, define $\alpha^\vee := \dfrac{2\alpha}{(\alpha, \alpha)}$ so that $(\alpha^\vee)^\vee = \alpha$. Let $s_\alpha : V \to V$ be the *reflection* in the hyperplane perpendicular to α given by

$$
s_\alpha(v) := v - (v, \alpha^\vee)\alpha.
$$

Then $s_\alpha(\alpha) = -\alpha$, and s_α fixes the hyperplane $\alpha^\perp$ pointwise. In particular, $s_\alpha^2 = \mathsf{id}_V$.

Definition 3.1. *A finite subset Φ of nonzero vectors in V is a root system if*

(i) Φ *spans V;*
(ii) $s_\alpha(\Phi) = \Phi$ *for $\alpha \in \Phi$;*
(iii) $\Phi \cap \mathbb{R}\alpha = \{\pm\alpha\}$ *for $\alpha \in \Phi$.*

Condition (iii) says that the root system is *reduced*. The finite root systems considered here will always be assumed to be reduced. If, in addition, Φ satisfies

(iv) $(\beta, \alpha^\vee) \in \mathbb{Z}$ *for all $\alpha, \beta \in \Phi$,*

then Φ is said to be *crystallographic.*

The reflections s_α for α in a root system Φ preserve $(\cdot, \cdot)$ and generate a subgroup $W := \langle s_\alpha : \alpha \in \Phi \rangle$ of the orthogonal group $\mathsf{O}_V((\cdot, \cdot))$ called the *Weyl group.* The set $\Phi^\vee := \{\alpha^\vee\}$ forms another root system, called the *dual root system* to Φ, with $s_{\alpha^\vee} = s_\alpha$ for all $\alpha \in \Phi$, so that Φ and $\Phi^\vee$ share the same Weyl group W.

It is well known that a root system Φ always contains an $\mathbb{R}$-basis $\{\alpha_1, \dots, \alpha_\ell\}$ for V of *simple roots* characterized by the property that Φ has a decomposition $\Phi = \Phi_+ \cup (-\Phi_+)$ into *positive* and *negative* roots, where $\Phi_+ := \{\alpha \in \Phi : \alpha = \sum_{i=1}^\ell c_i \alpha_i \text{ with } c_i \text{ in } \mathbb{N} \text{ for all } i\}$. The Weyl group acts (simply) transitively on the sets of simple roots.

The *Dynkin diagram* of Φ is the graph with ℓ vertices corresponding to the simple roots $\{\alpha_1, \dots, \alpha_\ell\}$, with the ith vertex joined to the jth by $(\alpha_i, \alpha_j^\vee)(\alpha_j, \alpha_i^\vee)$ edges for $i \neq j$. When α_i and α_j have different lengths (equivalently, when $(\alpha_i, \alpha_i) \neq (\alpha_j, \alpha_j)$), then an arrow is drawn on the edges connecting vertices i and j and pointing to the shorter of the two roots.

The root system Φ is *irreducible* if there does not exist a partition $\Phi = \Phi_1 \sqcup \Phi_2$ into two nonempty subsets Φ_1, Φ_2 having $(\alpha, \beta) = 0$ for every α in Φ_1 and β in Φ_2. It is a common occurrence in the study of root systems that in order to prove a result for all root systems, it is sufficient to prove the result in the irreducible case. Irreducibility is equivalent to connectedness of the Dynkin diagram.

If the inner product $(\cdot, \cdot)$ on V can be scaled so that $\alpha^\vee = \alpha$ for all α in Φ, then all roots have the same length, and Φ is said to be *simply laced.* Otherwise, a (reduced) irreducible root system has exactly two root lengths. The crystallographic condition ensures that every element of Φ is a $\mathbb{Z}$-linear combination of the simple roots and that the $\mathbb{Z}$-span of Φ determines a well-defined lattice structure $Q(\Phi)$, the *root lattice* of Φ.

The *weight lattice* of a root system Φ is

$$P(\Phi) := \{v \in V : (v, \alpha^\vee) \in \mathbb{Z} \text{ for all } \alpha \text{ in } \Phi\}.$$

For a crystallographic root system Φ, the weight lattice $P(\Phi)$ contains the root lattice $Q(\Phi)$ as a (full rank) sublattice and has $\{\alpha_1, \dots, \alpha_\ell\}$ as an ordered $\mathbb{Z}$-basis.

3.2. **Fundamental weights.** Fix an ordered set $\{\alpha_1, \dots, \alpha_\ell\}$ of simple roots for the root system Φ. Then the ordered set of *simple coroots* $\{\alpha_1^\vee, \dots, \alpha_\ell^\vee\}$ in $\Phi^\vee$ forms a basis for V, and the *fundamental weights* $(\lambda_1, \dots, \lambda_\ell)$ give the dual basis with respect to $(\cdot, \cdot)$; that is, $(\lambda_i, \alpha_j^\vee) = \delta_{i,j}$. Thus $\{\lambda_1, \dots, \lambda_\ell\}$ determines an ordered $\mathbb{Z}$-basis for $P(\Phi)$, and we will always identify $\mathbb{Z}^\ell$ with $P(\Phi)$ via these mutually inverse isomorphisms:

$$
\begin{array}{rcl}
\mathbb{Z}^\ell & \longrightarrow & P(\Phi) \\
v = [v_1, \dots, v_\ell]^t & \longmapsto & \lambda := \sum_{i=1}^\ell v_i \lambda_i
\end{array}
$$

(3.2)

$$
\begin{array}{rcl}
P(\Phi) & \longrightarrow & \mathbb{Z}^\ell \\
\lambda & \longmapsto & v := \left[(\lambda, \alpha_1^\vee), \dots, (\lambda, \alpha_\ell^\vee)\right]^t.
\end{array}
$$

Definition 3.3. *For a crystallographic root system Φ, its* Cartan matrix $C = (c_{ij})$ *in $\mathbb{Z}^{\ell \times \ell}$ relative to the choice of a set $\{\alpha_1, \dots, \alpha_\ell\}$ of ordered simple roots is given by $c_{ij} := (\alpha_i, \alpha_j^\vee)$ for $i, j = 1, 2, \dots, \ell$. Thus, the ith row of C or i^{th} column of C^t expresses the simple root α_i in the fundamental weights, that is,*

$$\alpha_i = \sum_{j=1}^\ell c_{ij} \lambda_j.$$

The cokernel $\mathsf{coker}(C^t)$ of the transpose C^t of the Cartan matrix is called the *fundamental group* of Φ, and can be reinterpreted as

$$\mathsf{coker}(C^t) \cong P(\Phi)/Q(\Phi),$$

the weight lattice modulo root lattice. Note $\widehat{\mathsf{coker}(C^t)} \cong \mathsf{coker}(C) \cong P(\Phi^\vee)/Q(\Phi^\vee)$. Their common cardinality, $f := |\mathsf{coker}(C^t)| = |\mathsf{coker}(C)|$, is called the *index of connection* for Φ.

The fundamental weights $\{\lambda_i\}_{i=1}^\ell$ span the extreme rays of a cone called the (closed) *fundamental chamber* or *dominant Weyl chamber*,

$$
\begin{aligned}
F &= \{v \in V : (v, \alpha) \geq 0 \text{ for all } \alpha \text{ in } \Phi_+\} \\
&= \{v \in V : (v, \alpha_i) \geq 0 \text{ for } i = 1, 2, \ldots, \ell\} \\
&= \{v \in V : (v, \alpha_i^\vee) \geq 0 \text{ for } i = 1, 2, \ldots, \ell\} \\
&= \left\{ \sum_{i=1}^\ell c_i \lambda_i : c_i \in \mathbb{R}_{\geq 0} \right\},
\end{aligned}
$$

and elements of $F \cap P(\Phi)$ are referred to as the *dominant weights*.

The cone F forms a fundamental domain for W, meaning that each W-orbit intersects F in exactly one point. For v in V, one can define a set that quantifies how "far" v is from F

$$
M(v) := \{\alpha \in \Phi_+ : (v, \alpha) < 0\}
$$

so that $F := \{v \in V : M(v) = \varnothing\}$. Then one has the following (see for example [4, Lemma 4.5.2]): for every v not in F, there is at least one simple root α_i in $M(v)$, and for any simple root α_i in $M(v)$ the inequality $|M(s_{\alpha_i}(v))| < |M(v)|$ holds.

3.3. **Root ordering and highest root.** There is an important partial ordering on the set Φ_+ of positive roots of a crystallographic root system, called the *root ordering*, defined by $\alpha \leq \beta$ if $\beta - \alpha = \sum_{i=1}^\ell c_i \alpha_i$ with $c_i \geq 0$. Here are some of its properties:

(a) Every β in Φ_+ has at least one simple root α_i with $(\beta, \alpha_i) > 0$, else the expansion $\beta = \sum_{i=1}^\ell k_i \alpha_i$ with $k_i \geq 0$ would give the contradiction $0 < (\beta, \beta) = \sum_{i=1}^\ell k_i(\beta, \alpha_i) \leq 0$. Then this α_i satisfies either $\beta = \alpha_i$ or $\beta - \alpha_i$ in Φ_+ (and $\beta - \alpha_i < \beta$ in root order); see [24, Lemma 9.4].

(b) There are (at most) two roots within the fundamental chamber F:
   - the *highest root* $\tilde{\alpha} := \tilde{\alpha}(\Phi)$, which is the unique maximum element relative to the root order on Φ_+;
   - the *highest short root* $\alpha^* := \alpha^*(\Phi)$ in Φ_+, characterized by the property that its corresponding coroot $(\alpha^*)^\vee = \tilde{\alpha}(\Phi^\vee)$ is the highest root in the dual root system $\Phi^\vee$, with respect to $\{\alpha_i^\vee\}_{i=1}^\ell$.

Moreover, $\alpha^* = \tilde{\alpha}$ if and only if Φ is simply laced.

Example 3.4. In the root system Φ of type B_ℓ ($\ell \geq 2$), if one chooses as the simple roots

$$
\{e_1 - e_2, e_2 - e_3, \ldots, e_{\ell-1} - e_\ell, e_\ell\}
$$

(using the identifications in (3.2)), then $\tilde{\alpha} = e_1 + e_2 \neq 2e_1 = \alpha^*$. The root order on B_4 is displayed below:

$$
\begin{array}{c}
\tilde{\alpha} = e_1 + e_2 \\
e_1 + e_3 \\
e_1 + e_4 \qquad e_2 + e_3 \\
\alpha^* = e_1 \qquad e_2 + e_4 \\
e_1 - e_4 \qquad e_2 \qquad e_3 + e_4 \\
e_1 - e_3 \qquad e_2 - e_4 \qquad e_3 \\
\alpha_1 = e_1 - e_2 \qquad \alpha_2 = e_2 - e_3 \qquad \alpha_3 = e_3 - e_4 \qquad \alpha_4 = e_4
\end{array}
$$
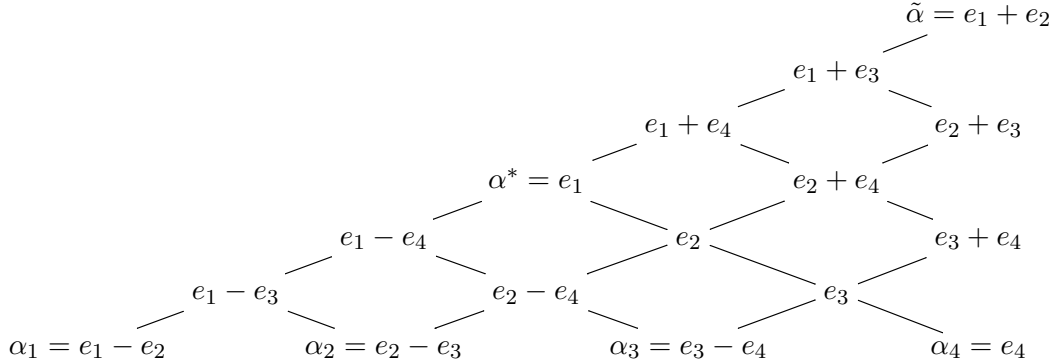
Note it is a consequence of the fact that $\tilde{\alpha}, \alpha^*$ belong to F that their expansions relative to fundamental weights have nonnegative coefficients:

$$
\text{(3.5)} \qquad
\begin{aligned}
\tilde{\alpha} &= \sum_{i=1}^{\ell} q_i \lambda_i, \quad \text{with } q := [q_1, \ldots, q_\ell]^t \geq \mathbf{0}, \\
\alpha^* &= \sum_{i=1}^{\ell} q_i^* \lambda_i, \quad \text{with } q^* := [q_1^*, \ldots, q_\ell^*]^t \geq \mathbf{0}.
\end{aligned}
$$

3.4. Extended Cartan matrix.

Definition 3.6. *For a finite, crystallographic, irreducible root system Φ, the* extended Cartan *matrix $\tilde{C}$ in $\mathbb{Z}^{(\ell+1) \times (\ell+1)}$ is given by*

$$
\tilde{C}_{ij} := (\alpha_i, \alpha_j^\vee)
$$

for $i, j = 0, 1, 2, \ldots, \ell$, where $\alpha_0 := -\tilde{\alpha}$.

While the Cartan matrix C is nonsingular, the extended Cartan matrix will, by its definition, have these left/right nullspaces:
(3.7)

$$
\begin{aligned}
\text{If } \tilde{\alpha} &= \tilde{\alpha}(\Phi) = \sum_{i=1}^{\ell} \delta_i \alpha_i, \quad \text{then } \mathsf{ker}(\tilde{C}^t) = \mathbb{R}\delta \quad \text{where } \delta := [1, \delta_1, \ldots, \delta_\ell]^t \in \mathbb{N}^{\ell+1}. \\
\text{If } \alpha^*(\Phi^\vee) &= \tilde{\alpha}(\Phi)^\vee = \sum_{i=1}^{\ell} \phi_i \alpha_i^\vee, \quad \text{then } \mathsf{ker}(\tilde{C}) = \mathbb{R}\phi \quad \text{where } \phi := [1, \phi_1, \ldots, \phi_\ell]^t \in \mathbb{N}^{\ell+1}.
\end{aligned}
$$

The coordinates of this last vector ϕ give the "marks" labeling the nodes on the affine diagrams in Kac [26, Ch. 4, Table Aff 1]. When Φ is simply laced, then $\tilde{\alpha}(\Phi) = \tilde{\alpha}(\Phi^\vee) = \alpha^*(\Phi) = \alpha^*(\Phi^\vee)$ and $\delta = \delta^\vee = \phi$.

3.5. Weyl vector and minuscule weights.
There is a distinguished element ϱ in the fundamental chamber F, the so-called *Weyl vector*, which is the half-sum of all the positive roots. Since for each simple root α_i, the corresponding reflection s_{α_i} permutes the set $\Phi_+ \setminus \{\alpha_i\}$, one has

$$
\varrho - (\varrho, \alpha_i^\vee)\alpha_i = s_{\alpha_i}(\varrho) = \frac{1}{2} \sum_{\alpha \in \Phi_+ \setminus \alpha_i} \alpha - \frac{1}{2}\alpha_i = \varrho - \alpha_i,
$$

implying that $(\varrho, \alpha_i^\vee) = 1$ for all i. Then it follows from the duality of the bases $\{\alpha_i^\vee\}$ and $\{\lambda_i\}$ that the expansion for ϱ in terms of the fundamental weights is given by

$$(3.8) \qquad \varrho := \varrho(\Phi) := \frac{1}{2} \sum_{\alpha \in \Phi_+} \alpha = \sum_{i=1}^{\ell} \lambda_i.$$

Definition 3.9. *A weight λ in $P(\Phi)$ is* minuscule *if $(\lambda, \alpha^\vee)$ lies in $\{-1, 0, 1\}$ for all α in Φ.*

The minuscule weights λ which are *dominant*, that is, lie in F, have important properties. We collect some of them here, compiling exercises from Bourbaki [6, Exer. VI.1.24(a,c), VI.2.2, VI.2.5(a,d)] and Humphreys [24, Exer. III.13.13], some worked out by Stembridge in [35, §1.2].

Proposition 3.10. *Let Φ be a finite, crystallographic, irreducible root system.*

(a) *The minuscule dominant weights are exactly the fundamental weights λ_i having coefficient $\delta_i^\vee = 1$ in this expansion:*

$$(3.11) \qquad \tilde{\alpha}(\Phi^\vee) = \sum_{i=1}^{\ell} \delta_i^\vee \alpha_i^\vee.$$

(b) *There are $f - 1$ minuscule dominant weights if $f = |P(\Phi)/Q(\Phi)|$ is the index of connection.*

(c) *The zero vector $\mathbf{0}$ together with the minuscule dominant weights give a system of coset representatives for the nonzero cosets in $\mathsf{coker}(C^t) = P(\Phi)/Q(\Phi)$.*

(d) *The zero vector $\mathbf{0}$ and the minuscule dominant weights can be characterized as follows: each is the unique element λ in $F \cap P(\Phi)$ in its coset $\lambda + Q(\Phi)$ which is minimal in the root ordering on $F \cap P(\Phi)$: if μ in $F \cap P(\Phi)$ has $\lambda - \mu = \sum_{i=1}^{\ell} z_i \alpha$ with z_i in $\mathbb{N}$, then $\mu = \lambda$.*

4. CHIP FIRING WITH CARTAN MATRICES

Fixing a finite, crystallographic, irreducible root system Φ, and a choice of simple roots $\alpha_1, \ldots, \alpha_\ell$, we wish to consider its Cartan matrix C in $\mathbb{Z}^{\ell \times \ell}$ as a Z-matrix, and do topplings with respect to C. If $e_1, \ldots, e_\ell$ are the standard unit basis vectors in $\mathbb{Z}^\ell$, then under our usual root system identification $\mathbb{Z}^\ell \longleftrightarrow P(\Phi)$ in (3.2) sending $v \mapsto \sum_{i=1}^{\ell} v_i \lambda_i$,

- one identifies $e_i = \lambda_i$, so that an avalanche operator acts as $X_i(v) = v + \lambda_i$,
- one identifies $q = \tilde{\alpha}$ and $q^* = \alpha^*$ where q, q^* are defined in (3.5),
- one identifies $\mathbf{1} = \varrho$ due to (3.8),
- a firing/toppling at some node $i \in \{1, \ldots, \ell\}$ attempts to replace v with $v - \alpha_i$, using the interpretation of the entries of row i of C as the coordinates of α_i with respect to the fundamental weights, but it gives a *valid* toppling if and only if $v_i = (v, \alpha_i^\vee) \geq 2 = c_{ii}$.

4.1. **Identifying the recurrent, superstable, and burning configurations of a Cartan matrix.** The following proposition seems well known, but is only implicit in Postnikov and Shapiro [31, Proof of Prop. 13.1] and proven by Kac [26, Chap. 4] in different language.

Proposition 4.1. *The Cartan matrix C of a finite, crystallographic, irreducible root system is an avalanche-finite matrix.*

Proof. By Proposition 2.2(ii,v), it suffices to note that the coefficient vector $r(> \mathbf{0})$ of $\varrho = \sum_{i=1}^{\ell} r_i \alpha_i$ has the entries of $r^t C$ expressing ϱ in the basis $\{\lambda_i\}_{i=1}^{\ell}$, and hence $r^t C = \mathbf{1}(> \mathbf{0})$ by (3.8).[1] $\square$

---

[1] An alternative way to prove this is to use the fact that the matrix C^{-1} expressing the λ_i in terms of the simple roots has nonnegative coefficients as seen in [24, Table 1, Sec. 13.2]. Of course, the two statements are equivalent by Proposition 2.2.

Proposition 4.2. *The burning configurations for a Cartan matrix C of a finite, crystallographic, irreducible root system Φ are the nonzero elements b of the fundamental chamber F that lie in the root lattice $Q(\Phi)$.*

Proof. If $b \in F \cap Q(\Phi)$, then b is the image under C^t of an element of $\mathbb{Z}^\ell$ (namely, its coefficient vector relative to the simple roots), and $b \geq \mathbf{0}$ because b is an element of F. When $b \neq \mathbf{0}$, there is a path from node i in $\mathsf{supp}(b)$ to any node j of the Dynkin diagram, because the diagram is connected. The conditions of Theorem 2.16 are satisfied, and b is a burning configuration. Conversely, if $b \in \mathbb{Z}^\ell$ is a burning configuration for C, then $b \geq \mathbf{0}$, and $b = C^t z$ for some $z \in \mathbb{Z}^\ell$ so that $b \in F \cap Q(\Phi)$. $\square$

Remark 4.3. The highest root $\tilde{\alpha}$ and highest short root α^* of Φ lie in $F \cap Q(\Phi)$ and hence give burning configurations for any finite, crystallographic, irreducible root system Φ. When $\mathsf{det}(C) = 1$, (i.e. when Φ is of type E_8, F_4, or G_2), so that $P(\Phi) = Q(\Phi)$, the burning configurations for Φ are exactly the dominant weights $\left(F \cap P(\Phi) \right) \setminus \{\mathbf{0}\}$.

When using the highest root $\tilde{\alpha} = \sum_{i=1}^\ell \delta_i \alpha_i$ as a burning configuration, Remark 2.17 implies that the number of topplings from $v + \tilde{\alpha}$ to v will be $\mathsf{ht}(\tilde{\alpha}) := \sum_{i=1}^\ell \delta_i = h - 1$, where h is the *Coxeter number*,

$$h := \frac{|\Phi|}{\ell},$$

which is also equal to the order of a *Coxeter element* $w = s_{\alpha_1} \cdots s_{\alpha_\ell} \in W$. (See for example, [25, Sec. 3.18], for some of its other incarnations.) Similarly, when α^* is used as a burning configuration, the number of topplings from $v + \alpha^*$ to v is given by the height $\mathsf{ht}(\alpha^*)$ of α^*.

Our main result of this section is Theorem 1.1, which identifies the recurrent and superstable configurations for a Cartan matrix C.

Theorem 1.1. *The Cartan matrix C of a finite, crystallographic, irreducible root system has as its*

    (i) *superstable configurations the zero vector $\mathbf{0}$ and the minuscule dominant weights λ_i,*
    (ii) *recurrent configurations $\mathbf{1} = \varrho$ and $\mathbf{1} - e_i = \varrho - \lambda_i$ for all minuscule dominant weights λ_i.*

The assertions (i) and (ii) in the theorem are equivalent via Theorem 2.13, since $c_{ii} = 2$ for $i = 1, 2, \ldots, \ell$, so that $v^C = \mathbf{1} = \varrho$. We will give two proofs: the first proves (i) and has the advantage of brevity, while the second proves (ii) and has the advantage of identifying certain stabilization sequences with other known combinatorial objects– see Remarks 4.10 and 4.11 below.

Proof of Theorem 1.1 using (i). Our identification of $\mathbb{Z}^\ell$ with the weight lattice $P(\Phi)$ identifies $\mathbb{N}^\ell$ with the set $F \cap P(\Phi)$ of dominant weights. Recall from Definition 2.11 that the superstable configurations are the u in $\mathbb{N}^\ell$ for which $z \in \mathbb{N}^\ell$ together with $u - C^t z$ lying in $\mathbb{N}^\ell$ forces $z = \mathbf{0}$. Such u then correspond to λ in $F \cap P(\Phi)$ for which $z = [z_1, \ldots, z_\ell]^t \in \mathbb{N}^\ell$ together with $\lambda - \sum_{i=1}^\ell z_i \alpha_i =: \mu$ lying in $F \cap P(\Phi)$ forces $\lambda = \mu$. But this is exactly the characterization of $\mathbf{0}$ and the minuscule dominant weights given by Proposition 3.10 (d). $\square$

Proof of Theorem 1.1 using (ii). We first explain why the assertions in (ii) follow once we show

$$(4.4) \qquad\qquad \varrho = \mathsf{stab}_C(\varrho + \tilde{\alpha}), \quad \text{and}$$

$$(4.5) \qquad\qquad \varrho - \lambda_i = \mathsf{stab}_C\left((\varrho - \lambda_i) + \alpha^* \right)$$

for λ_i any minuscule dominant weight.

Indeed, since Remark 4.3 shows that $\tilde{\alpha}$ and α^* are both burning configurations, (4.4) and (4.5) would show that the elements in (ii) are C-recurrent. As the set of elements in (ii) has cardinality $f = |\mathsf{coker}(C^t)|$ from Proposition 3.10(b), we would then know from Theorem 2.10 that there are no other C-recurrent configurations.

Proof of (4.4). Use Property (a) of the root ordering from Section 3.3 repeatedly to express the highest root as

$$
\begin{aligned}
\varrho + \tilde{\alpha} \;=\; & \varrho + \beta_{h-1} \;=\; \varrho + \alpha_{i_1} + \alpha_{i_2} + \cdots + \alpha_{i_{h-2}} + \alpha_{i_{h-1}} \\
& \xrightarrow{\text{topple } i_{h-1}} \quad \varrho + \beta_{h-1} \;=\; \varrho + \alpha_{i_1} + \alpha_{i_2} + \cdots + \alpha_{i_{h-2}} \\
& \;\;\vdots \\
& \xrightarrow{\text{topple } i_3} \quad \varrho + \beta_2 \;=\; \varrho + \alpha_{i_1} + \alpha_{i_2} \\
& \xrightarrow{\text{topple } i_2} \quad \varrho + \beta_1 \;=\; \varrho + \alpha_{i_1} \\
& \xrightarrow{\text{topple } i_1} \quad \varrho.
\end{aligned}
$$

The reason this works is $v = \varrho + \beta_k$ has $v_{i_k} = (\varrho + \beta_k, \alpha_{i_k}^\vee) = (\varrho, \alpha_{i_k}^\vee) + (\beta_k, \alpha_{i_k}^\vee) \geq 2$, since $(\varrho, \alpha_{i_k}^\vee) = 1$ by (3.8), and the integer $(\beta_k, \alpha_{i_k}^\vee)$ is strictly positive by construction. Thus, $\varrho = \mathsf{stab}_C(\varrho + \tilde{\alpha})$ as claimed.

Proof of (4.5). Fix a minuscule dominant weight λ, that is, a minuscule weight lying in the fundamental chamber F. Section 3.3 characterizes λ as a fundamental weight λ_i for which (3.11) gives $1 = \delta_i^\vee = (\lambda_i, \tilde{\alpha}(\Phi^\vee)) = (\lambda_i, (\alpha^*)^\vee)$. Consequently, one has

$$
s_{\alpha^*}(\lambda) = \lambda - (\lambda, (\alpha^*)^\vee)\alpha^* = \lambda - \alpha^*.
$$

Now set $u^{(0)} := s_{\alpha^*}(\lambda)$. Using induction on the cardinality of the set $M(u^{(k)})$ defined in Section 3.2, we can create a sequence $u^{(0)}, u^{(1)}, \ldots, u^{(m)}$ eventually ending with $u^{(m)}$ in F, where $m = \mathsf{ht}(\alpha^*)$ and $u^{(k)} = s_{\alpha_{i_k}}(u^{(k-1)})$ for some simple root α_{i_k} having $(u^{(k-1)}, \alpha_{i_k}) < 0$. Note that at each stage $k = 1, 2, \ldots, m$, one has $u^{(k)} = w_{(k)}(\lambda)$ for an element $w_{(k)} := s_{\alpha_{i_k}} \cdots s_{\alpha_{i_2}} s_{\alpha_{i_1}} s_{\alpha^*}$ of the Weyl group W. Therefore, the inner product

$$
(\alpha_{i_k}^\vee, u^{(k-1)}) = (\alpha_{i_k}^\vee, w_{(k)}(\lambda)) = (w_{(k)}^{-1}(\alpha_{i_k}^\vee), \lambda)
$$

always lies in $\{-1, 0, 1\}$, because λ is minuscule. However, $(\alpha_{i_k}, u^{(k-1)}) < 0$ by construction, so

$$
(4.6) \qquad\qquad\qquad (\alpha_{i_k}^\vee, u^{(k-1)}) = -1.
$$

Therefore $u^{(k)} = s_{\alpha_{i_k}}(u^{(k-1)}) = u^{(k-1)} - (u^{(k-1)}, \alpha_{i_k}^\vee)\alpha_{i_k} = u^{(k-1)} + \alpha_{i_k}$. Thus, we obtain a sequence

$$
(4.7) \qquad
\begin{aligned}
u^{(0)} &:= s_{\alpha^*}(\lambda) &&= \lambda - \alpha^* \\
u^{(1)} &:= s_{i_1} s_{\alpha^*}(\lambda) &&= \lambda - \alpha^* + \alpha_{i_1} \\
u^{(2)} &:= s_{i_2} s_{i_1} s_{\alpha^*}(\lambda) &&= \lambda - \alpha^* + (\alpha_{i_1} + \alpha_{i_2}) \\
&\;\;\vdots \\
u^{(m)} &:= s_{i_m} \cdots s_{i_2} s_{i_1} s_{\alpha^*}(\lambda) &&= \lambda - \alpha^* + (\alpha_{i_1} + \alpha_{i_2} + \cdots + \alpha_{i_m}) = \lambda,
\end{aligned}
$$

where $u^{(m)} = \lambda$ follows, because $u^{(m)}$ lies in both the W-orbit of λ and the fundamental chamber F.

We then claim $v^{(k)} := \varrho - u^{(k)}$ for $k = 0, 1, 2, \ldots, m$ gives this valid C-toppling sequence, showing (4.5):

$$
\begin{aligned}
v^{(0)} &= \varrho - \lambda + \alpha^*, \\
\xrightarrow{\text{topple } i_1} v^{(1)} &= \varrho - \lambda + \alpha^* - \alpha_{i_1}, \\
\xrightarrow{\text{topple } i_2} v^{(2)} &= \varrho - \lambda + \alpha^* - (\alpha_{i_1} + \alpha_{i_2}), \\
&\;\;\vdots \\
\xrightarrow{\text{topple } i_m} v^{(m)} &= \varrho - \lambda + \alpha^* - (\alpha_{i_1} + \alpha_{i_2} + \cdots + \alpha_{i_m}) = \varrho - \lambda.
\end{aligned}
$$

These are valid topplings, because for each $k = 1, 2, \ldots, m$, one has using (4.6)

$$v_{i_k}^{(k-1)} = (\alpha_{i_k}^\vee, v^{(k-1)}) = (\alpha_{i_k}^\vee, \varrho - u^{(k-1)}) = (\alpha_{i_k}^\vee, \varrho) - (\alpha_{i_k}^\vee, u^{(k-1)}) = 1 - (-1) = 2. \quad \square$$

Example 4.8. When Φ is of type E_6, there are two minuscule dominant weights λ_i, whose associated nodes in the Dynkin diagram are darkened here:

$$\begin{array}{c} \circ_2 \\ | \\ \bullet_1 - \circ_3 - \circ_4 - \circ_5 - \bullet_6 \end{array} \quad .$$

This labeling of the Dynkin diagram corresponds to the following Cartan matrix:

$$C = \begin{pmatrix} 2 & 0 & -1 & 0 & 0 & 0 \\ 0 & 2 & 0 & -1 & 0 & 0 \\ -1 & 0 & 2 & -1 & 0 & 0 \\ 0 & -1 & -1 & 2 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 2 \end{pmatrix}$$

Let λ be the left minuscule dominant weight, represented by the vector $[1, 0, 0, 0, 0, 0]^t$. We exhibit the C-toppling sequence $v^{(0)}, v^{(1)}, \ldots, v^{(m)}$ showing $\varrho - \lambda = \mathsf{stab}_C((\varrho - \lambda) + \alpha^*)$, where

$$\alpha^* = \tilde\alpha = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6 = \lambda_2.$$

We display the coordinates relative to the basis of fundamental weights $\{\lambda_i\}_{i=1}^6$ using the Dynkin diagram. The number of steps (including addition of $\alpha^* = \tilde\alpha$ is 12, which is the Coxeter number for E_6.

$$\varrho - \lambda = \begin{bmatrix} & & 1 & & \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix} \xrightarrow{+\alpha^*} \begin{bmatrix} & & 2 & & \\ 0 & 1 & 1 & 1 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} & & 0 & & \\ 0 & 1 & 2 & 1 & 1 \end{bmatrix}$$

$$\uparrow \qquad\qquad\qquad\qquad\qquad \downarrow$$

$$\begin{bmatrix} & & 1 & & \\ 2 & 0 & 1 & 1 & 1 \end{bmatrix} \qquad\qquad\qquad \begin{bmatrix} & & 1 & & \\ 0 & 2 & 0 & 2 & 1 \end{bmatrix}$$

$$\uparrow \qquad\qquad\qquad\qquad\qquad \downarrow$$

$$\begin{bmatrix} & & 1 & & \\ 1 & 2 & 0 & 1 & 1 \end{bmatrix} \qquad\qquad\qquad \begin{bmatrix} & & 1 & & \\ 1 & 0 & 1 & 2 & 1 \end{bmatrix}$$

$$\uparrow \qquad\qquad\qquad\qquad\qquad \downarrow$$

$$\begin{bmatrix} & & 0 & & \\ 1 & 1 & 2 & 0 & 1 \end{bmatrix} \qquad\qquad\qquad \begin{bmatrix} & & 1 & & \\ 1 & 0 & 2 & 0 & 2 \end{bmatrix}$$

$$\uparrow \qquad\qquad\qquad\qquad\qquad \downarrow$$

$$\begin{bmatrix} & & 2 & & \\ 1 & 1 & 1 & 0 & 1 \end{bmatrix} \longleftarrow \begin{bmatrix} & & 2 & & \\ 1 & 1 & 0 & 2 & 0 \end{bmatrix} \longleftarrow \begin{bmatrix} & & 1 & & \\ 1 & 0 & 2 & 1 & 0 \end{bmatrix}$$

Remark 4.9. Since the elements of $\mathsf{coker}(C^t) = P(\Phi)/Q(\Phi)$ are represented by the set of all recurrent configurations $\{\varrho, \ \varrho - \lambda_i : i \text{ a minuscule node}\}$, one might guess that the zero coset

$\mathbf{0} + \mathsf{im}(C^t)$ is represented by ϱ. This is *not* always true, e.g. in type A_ℓ for ℓ odd, the zero coset is represented by $\varrho - \lambda_{\frac{1}{2}(\ell+1)}$. The question of which recurrent configuration represents the zero coset is equivalent to determining which element among $\mathbf{0}$ and the minuscule dominant weights λ_i is equivalent to ϱ in $P(\Phi)$ modulo $Q(\Phi)$.

Remark 4.10. The proof of (4.4) shows that stabilization sequences from $\varrho + \tilde{\alpha}$ to ϱ have an obvious bijection with maximal chains

$$\beta_1 < \beta_2 < \cdots < \beta_{h-1} = \tilde{\alpha}$$

in the root ordering on Φ_+.

Remark 4.11. The proof of (4.5) exhibits stabilization sequences from $(\varrho - \lambda) + \alpha^*$ to $\varrho - \lambda$, in which a sequence of vectors in (4.7),

$$s_{\alpha^*}(\lambda) = u^{(0)}, u^{(1)}, \ldots, u^{(m-1)}, u^{(m)} = \lambda,$$

($m = \mathsf{ht}(\alpha^*)$) is subtracted. We explain here how this sequence (4.7) may be viewed as a *winning sequence* in Mozes's *numbers game* using the Cartan matrix C for Φ, starting from the play position $u^{(0)}$. In this game, if $u = [u_1, \ldots, u_\ell]^t = \sum_{i=1}^{\ell} u_i \lambda_i$ has any coordinate $u_i = (u, \alpha_i^\vee) < 0$, then one is permitted to do a *numbers firing* at node i that replaces $u \longmapsto u' = s_{\alpha_i}(u)$, (or in coordinates, $u'_j = u_j - c_{ij}u_i$). (See for example, Björner and Brenti [4, §4.3] and Eriksson [15].) One *wins* the game when one reaches u having all nonnegative coordinates, that is, when u lies in the fundamental chamber F. The survey by Eriksson [15] discusses both the chip-firing and numbers games when played with Cartan matrices for finite root systems and points out that generally the two games are unrelated [15, p. 118, ¶2].

However, something very special happens in the sequence (4.7). Because λ is minuscule, the entries in each $u^{(k)}$ lie in $\{-1, 0, 1\}$, and one is always firing $u^{(k)}$ at a node i having $u_i^{(k)} = -1$. Since C has all its diagonal entries $c_{ii} = 2$, one can check that in this setting the involution swapping $u \leftrightarrow v$ whenever $u + v = \mathbf{1} = \varrho$ sends a numbers game configuration u in $\{-1, 0, 1\}^\ell$ to a chip configuration v in $\{0, 1, 2\}^\ell$ in such a way that the following diagram commutes:

$$
\begin{array}{ccc}
u & \longleftrightarrow & v \\
\downarrow & & \downarrow \\
u + \alpha_i & \longleftrightarrow & v - \alpha_i
\end{array}
$$

Here the left (resp. right) vertical arrow is a valid numbers firing (resp. valid toppling) at node i.

One obtains a further interpretation by slightly altering the sequence (4.7) in two ways:

- Augment the sequence with $u^{(-1)} := \lambda$, giving the longer sequence

$$
\begin{array}{ccccc}
u^{(-1)}, & u^{(0)} & , u^{(1)}, \ldots, u^{(m-1)}, & u^{(m)} \\
\| & \| & & \| \\
\lambda & s_{\alpha^*}(\lambda) & & \lambda.
\end{array}
$$

- Artificially pad each element $u = u^{(k)} = [u_1, \ldots, u_\ell]^t$ in $\mathbb{Z}^\ell$ to a vector $\tilde{u} = \tilde{u}^{(k)} := [u_0, u_1, \ldots, u_\ell]^t$ in $\mathbb{Z}^{\ell+1}$ having $u_0 := -\sum_{i=1}^{\ell} \phi_i u_i$ where $\phi = [1, \phi_1, \ldots, \phi_\ell]^t$ gives the coefficients expressing $\alpha^*(\Phi) = \tilde{\alpha}(\Phi)^\vee = \sum_{i=1}^{\ell} \phi_i \alpha_i^\vee$ as in (3.7). This is *forcing* each $\tilde{u}^{(k)}$ to lie in $\phi^\perp$.

The result $\tilde{u}^{(-1)}, \tilde{u}^{(0)}, \cdots, \tilde{u}^{(m)}$ turns out to be a *looping* sequence for the *numbers game* played using the *extended Cartan matrix* $\tilde{C}(\Phi^\vee)$ for the *dual* (!) root system $\Phi^\vee$. Looping sequences were characterized by Eriksson [15, §3.2], and studied further by Gashi and Schedler [19], and by Gashi, Schedler and Speyer [20].

Example 4.12. The root system of type E_6 is simply laced, so that $\tilde{\alpha}(\Phi) = \tilde{\alpha}(\Phi^\vee) = \alpha^*(\Phi) = \alpha^*(\Phi^\vee)$. Therefore, one has $\delta = \delta^\vee = \phi = [1, \phi_1, \ldots, \phi_6]^t$, with ϕ shown here labeling the extended Dynkin diagram for $\tilde{\mathsf{E}}_6$:

$$\begin{array}{c}
1 \\
| \\
2 \\
| \\
1 \text{-} 2 \text{-} 3 \text{-} 2 \text{-} 1
\end{array} \qquad .$$

The looping sequence $(\tilde{u}^{(k)})_{k=-1}^m$ inside $\phi^\perp$ for the numbers game played with the extended Cartan matrix $\tilde{C}(\mathsf{E}_6^\vee) = \tilde{C}(\mathsf{E}_6)$ for E_6, is displayed here, to be compared with Example 4.8:

$$\tilde{u}^{(-1)} = \begin{bmatrix} & & -1 & & \\ & & 0 & & \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} & & 1 & & \\ & & -1 & & \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} & & 0 & & \\ & & 1 & & \\ 1 & 0 & -1 & 0 & 0 \end{bmatrix}$$

$$\uparrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow$$

$$\begin{bmatrix} & & -1 & & \\ & & 0 & & \\ -1 & 1 & 0 & 0 & 0 \end{bmatrix} \qquad\qquad\qquad \begin{bmatrix} & & 0 & & \\ & & 0 & & \\ 1 & -1 & 1 & -1 & 0 \end{bmatrix}$$

$$\uparrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow$$

$$\begin{bmatrix} & & -1 & & \\ & & 0 & & \\ 0 & -1 & 1 & 0 & 0 \end{bmatrix} \qquad\qquad\qquad \begin{bmatrix} & & 0 & & \\ & & 0 & & \\ 0 & 1 & 0 & -1 & 0 \end{bmatrix}$$

$$\uparrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow$$

$$\begin{bmatrix} & & -1 & & \\ & & 1 & & \\ 0 & 0 & -1 & 1 & 0 \end{bmatrix} \qquad\qquad\qquad \begin{bmatrix} & & 0 & & \\ & & 0 & & \\ 0 & 1 & -1 & 1 & -1 \end{bmatrix}$$

$$\uparrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad \downarrow$$

$$\begin{bmatrix} & & 0 & & \\ & & -1 & & \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \longleftarrow \begin{bmatrix} & & 0 & & \\ & & -1 & & \\ 0 & 0 & 1 & -1 & 1 \end{bmatrix} \longleftarrow \begin{bmatrix} & & 0 & & \\ & & 0 & & \\ 0 & 1 & -1 & 0 & 1 \end{bmatrix}$$

Example 4.13. The root system Φ of type C_ℓ ($\ell \geq 2$) is not simply laced. Choosing simple roots

$$\{\alpha_1 = e_1 - e_2, \ \alpha_2 = e_2 - e_3, \ \ldots, \ \alpha_{\ell-1} = e_{\ell-1} - e_\ell, \ \alpha_\ell = 2e_\ell\},$$

we have that the simple coroots are

$$\{\alpha_1^\vee = e_1 - e_2, \ \alpha_2^\vee = e_2 - e_3, \ \ldots, \ \alpha_{\ell-1}^\vee = e_{\ell-1} - e_\ell, \ \alpha_\ell^\vee = e_\ell\},$$

and the fundamental dominant weights for C_ℓ are

$$\{\lambda_i = e_1 + \cdots + e_i \mid i = 1, \ldots, \ell\}.$$

Moreover,

$$\tilde{\alpha}(\Phi) = 2e_1,$$

$$\tilde{\alpha}(\Phi^\vee) = e_1 + e_2 = \sum_{i=1}^{\ell} \delta_i^\vee \alpha_i^\vee = \alpha_1^\vee + 2\alpha_2^\vee + 2\alpha_3^\vee + \cdots + 2\alpha_\ell^\vee,$$

$$\alpha^*(\Phi) = e_1 + e_2 = \sum_{i=1}^{\ell} \phi_i \alpha_i^\vee = \alpha_1^\vee + 2\alpha_2^\vee + 2\alpha_3^\vee + \cdots + 2\alpha_\ell^\vee = \lambda_2.$$

Hence, λ_1 is the only minuscule dominant weight, and its associated node is darkened in the Dynkin diagram for the root system Φ of type C_ℓ:

$$\bullet_1 - \circ_2 - \circ_3 - \ldots - \circ_{\ell-1} \Leftarrow \circ_\ell.$$

The extended Dynkin diagram for the dual root system $\Phi^\vee$ of type B_ℓ labelled by ϕ is given here:

$$1 - 2 - 2 - \ldots - 2 \Rightarrow 2$$
$$\overset{|}{1}$$

Taking $\ell = 4$, we display on the left the C-topplings showing that $\varrho - \lambda_1 = \mathsf{stab}_C((\varrho - \lambda_1) + \alpha^*)$, and on the right, the corresponding looping sequence $(\tilde{u}^{(k)})_{k=-1}^m$ in $\phi^\perp$ for the numbers game played with the extended Cartan matrix $\tilde{C}(\Phi^\vee) = \tilde{C}(\mathsf{B}_4)$ for the dual root system:

$$\varrho - \lambda_1 = \begin{bmatrix} 0 & 1 & 1 & 1 \end{bmatrix} \qquad \tilde{u}^{(-1)} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ & -1 & & \end{bmatrix}$$

$$\downarrow {+}\alpha^* \qquad\qquad\qquad \downarrow$$

$$\begin{bmatrix} 0 & 2 & 1 & 1 \end{bmatrix} \qquad\qquad \begin{bmatrix} 1 & -1 & 0 & 0 \\ & 1 & & \end{bmatrix}$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$\begin{bmatrix} 1 & 0 & 2 & 1 \end{bmatrix} \qquad\qquad \begin{bmatrix} 0 & 1 & -1 & 0 \\ & 0 & & \end{bmatrix}$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$\begin{bmatrix} 1 & 1 & 0 & 2 \end{bmatrix} \qquad\qquad \begin{bmatrix} 0 & 0 & 1 & -1 \\ & 0 & & \end{bmatrix}$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$\begin{bmatrix} 1 & 1 & 2 & 0 \end{bmatrix} \qquad\qquad \begin{bmatrix} 0 & 0 & -1 & 1 \\ & 0 & & \end{bmatrix}$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$\begin{bmatrix} 1 & 2 & 0 & 1 \end{bmatrix} \qquad\qquad \begin{bmatrix} 0 & -1 & 1 & 0 \\ & 0 & & \end{bmatrix}$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$\begin{bmatrix} 2 & 0 & 1 & 1 \end{bmatrix} \qquad\qquad \begin{bmatrix} -1 & 1 & 0 & 0 \\ & -1 & & \end{bmatrix}$$

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$\begin{bmatrix} 0 & 1 & 1 & 1 \end{bmatrix} \qquad\qquad \begin{bmatrix} 1 & 0 & 0 & 0 \\ & -1 & & \end{bmatrix}.$$

5. McKay quivers

We now discuss another source of avalanche-finite matrices from McKay quivers and their associated matrices. We first define the quivers and then review some of their basic properties, most of which can be found, for example, in Steinberg [34, §1(2)].

Definitions 5.1. For a complex representation $\gamma : G \to \mathsf{GL}_n(\mathbb{C})$ of a finite group G, denote its *character* by $\chi_\gamma : G \to \mathbb{C}$. The *McKay quiver* of γ is the digraph (Q_0, Q_1) whose node set $Q_0 = \{\chi_0, \chi_1, \ldots, \chi_\ell\}$ is the set of inequivalent irreducible complex G-representations χ_i, with the convention that $\chi_0 = \mathbf{1}_G$ is the trivial representation, and whose arrow set Q_1 has m_{ij} arrows from χ_i to χ_j if one has the irreducible expansions:

$$(5.2) \qquad\qquad \chi_\gamma \cdot \chi_i \cong \sum_{j=0}^{\ell} m_{ij} \chi_j.$$

Record the coefficients as the matrix $M = (m_{ij})$ in $\mathbb{Z}^{(\ell+1) \times (\ell+1)}$, and define the

- *extended McKay-Cartan matrix* $\tilde{C} := nI - M$, where I is the identity matrix of size $\ell + 1$,
- *McKay-Cartan matrix* C as the submatrix of $\tilde{C}$ obtained by removing the row and column corresponding to χ_0.

Proposition 5.3. *Fix a representation* $\gamma : G \to \mathsf{GL}_n(\mathbb{C})$ *of a finite group* G.

(a) *A full set of orthogonal eigenvectors for* $\tilde{C}$ *is given by the column vectors in the character table of* G,

$$\delta^{(g)} := [\chi_0(g), \chi_1(g), \ldots, \chi_\ell(g)]^t,$$

as g *ranges over a set of conjugacy class representatives. These vectors satisfy the eigenvector equations*

$$(5.4) \qquad\qquad \tilde{C}\delta^{(g)} = (n - \chi_\gamma(g)) \cdot \delta^{(g)}.$$

(b) *In particular, the nullspace of* $\tilde{C}$ *contains*

$$(5.5) \qquad\qquad \delta^{(e)} = [\chi_0(e), \chi_1(e), \ldots, \chi_\ell(e)]^t,$$

where e *is the identity element of* G, *and a basis for this nullspace is given by the column vectors* $\{\delta^{(g)}\}$ *indexed by* G-*conjugacy class representatives* g *lying in the normal subgroup* $\mathsf{ker}(\gamma)$ *of* G. *In particular, when* γ *is faithful (injective), the vector* $\delta^{(e)}$ *is a basis for the nullspace.*

(c) *Consequently,* $\tilde{C}$ *has rank at most* $n - 1$, *with equality if and only if* γ *is faithful.*

Proof. For (a), the eigenvector equation $M\delta^{(g)} = \chi_\gamma(g) \cdot \delta^{(g)}$, which is equivalent to (5.4), follows from evaluating both sides of (5.2) on g. The rest of (a) is a consequence of the orthogonality of the columns of the character table.

For (b), use the well-known fact that $\chi_\gamma(g) = n$ if and only if $\gamma(g) = \mathrm{id}_{\mathbb{C}^n}$: if $\gamma(g)$ has eigenvalues $\zeta_1, \ldots, \zeta_n$ and if $\chi_\gamma(g) = \sum_{i=1}^n \zeta_i = n$, then $n = |\chi_\gamma(g)| = |\sum_{i=1}^n \zeta_i| \leq \sum_{i=1}^n |\zeta_i| = n$, with equality forcing all the ζ_i to be equal by the Cauchy-Schwartz inequality, and hence, all equal to 1.

Then (c) is immediate from (b). $\qquad\qquad\square$

In defining C from $\tilde{C}$, the choice of the row/column indexed by χ_0, compared to the row/column indexed by any other one-dimensional representation, affects C only up to similarity, as shown by the first part of the next proposition. Recall that one-dimensional G-representations are simply group homomorphisms φ in $\widehat{G} = \mathsf{Hom}(G, \mathbb{C}^\times)$, and for every irreducible character χ_i of G, the complex conjugate $g \mapsto \overline{\chi_i}(g) := \overline{\chi_i(g)} = \chi_i(g^{-1})$ is also an irreducible character.

Proposition 5.6. *Fix a representation* $\gamma : G \to \mathsf{GL}_n(\mathbb{C})$ *as before.*

(a) *For* φ *in* $\widehat{G}$, *define a permutation* $i \mapsto \varphi(i)$ *on* $\{0, 1, 2, \ldots, \ell\}$ *so that* $\chi_i \cdot \varphi = \chi_{\varphi(i)}$. *Then*

$$\tilde{C}_{ij} = \tilde{C}_{\varphi(i)\varphi(j)}.$$

(b) *Define an involution $i \mapsto i^*$ on $\{0, 1, 2, \ldots, \ell\}$ so that $\overline{\chi_i} \cong \chi_{i^*}$. Then*

$$\tilde{C}_{ij} = \tilde{C}_{j^* i^*}.$$

(c) *The involution on $\mathbb{Z}^{\ell+1}$ sending e_i to e_{i^*} fixes the vector $\delta^{(e)}$ and induces an isomorphism* $\mathsf{ker}(\tilde{C}) \cong \mathsf{ker}(\tilde{C}^t)$. *In particular, when γ is faithful, $\delta^{(e)}$ spans both $\mathsf{ker}(\tilde{C})$ and $\mathsf{ker}(\tilde{C}^t)$.*

Proof. For (a), note that multiplying (5.2) by φ gives

$$(\chi_\gamma \cdot \chi_i) \cdot \varphi = \sum_{j=0}^{\ell} m_{ij} \chi_j \cdot \varphi = \sum_{j=0}^{\ell} m_{ij} \chi_{\varphi(j)}$$

$$\chi_\gamma \cdot (\chi_i \cdot \varphi) = \sum_{j=0}^{\ell} m_{\varphi(i)j} \chi_j,$$

implying $m_{ij} = m_{\varphi(i)\varphi(j)}$ for all i, j.

For (b), use

$$(5.7) \qquad m_{ij} = \frac{1}{|G|} \sum_{g \in G} \chi_\gamma(g) \chi_i(g) \chi_j(g^{-1}) = \frac{1}{|G|} \sum_{g \in G} \chi_\gamma(g) \overline{\chi_i}(g^{-1}) \overline{\chi_j}(g)$$

$$= \frac{1}{|G|} \sum_{g \in G} \chi_\gamma(g) \chi_{j^*}(g) \chi_{i*}(g^{-1}) = m_{j^* i^*}. \qquad \square$$

For (c), note that $\delta_i^{(e)} = \chi_i(e) = \overline{\chi_i}(e) = \delta_{i^*}^{(e)}$, and the rest follows from (b).

Proposition 5.6 (b) has a convenient rephrasing. Let P be the $\ell \times \ell$ permutation matrix for the involution $i \leftrightarrow i^*$ having $\overline{\chi_i} \cong \chi_{i^*}$, restricted to the nontrivial irreducible G-characters $\{\chi_1, \ldots, \chi_\ell\}$, so $P = P^{-1} = P^t$. Then

$$(5.8) \qquad\qquad C^t = PCP = P^t CP.$$

This, combined with an observation of Steinberg [34], will prove very useful in showing that C is an avalanche-finite matrix. To state the ideas in [34, (3)], recall that any square matrix C in $\mathbb{R}^{\ell \times \ell}$, whether symmetric or not, gives rise to a quadratic form $Q(x) = x^t C x$ on $\mathbb{R}^\ell$. The *radical* of Q is $\mathsf{rad}(Q) := \{x \in \mathbb{R}^\ell : Q(x) = 0\}$. The form Q is said to be *nonnegative semidefinite* if $Q(x) \geq 0$ for all x in $\mathbb{R}^\ell$, and *positive definite* if additionally $\mathsf{rad}(Q) = \{\mathbf{0}\}$.

Proposition 5.9. [34, (3)] *For a representation $\gamma : G \to \mathsf{GL}_n(\mathbb{C})$, consider the quadratic form $\tilde{Q}(y) := y^t \tilde{C} y$ on $\mathbb{R}^{\ell+1}$ defined by the extended McKay-Cartan matrix $\tilde{C}$. Then $\tilde{Q}$ is nonnegative semidefinite, with $\mathsf{rad}(\tilde{Q}) \supseteq \mathbb{R}\delta^{(e)}$, and equality holds if and only if γ is faithful.*

Corollary 5.10. *For any representation $\gamma : G \to \mathsf{GL}_n(\mathbb{C})$, the quadratic form $Q(x) := x^t C x$ on $\mathbb{R}^\ell$ defined by the McKay-Cartan matrix C is also nonnegative semidefinite, and positive definite whenever γ is faithful.*

Proof. Include $\mathbb{R}^\ell \hookrightarrow \mathbb{R}^{\ell+1}$ by sending $x = [x_1, \ldots, x_\ell]^t \mapsto y = [0, x_1, \ldots, x_\ell]^t$. Then the quadratic form Q is the restriction of $\tilde{Q}$ from $\mathbb{R}^{\ell+1}$ to $\mathbb{R}^\ell$. Therefore Q is nonnegative semidefinite with

$$\mathsf{rad}(Q) = \mathbb{R}^\ell \cap \mathsf{rad}(\tilde{Q}) = \mathbb{R}^\ell \cap \mathbb{R}\delta^{(e)} = \{\mathbf{0}\}. \quad \square$$

This leads to our proof of Theorem 1.2, which we restate here.

Theorem 1.2. *The McKay-Cartan matrix C of a faithful representation $\gamma : G \hookrightarrow \mathsf{GL}_n(\mathbb{C})$ of a finite group G is an avalanche-finite matrix.*

Proof. By Proposition 2.2(iii), it suffices to check $C + C^t$ is positive definite. As $Q(x) := x^t C x$ is positive definite by Corollary 5.10, and $C^t = P^t C P$ by (5.8), then since P is a permutation matrix,

$$Q'(x) := x^t C^t x = (P^t x)^t P^t C P (P^t x) = x^t P^t C P x = (Px)^t C (Px)$$

is positive definite. Thus $x^t (C + C^t) x = Q(x) + Q'(x)$ is positive definite; that is, $C + C^t$ is positive definite. $\square$

Definition 5.11. Given a faithful representation $\gamma : G \hookrightarrow \mathsf{GL}_n(\mathbb{C})$, with McKay-Cartan and extended McKay-Cartan matrices $C, \tilde{C}$, define its *critical group* in any of the following four ways, which are equivalent by Proposition 2.19 and Proposition 5.6 (c):

$$\mathsf{K}(\gamma) := \mathsf{coker}(C^t) \left(= \mathbb{Z}^\ell / \mathsf{im}(C^t) = \mathsf{K}(C) \right),$$
$$:= (\delta^{(e)})^\perp / \mathsf{im}(\tilde{C}^t),$$
$$:= \mathbb{Z}^{\ell+1} / (\mathbb{Z} e_0 + \mathsf{im}(\tilde{C}^t)), \quad \text{or}$$
$$\mathbb{Z} \oplus \mathsf{K}(\gamma) := \mathsf{coker}(\tilde{C}^t).$$

Example 5.12. Let $G = \mathfrak{A}_4$ be the alternating subgroup of the symmetric group $\mathfrak{S}_4$, and consider its faithful irreducible representation $\gamma : \mathfrak{A}_4 \to \mathsf{SO}_3(\mathbb{R}) \subset \mathsf{GL}_3(\mathbb{C})$ as the rotational symmetries of a regular tetrahedron, where χ_γ is labeled χ_3 in the character table below, and $\omega := e^{2\pi i/3}$:

$\mathfrak{A}_4$	e	$(123), (134),$ $(142), (243),$	$(132), (143)$ $(124), (234)$	$(12)(34), (13)(24),$ $(14)(23)$
χ_0	1	1	1	1
χ_1	1	ω	ω^2	1
χ_2	1	ω^2	ω	1
χ_3	3	0	0	-1

The matrices $M, \tilde{C}, C$ associated with γ are

$$M = \begin{array}{c} \\ \chi_0 \\ \chi_1 \\ \chi_2 \\ \chi_3 \end{array} \begin{array}{cccc} \chi_0 & \chi_1 & \chi_2 & \chi_3 \\ \left(\begin{array}{cccc} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 2 \end{array} \right) \end{array}, \quad \tilde{C} = \begin{array}{c} \\ \chi_0 \\ \chi_1 \\ \chi_2 \\ \chi_3 \end{array} \begin{array}{cccc} \chi_0 & \chi_1 & \chi_2 & \chi_3 \\ \left(\begin{array}{cccc} 3 & 0 & 0 & -1 \\ 0 & 3 & 0 & -1 \\ 0 & 0 & 3 & -1 \\ -1 & -1 & -1 & 1 \end{array} \right) \end{array}, \quad C = \begin{array}{c} \\ \chi_1 \\ \chi_2 \\ \chi_3 \end{array} \begin{array}{ccc} \chi_1 & \chi_2 & \chi_3 \\ \left(\begin{array}{ccc} 3 & 0 & -1 \\ 0 & 3 & -1 \\ -1 & -1 & 1 \end{array} \right) \end{array}$$

Note that $\tilde{C}$ is unchanged by simultaneous cyclic permutations of the rows/columns indexed by χ_0, χ_1, χ_2, as predicted by Proposition 5.6(a), since $\{\chi_0, \chi_1, \chi_2\} = \widehat{G} \cong \mathbb{Z}/3\mathbb{Z}$.

Since $\mathsf{det}(C) = 3$, one concludes that γ has critical group

$$\mathsf{K}(\gamma) := \mathsf{coker}(C^t : \mathbb{Z}^3 \to \mathbb{Z}^3) = \mathbb{Z}^3 / \mathsf{im}(C^t) \cong \mathbb{Z}/3\mathbb{Z}.$$

Toppling with C gives these superstable configurations u and corresponding (via Theorem 2.13) recurrent configurations $v^C - u$, where $v^C = [2, 2, 0]^t$:

superstables	$[0,0,0]^t$	$[1,0,0]^t$	$[0,1,0]^t$
recurrents	$[2,2,0]^t$	$[1,2,0]^t$	$[2,1,0]^t$

Remark 5.13. The equivalence of the various definitions of $\mathsf{K}(\gamma)$ given in Definition 5.11 is sensitive to the choice of the row/column removed to form C from $\tilde{C}$. For example, in the representation $\gamma : \mathfrak{A}_4 \to \mathsf{SO}_3(\mathbb{R})$ considered in Example 5.12, deleting the row/column corresponding to χ_0 gives $\mathsf{coker}(\tilde{C}^t) \cong \mathbb{Z} \oplus \mathbb{Z}/3\mathbb{Z}$. However, the submatrix of $\tilde{C}$ obtained by striking out the row/column indexed by χ_3 is the 3×3 matrix $3I$, whose cokernel is isomorphic to $(\mathbb{Z}/3\mathbb{Z})^3$. Hence, toppling with respect to this matrix gives 27 distinct recurrent configurations, which are all the integer configurations of the form $[a_1, a_2, a_3]^t$ such that $0 \le a_i \le 2$.

5.1. A burning configuration for McKay-Cartan matrices.

For a faithful representation $\gamma : G \hookrightarrow \mathsf{GL}_n(\mathbb{C})$, the 0th row vector $[m_{00}, m_{01}, m_{02}, \ldots, m_{0\ell}]^t$ of M gives the (nonnegative) coefficients in the irreducible expansion $\chi_\gamma = \sum_{j=0}^{\ell} m_{0j}\chi_j$. Next we observe that the projection $\pi : \mathbb{Z}^{\ell+1} \to \mathbb{Z}^{\ell}$ which forgets the 0th coordinate sends it to a vector

$$b_0 := [m_{01}, m_{02}, \ldots, m_{0\ell}]^t$$

which is a burning configuration for C.

Proposition 5.14. *In the above setting, b_0 is a burning configuration for C.*

Proof. It is apparent that b_0 lies in $\mathbb{N}^{\ell}$, because the multiplicities m_{ij} of the irreducible summands are nonnegative. Now we check that b_0 satisfies conditions (i) and (ii) in Definition 2.15 of a burning configuration.

For (i), since $\delta^{(e)} \in \mathsf{ker}(\tilde{C})$ and $\delta_0^{(e)} = 1$, the matrix $\tilde{C}$ has its 0th row $\tilde{C}_{0*}$ lying in the $\mathbb{Z}$-span of the other rows: $\tilde{C}_{0*} = -\sum_{i=1}^{\ell} \delta_i^{(e)}\tilde{C}_{i*}$. Applying $\pi : \mathbb{Z}^{\ell+1} \to \mathbb{Z}^{\ell}$ gives the second equality here

$$(5.15) \qquad b_0 = -\pi(\tilde{C}_{0*}) = \sum_{i=1}^{\ell} \delta_i^{(e)}\pi(\tilde{C}_{i*}) = \sum_{i=1}^{\ell} \delta_i^{(e)}C_{i*}$$

which shows that b_0 lies in $\mathsf{im}(C^t)$.

For (ii), Burnside's theorem says that for a faithful representation γ of a finite group G on $V = \mathbb{C}^n$, any irreducible character χ_j of G occurs in $V^{\otimes m}$ for some m (in fact, for $0 \leq m \leq |\gamma(G)|$ by Brauer's strengthening of that result ([12, Thm. 9.34]) - see also [16, Problem 4.12.10]). This implies there is a directed path with m steps from χ_0 to χ_j in the McKay quiver determined by γ; hence, it also gives such a directed path from some node in $\mathsf{supp}(b_0)$ to χ_j in the digraph $D(C)$ with at most $m - 1$ steps. Consequently, b_0 is a burning configuration for C. $\qquad \square$

Remark 5.16. *Equation (5.15) shows that when stabilizing $v + b_0$ to v for any recurrent configuration v, node i will topple $\delta_i^{(e)}$ times for each $i = 1, 2, \ldots, \ell$, giving a total of $\sum_{i=1}^{\ell} \delta_i^{(e)}$ topplings.*

Example 5.17. For $G = \mathfrak{A}_4$ as in Example 5.12, one has $\delta^{(e)} = [1, 1, 1, 3]^t$. Then

$$b_0 = [0, 0, 1]^t = C^t[1, 1, 3]^t$$

is a burning configuration, and if b_0 is added to any of the recurrent configurations $v = [2, 2, 0]^t$, $[1, 2, 0]^t$ or $[2, 1, 0]^t$, we obtain back v after $5 = 1 + 1 + 3$ topplings.

5.2. Ring and rng structures from tensor product.
As we explain next, besides their additive structure, the abelian groups $\mathsf{K}(C)$ coming from McKay-Cartan matrices carry a *multiplicative* structure as a *rng* (=ring without unit). This transpires by viewing $\mathsf{K}(C)$ as an *ideal* inside the ring $\mathsf{coker}(\tilde{C})$ viewed as a certain quotient of the usual *(virtual) representation ring* $R(G)$ for G.

For a finite group G with irreducible complex characters $\mathsf{Irr}(G) = \{\mathbf{1}_G = \chi_0, \chi_1, \ldots, \chi_\ell\}$, recall that direct sum $\oplus$ and tensor product $\otimes$ of G-representations correspond to pointwise addition and pointwise multiplication of characters, considered as functions $G \to \mathbb{C}$.

Definitions 5.18. For a finite group G, the *ring of virtual characters* or *representation ring* $R(G)$ is a commutative $\mathbb{Z}$-algebra whose additive structure is a free $\mathbb{Z}$-module $\mathbb{Z}^{\ell+1}$ having the ordered $\mathbb{Z}$-basis $\{e_0, e_1, \ldots, e_\ell\}$ in bijection with $\mathsf{Irr}(G)$, and whose multiplication extends $\mathbb{Z}$-linearly the rule $e_i e_j = \sum_{k=0}^{\ell} c_k e_k$ if $\chi_i \cdot \chi_j = \sum_{k=0}^{\ell} c_k \chi_k$ as a pointwise equality of functions $G \to \mathbb{C}$. The unit element 1 of $R(G)$ is e_0. The *(virtual) degree* function is a $\mathbb{Z}$-algebra homomorphism defined via

$$\begin{aligned} \deg : R(G) &\longrightarrow \mathbb{Z} \\ e_i &\longmapsto \delta_i^{(e)}(= \chi_i(e)). \end{aligned}$$

One can think of this as taking the dot product with $\delta^{(e)}$ for vectors in $\mathbb{Z}^{\ell+1}$.

For a faithful representation $\gamma : G \hookrightarrow GL_n(\mathbb{C})$ of degree n, with irreducible decomposition $\chi_\gamma = \sum_{j=0}^\ell m_{0j} \chi_j$, its corresponding element $e_\gamma := \sum_{j=0}^\ell m_{0j} e_j$ in $R(G)$ has $\mathsf{deg}(e_\gamma) = n$. Hence the element $n - e_\gamma$ in $R(G)$ has degree 0, and generates a principal subideal $\langle n - e_\gamma \rangle$ inside the ideal $\mathsf{ker}(\mathsf{deg}) = (\delta^{(e)})^\perp$.

Definition 5.19. In the above context, define the quotient ring

$$R(\gamma) := R(G)/\langle n - e_\gamma \rangle.$$

Since $\langle n - e_\gamma \rangle \subset \mathsf{ker}(\mathsf{deg})$, there is an induced $\mathbb{Z}$-algebra homomorphism $\overline{\mathsf{deg}} : R(\gamma) \longrightarrow \mathbb{Z}$. Define its kernel, the ideal within the quotient ring $R(\gamma)$,

$$I(\gamma) := \mathsf{ker}(\overline{\mathsf{deg}} : R(\gamma) \longrightarrow \mathbb{Z}).$$

Proposition 5.20. *For a faithful representation* $\gamma : G \hookrightarrow GL_n(\mathbb{C})$, *there are additive isomorphisms*

(5.21) $$R(\gamma) \cong \mathsf{coker}(\tilde{C}) \qquad (\cong \mathbb{Z} \oplus \mathsf{K}(C))$$

(5.22) $$I(\gamma) \cong \mathsf{coker}(C) \qquad (\cong \mathsf{K}(C)),$$

which endow

- $\mathbb{Z} \oplus \mathsf{K}(C)$ *with the extra structure of a* $\mathbb{Z}$*-algebra as* $R(\gamma)$, *and*
- $\mathsf{K}(C)$ *with the extra structure of a ring-without-unit (rng), as the ideal* $I(\gamma)$ *in* $R(\gamma)$.

Proof. The isomorphism (5.21) follows since the matrix $\tilde{C}^t$ expresses multiplication by $n - e_\gamma$ in the ring $R(G)$, when using the ordered $\mathbb{Z}$-basis $\{e_0, e_1 \ldots, e_\ell\}$. Then (5.22) comes from restricting this multiplication by $n - \chi_\gamma$ to its action on the ideal $(\delta^{(e)})^\perp = \mathsf{ker}(\mathsf{deg})$ within $R(G)$, and using the equivalent definition $\mathsf{K}(C) = (\delta^{(e)})^\perp/\mathsf{im}(\tilde{C})$ from Definition 5.11. □

Here are three examples of these ring and rng structures.

Example 5.23. Continue Example 5.12 of the alternating group $G = \mathfrak{A}_4$. It has irreducible characters $\mathsf{Irr}(G) = \{\chi_0, \chi_1, \chi_2, \chi_3\}$, and its character table shown there allows one to check these relations:

$$\begin{cases} \chi_k & = \chi_1^k, \\ \chi_k \cdot \chi_3 & = \chi_3, \end{cases} \quad \text{for } k = 0, 1, 2, \qquad \text{and} \qquad \begin{aligned} \chi_1^3 &= \mathbf{1}_G = \chi_0 \\ \chi_3^2 &= 2\chi_3 + \chi_0 + \chi_1 + \chi_2. \end{aligned}$$

Therefore letting $x := e_1, y := e_3$, with $\mathsf{deg}(x) = 1, \mathsf{deg}(y) = 3$, one has

$$R(G) \cong \mathbb{Z}[x, y]/\langle x^3 - 1, \ xy - y, \ y^2 - (2y + 1 + x + x^2) \rangle.$$

Now consider the faithful representation $\gamma : G = \mathfrak{A}_4 \hookrightarrow \mathsf{SO}_3(\mathbb{R}) \subset \mathsf{GL}_3(\mathbb{C})$ from Example 5.12 representing $G = \mathfrak{A}_4$ as the rotational symmetries of a regular tetrahedron. Since $\chi_\gamma = \chi_3$, one has

$$\begin{aligned} R(\gamma) &:= R(G)/\langle 3 - y \rangle \\ &\cong \mathbb{Z}[x, y]/\langle x^3 - 1, \ xy - y, \ y^2 - (2y + 1 + x + x^2), \ 3 - y \rangle \\ &\cong \mathbb{Z}[x]/\langle x^3 - 1, \ 3(x - 1), \ x^2 + x - 2 \rangle \\ &\cong \mathbb{Z}[x]/\langle 3(x - 1), \ (x - 1)^2 \rangle \\ &\cong \mathbb{Z}[u]/\langle 3u, \ u^2 \rangle, \end{aligned}$$

where the last isomorphism comes from a change of variable $u := x - 1$. In fact, u principally generates the ideal $I(\gamma)$ in $R(\gamma)$, and additively one has

$$R(\gamma) \cong \mathbb{Z} \oplus I(\gamma)$$

in which

$$I(\gamma) = (\mathbb{Z}/3\mathbb{Z})u \cong \mathbb{Z}/3\mathbb{Z},$$

with trivial rng (multiplicative) structure on $I(\gamma)$ since $u^2 = 0$.

Example 5.24. The cyclic group $G = \langle g : g^m = e \rangle \cong \mathbb{Z}/m\mathbb{Z}$ has m irreducible characters $\mathsf{Irr}(G) = \{\chi_0, \chi_1, \chi_2, \cdots, \chi_{m-1}\}$, all one-dimensional of the form $\chi_k(g^j) = \omega_m^{jk}$ with $\omega_m := e^{2\pi i/m}$. Hence $\chi_k = \chi_1^k$ for $k = 0, 1, 2, \ldots, m-1$ with $\chi_1^m = \chi_0 = \mathbf{1}_G$, and

$$R(G) \cong \mathbb{Z}[x]/\langle x^m - 1 \rangle,$$

where $x := e_1$ and $\deg(x) = 1$. Choosing the faithful representation $\gamma : G \hookrightarrow \mathsf{SL}_2(\mathbb{C})$ that sends $g \longmapsto \begin{bmatrix} \omega_m & 0 \\ 0 & \omega_m^{-1} \end{bmatrix}$, one has $\chi_\gamma = \chi_1 + \chi_{m-1} = \chi_1 + \chi_1^{m-1}$, and hence $e_\gamma = x + x^{m-1}$. Therefore,

$$\begin{aligned} R(\gamma) &:= R(G)/\langle 2 - (x + x^{m-1}) \rangle \\ &\cong \mathbb{Z}[x]/\langle x^m - 1, \ x^{m-1} + x - 2 \rangle \\ &\cong \mathbb{Z}[x]/\langle x^2 - 2x + 1, \ x^{m-1} + x - 2 \rangle \\ &\cong \mathbb{Z}[x]/\langle x^2 - 2x + 1, \ m(x - 1) \rangle \\ &\cong \mathbb{Z}[u]/\langle u^2, \ mu \rangle, \end{aligned}$$

where

- the second isomorphism used the fact that in $\mathbb{Z}[x]$ dividing $x^m - 1$ by $x^{m-1} + x - 2$ leaves a remainder of $-(x^2 - 2x + 1) = -(x-1)^2$, when $m \geq 3$,
- the third isomorphism used the fact that in $\mathbb{Z}[x]$ dividing $x^{m-1} + x - 2$ by $x^2 - 2x + 1$ leaves a remainder of $m(x-1)$,

and the final isomorphism comes from the change of variable $u := x - 1$. Thus, additively one has

$$R(\gamma) \cong \mathbb{Z} \oplus I(\gamma)$$
$$I(\gamma) \cong (\mathbb{Z}/m\mathbb{Z})\, u \cong \mathbb{Z}/m\mathbb{Z}$$

with trivial rng (multiplicative) structure on $I(\gamma)$ since $u^2 = 0$.

Example 5.25. To illustrate some nontrivial multiplicative structure on $I(\gamma)$, we consider a different family of faithful representations for the same cyclic group G of order m. Fix $n \geq 1$, and let $\gamma : G \hookrightarrow \mathsf{GL}_n(\mathbb{C})$ send g to the $n \times n$ scalar matrix $\omega_m I$. Therefore, $\chi_\gamma = n\chi_1$, and $e_\gamma = nx$, so that

$$\begin{aligned} R(\gamma) &:= R(G)/\langle n - nx \rangle \\ &\cong \mathbb{Z}[x]/\langle x^m - 1, \ n - nx \rangle \\ &\cong \mathbb{Z}[u]/\langle (u+1)^m - 1, \ nu \rangle, \end{aligned}$$

where the last isomorphism comes from our usual change of variable $u := x - 1$. Again, u principally generates the ideal $I(\gamma)$ in $R(\gamma)$, and additively,

$$R(\gamma) \cong \mathbb{Z} \oplus I(\gamma)$$

in which

$$I(\gamma) = (\mathbb{Z}/n\mathbb{Z})u \oplus (\mathbb{Z}/n\mathbb{Z})u^2 \oplus \cdots \oplus (\mathbb{Z}/n\mathbb{Z})u^{m-1} \cong (\mathbb{Z}/n\mathbb{Z})^{m-1}$$

with rng (multiplicative) structure determined by $(u+1)^m - 1 = 0$, that is, $u^m = -\sum_{k=1}^{m-1} \binom{m}{k} u^k$.

6. RELATION TO THE ABELIANIZATION

Our goal here is to prove Theorem 1.3, relating the critical group $\mathsf{K}(\gamma)$ to the *abelianization* $G^{\mathsf{ab}} := G/[G, G]$, whenever γ is a faithful representation that maps G into the *special linear group* $\mathsf{SL}_n(\mathbb{C})$ of complex $n \times n$-matrices of determinant 1.

Recall that every group homomorphism in $\widehat{G}$ factors as a composite $G \twoheadrightarrow G^{\mathsf{ab}} \to \mathbb{C}^\times$, that is, $\widehat{G} \cong \widehat{G^{\mathsf{ab}}}$. Hence for G finite, $\widehat{G}$ is Pontrjagin dual to the abelianization G^{ab}, and (non-canonically) isomorphic to it.

Definition 6.1. *For $\gamma : G \to \mathsf{GL}_n(\mathbb{C})$, let $\mathsf{det}_\gamma := \mathsf{det} \circ \gamma$ in $\widehat{G}$, that is, $\mathsf{det}_\gamma(g) := \mathsf{det}(\gamma(g))$.*

The following is a more precise version of Theorem 1.3. In stating it, we are considering $\mathbb{Z}^{\ell+1}$ and $\widehat{G}$ as abelian groups under addition and pointwise multiplication, respectively.

Theorem 6.2. *For a faithful representation $\gamma : G \hookrightarrow \mathsf{SL}_n(\mathbb{C})$ of a finite group G, the homomorphism*

$$
\begin{array}{ccc}
\mathbb{Z}^{\ell+1} & \xrightarrow{\pi} & \widehat{G} \\
e_i & \longmapsto & \mathsf{det}_{\chi_i}
\end{array}
$$

induces a surjective homomorphism of abelian groups $\mathsf{K}(\gamma) := \mathbb{Z}^{\ell+1}/(\mathbb{Z}e_0 + \mathsf{im}(\tilde{C}^t)) \twoheadrightarrow \widehat{G}$.

Proof. Surjectivity of π follows because every homomorphism φ in $\widehat{G}$ is itself a one-dimensional irreducible character χ_i of G for some i, and hence $\pi(e_i) = \varphi$. Since $\mathsf{K}(\gamma) = \mathbb{Z}^{\ell+1}/(\mathbb{Z}e_0 + \mathsf{im}(\tilde{C}^t))$, it suffices to show that e_0 and $\mathsf{im}(\tilde{C}^t)$ lie in $\mathsf{ker}(\pi)$. For e_0 this is clear, since $\chi_0 = \mathbf{1}_G$ and $\mathsf{det}_{\mathbf{1}_G} = \mathbf{1}_G$.

To show that $\mathsf{im}(\tilde{C}^t) \subset \mathsf{ker}(\pi)$, we compare the expressions for det_γ on the two sides of

$$
\tag{6.3} \chi_\gamma \cdot \chi_i = \sum_{j=0}^\ell m_{ij} \chi_j.
$$

On the left side of (6.3), note that for any two linear operators $T_i : U_i \to U_i$ acting on finite-dimensional vector spaces U_i for $i = 1, 2$, we have

$$
\tag{6.4} \mathsf{det}_{U_1 \otimes U_2}(T_1 \otimes T_2) = \mathsf{det}_{U_1}(T_1)^{\mathsf{dim}(U_2)} \mathsf{det}_{U_2}(T_2)^{\mathsf{dim}(U_1)}.
$$

Consequently, for any two genuine characters χ, ψ of G, one has

$$
\tag{6.5} \mathsf{det}_{\chi \cdot \psi} = (\mathsf{det}_\chi)^{\psi(e)} (\mathsf{det}_\psi)^{\chi(e)}.
$$

Since $\gamma : G \to \mathsf{SL}_n(\mathbb{C})$ means that $\mathsf{det}_\gamma(-) = \mathbf{1}_G$, (6.4), this shows that

$$
\mathsf{det}_{\chi_\gamma \cdot \chi_i} = (\mathsf{det}_{\chi_i})^n (\mathsf{det}_\gamma)^{\delta_i^{(e)}} = (\mathsf{det}_{\chi_i})^n.
$$

Comparing this with the right side of (6.3), we conclude that for each $i = 0, 1, 2, \ldots, \ell$, one has

$$
(\mathsf{det}_{\chi_i})^n = \prod_{j=0}^\ell (\mathsf{det}_{\chi_j})^{m_{ij}}.
$$

This says that row i of $\tilde{C}$ (= column i of $\tilde{C}^t$) lies in $\mathsf{ker}(\pi)$ for each i, and hence $\mathsf{im}(\tilde{C}^t) \subset \mathsf{ker}(\pi)$. $\quad\square$

Example 6.6. Example 5.12 considered a faithful representation $\gamma : G = \mathfrak{A}_4 \hookrightarrow \mathsf{SL}_3(\mathbb{C})$ with

$$
\mathsf{K}(\gamma) \cong \mathbb{Z}/3\mathbb{Z} \cong G^{\mathsf{ab}}.
$$

Example 6.7. Example 5.24, considered a faithful representation $\gamma : G = \mathbb{Z}/m\mathbb{Z} \hookrightarrow \mathsf{SL}_2(\mathbb{C})$ with

$$
\mathsf{K}(\gamma) \cong \mathbb{Z}/m\mathbb{Z} \cong G(= G^{\mathsf{ab}}).
$$

Example 6.8. On the other hand, Example 5.25 considered a different family of faithful representations $\gamma : G = \mathbb{Z}/m\mathbb{Z} \hookrightarrow \mathsf{GL}_n(\mathbb{C})$ that sent g to $\omega_m I \in \mathsf{GL}_n(\mathbb{C})$, with

$$
\mathsf{K}(\gamma) \cong (\mathbb{Z}/n\mathbb{Z})^m.
$$

Note that $\gamma(G) \subset \mathsf{SL}_n(\mathbb{C})$ if and only if m divides n, which is exactly the same condition under which $\mathsf{K}(\gamma) \cong (\mathbb{Z}/n\mathbb{Z})^m$ can surject onto $\mathbb{Z}/m\mathbb{Z} = G(= G^{\mathsf{ab}})$, as Theorem 6.2 would predict.

Theorem 6.2 interacts in an interesting way with multiplication in $R(G)$.

Proposition 6.9. *The surjection π from Theorem 6.2, considered as a homomorphism*

$$(R(G), +) \longrightarrow \widehat{G}$$

annihilates all products xy with x, y, in $I(G)$, that is, it factors through the quotient $(R(G)/I(G)^2, +)$.

Proof. Note that $I(G) := \mathsf{ker}(\mathsf{deg})$ is the $\mathbb{Z}$-linear span of the elements $\{e_i - \delta_i^{(e)}\}_{i=1,2,\dots,\ell}$:

$$x = \sum_{i=0}^{\ell} c_i e_i \text{ lies in } \mathsf{ker}(\mathsf{deg}) \quad \Longleftrightarrow \quad \sum_{i=0}^{\ell} c_i \delta_i^{(e)} = 0 \quad \Longleftrightarrow \quad x = \sum_{i=0}^{\ell} c_i (e_i - \delta_i^{(e)}).$$

Therefore it suffices to show that π annihilates all products of the form $(e_i - \delta_i^{(e)})(e_j - \delta_j^{(e)})$:

$$\pi\left((e_i - \delta_i^{(e)})(e_j - \delta_j^{(e)}) \right) = \pi\left(e_i e_j - (\delta_i^{(e)} e_j + \delta_j^{(e)} e_i) + \delta^{(e)} \delta_j^{(e)} \right)$$

$$= \left(\mathsf{det}_{\chi_i \cdot \chi_j} \cdot \mathbf{1}_G^{\delta_i^{(e)} \delta_j^{(e)}} \right) / \left((\mathsf{det}_{\chi_i})^{\delta_j^{(e)}} (\mathsf{det}_{\chi_j})^{\delta_i^{(e)}} \right) = \mathbf{1}_G,$$

where the last equality used identity (6.5). $\square$

Corollary 6.10. *The surjection π from Theorem 6.2 induces a surjection of abelian groups*

$$\mathsf{K}(\gamma) \cong (I(\gamma), +) \twoheadrightarrow \widehat{G}$$

which annihilates all products in $I(\gamma)^2$. In particular, whenever there is an isomorphism $\mathsf{K}(\gamma) \cong \widehat{G}$, all products in $I(\gamma)$ vanish, that is, $I(\gamma)^2 = 0$.

Proof. Since $\pi(e_i - \delta_i^{(e)}) = \mathsf{det}_{\chi_i} = \pi(e_i)$, the map π is surjective when restricted from $R(G)$ to $I(G)$. Since π descends to the quotient $I(\gamma)$ of $I(G)$, the rest follows from Proposition 6.9. $\square$

Question 6.11. For which faithful representations $\gamma : G \hookrightarrow \mathsf{SL}_n(\mathbb{C})$ of a finite group G is the surjection in Theorem 6.2 an *isomorphism* $\mathsf{K}(\gamma) \cong \widehat{G}$?

The answer for *abelian groups* will be given in Proposition 6.19 below.
    We mention here two general reductions in Question 6.11.

Proposition 6.12. *The critical group $\mathsf{K}(\gamma)$ of a representation $\gamma : G \to \mathsf{GL}_n(\mathbb{C})$ and the ring and rng structures on $R(\gamma), I(\gamma)$ are unchanged by*

   (a) *adding copies of the trivial representation, that is, replacing $\chi_\gamma \mapsto \chi_\gamma + d \cdot \chi_0$, or*
   (b) *precomposing with any group automorphism $\sigma : G \to G$, that is, replacing $\gamma \mapsto \gamma \circ \sigma$.*

Proof. For (a), note that replacing χ_γ with $\chi_{\gamma'} := \chi_\gamma + d \cdot \chi_0$, replaces M with $M' = M + dI$, and hence $\tilde{C}$ with $\tilde{C}' = (n + d)I - (M + dI) = nI - M = \tilde{C}$. It also replaces $n - e_\gamma$ with $(n + d) - (e_\gamma + d) = n - e_\gamma$, so that $R(\gamma), I(\gamma), \mathsf{coker}(\tilde{C})$, and $\mathsf{coker}(C)$ are unchanged. For (b), note that precomposing G-representations with σ permutes $\{\chi_0, \chi_1, \dots, \chi_\ell\}$, replacing $\tilde{C}$ by $P\tilde{C}P^{-1}$ for some permutation matrix P, and inducing simultaneous ring automorphisms of $R(G)$ and $R(\gamma)$. $\square$

6.1. Example: McKay correspondence for subgroups of $\mathsf{SL}_2(\mathbb{C})$.

For a finite subgroup G of $\mathsf{SL}_2(\mathbb{C})$, the action of G on $\mathbb{C}^2$ (by matrix multiplication) gives a faithful representation γ. McKay originally observed in [27] (see also Steinberg [34, §1(4)]) that the McKay quiver (Q_0, Q_1) has

   • *undirected* arcs, implying $m_{ij} = m_{ji}$,
   • no multiple edges, meaning $m_{ij} = m_{ji} \in \{0, 1\}$,
   • no self-loops, meaning $m_{ii} = 0$,
   • $\tilde{C} = 2I - M$, the extended Cartan matrix of a simply-laced root system, as in Section 3.4.

In particular, $\mathsf{K}(\gamma) = \mathbb{Z}^\ell/\mathrm{im}(C^t) = P(\Phi)/Q(\Phi)$ is the *fundamental group* of the finite root system Φ, the weight lattice modulo the root lattice, discussed in Section 3.3.

Note that here $\delta^{(e)} = \delta = \delta^\vee$, where $\delta^{(e)}$ was defined in (5.5), and $\delta, \delta^\vee$ were defined in Section 3.3: Φ being simply laced implies $\delta^\vee = \delta$ is the basis for the left or right-nullspace of $\tilde{C}$, and since γ is faithful, Proposition 5.3(c) tells us that this nullspace is one-dimensional, spanned by $\delta^{(e)}$ from (5.5). Since $\delta_0 = 1 = \delta_0^{(e)}$, it must be that $\delta = \delta^{(e)}$.

We have the following more precise version of Theorem 1.4.

Theorem 6.13. *For G a finite subgroup of $\mathsf{SL}_2(\mathbb{C})$ and $\gamma : G \hookrightarrow \mathsf{SL}_2(\mathbb{C})$ (the natural representation), the surjection in Theorem 6.2 is an isomorphism $\mathsf{K}(\gamma) \cong \widehat{G}$. Thus the critical group $\mathsf{K}(\gamma)$ is Pontrjagin dual (so non-canonically isomorphic) to the abelianization G^{ab}.*

Proof of Theorem 6.13. It suffices to show $|\mathsf{K}(\gamma)| = |\widehat{G}|$. As discussed above, $\mathsf{K}(\gamma) = P(\Phi)/Q(\Phi)$, and hence $|\mathsf{K}(\gamma)| = f$, the *index of connection* for Φ. Section 3.3 asserted that $f - 1$ is the number of indices $i = 1, 2, \ldots, \ell$ for which $\delta_i^\vee = 1$ in (3.11). Since $\delta^\vee = \delta = \delta^{(e)}$, it must be that $f - 1$ is the number of i for which $\delta_i^{(e)} = 1$, that is, the number of one-dimensional characters in $\widehat{G} \setminus \{\mathbf{1}_G\}$. Thus $|\widehat{G}| = f = |\mathsf{K}(\gamma)|$. $\qquad\square$

root system Φ	presentation of $G \subset \mathsf{SL}_2(\mathbb{C})$	$\mathsf{K}(\gamma) \cong P(\Phi)/Q(\Phi)$ $\cong G^{\mathsf{ab}}$	affine diagram labeled by $\delta = \delta^\vee = \delta^{(e)}$
A_{m-1}	$\langle r : r^m = 1 \rangle$	$\mathbb{Z}/m\mathbb{Z}$	$\begin{array}{c} 1 \\ 1 = 1 - \cdots - 1 = 1 \end{array}$
D_{m+2}	$\langle r, s, t : r^2 = s^2 = t^m = rst \rangle$	$\mathbb{Z}/4\mathbb{Z}$ if m odd $(\mathbb{Z}/2\mathbb{Z})^2$ if m even	$\begin{array}{c} 1 \qquad\qquad 1 \\ \;\;\diagdown \qquad\quad \diagup \\ 2 - 2 - \cdots - 2 \\ \;\;\diagup \qquad\quad \diagdown \\ 1 \qquad\qquad 1 \end{array}$
E_6	$\langle r, s, t : r^2 = s^3 = t^3 = rst \rangle$	$\mathbb{Z}/3\mathbb{Z}$	$\begin{array}{c} 1 \\ 2 \\ 1 - 2 - 3 - 2 - 1 \end{array}$
E_7	$\langle r, s, t : r^2 = s^3 = t^4 = rst \rangle$	$\mathbb{Z}/2\mathbb{Z}$	$\begin{array}{c} 2 \\ 1 - 2 - 3 - 4 - 3 - 2 - 1 \end{array}$
E_8	$\langle r, s, t : r^2 = s^3 = t^5 = rst \rangle$	0	$\begin{array}{c} 3 \\ 1 - 2 - 3 - 4 - 5 - 6 - 4 - 2 \end{array}$

For example, A_{m-1} in the table corresponds to the representation of $\mathbb{Z}/m\mathbb{Z}$ from Example 5.24.

Remark 6.14. We explain next how Theorem 6.13 is Pontrjagin dual to a result of Steinberg [34, §1(6)].

A finite, crystallographic, irreducible root system Φ has associated to it various compact real Lie groups $\mathfrak{G}$ that all share the same Lie algebra $\mathfrak{g}$ and root system Φ. The two extremes among them are the *adjoint group* $\mathfrak{G}_{\mathsf{ad}}$ and its simply-connected universal cover $\mathfrak{G}_{\mathsf{sc}} \to \mathfrak{G}_{\mathsf{ad}}$, a Galois covering with covering group

$$\pi_1(\mathfrak{G}_{\mathsf{ad}}) \cong P(\Phi^\vee)/Q(\Phi^\vee) \cong \mathfrak{Z}(\mathfrak{G}_{\mathsf{sc}})$$

where $\mathfrak{Z}(\mathfrak{G}_{\mathsf{sc}})$ denotes the center of the group $\mathfrak{G}_{\mathsf{sc}}$. This explains the terminology "fundamental group"; see Bröcker-tom Dieck [8, Chap. V §7, Thms. 7.1, 7.16] and Bourbaki [7, Chap. IX §9, Thm.2ff]. For a finite group $G \subset \mathsf{SL}_2(\mathbb{C})$, Steinberg [34, §1(6)] describes an isomorphism $G^{\mathsf{ab}} \xrightarrow{\sim} \mathfrak{Z}(\mathfrak{G}_{\mathsf{sc}})$. Applying the contravariant Pontrjagin duality functor $A \mapsto \widehat{A}$ then gives this

disguised version of the isomorphism in Theorem 6.13:

$$
\begin{array}{ccc}
\widehat{\mathfrak{Z}(\mathfrak{G}_{\mathsf{sc}})} & \xrightarrow{\ \sim\ } & \widehat{G^{\mathsf{ab}}} \\
\| & & \| \\
P(\Phi)/Q(\Phi) & & \widehat{G} \\
\| & & \\
\mathsf{K}(\gamma) & &
\end{array}
$$

Remark 6.15. Theorem 6.13 disagrees slightly with Brylinski [9, Cor. 5.9], which says that the *index of connection* $f := |\mathsf{K}(\gamma)|$ is the *exponent* of the abelianization G^{ab}. For example, in the McKay correspondence for type D_ℓ with ℓ even, the above table shows $\mathsf{K}(\gamma) = G^{\mathsf{ab}} = (\mathbb{Z}/2\mathbb{Z})^2$, so that one has $f = 4$, but the exponent of G^{ab} is 2.

Remark 6.16. In McKay's original setting of $G \hookrightarrow \mathsf{SL}_2(\mathbb{C})$, the interpretation of minuscule dominant weights from Proposition 3.10 (a) enables us to reinterpret the assertion of Theorem 1.1(i) as follows: the (nonzero) superstable configurations for C are exactly the basis vectors e_i corresponding to the (nontrivial) 1-dimensional characters χ_i of G. In light of Theorem 1.3, this suggests the following question.

Question 6.17. Given a faithful representation $G \hookrightarrow \mathsf{SL}_n(\mathbb{C})$, and its McKay-Cartan matrix C, do the basis vectors e_i corresponding to the nontrivial one-dimensional characters χ_i of G always form a subset of the superstable configurations for C?

6.2. **Example: abelian groups.** We explain here why any faithful representation $\gamma : G \hookrightarrow \mathsf{GL}_n(\mathbb{C})$ of a finite *abelian* group G always has critical group $\mathsf{K}(\gamma)$ equal to the usual critical group $K(D)$ for the directed graph D which is its McKay quiver, as in Remark 2.18.

For G abelian, all irreducible representations of G are one-dimensional, that is, $\{\chi_0, \chi_1, \dots, \chi_\ell\} = \widehat{G}$. Evaluating the characters on both sides of $\chi_\gamma \cdot \chi_i = \sum_{j=0}^{\ell} m_{ij}\chi_j$ at e shows that $n = \sum_{j=0}^{\ell} m_{ij}$ is the outdegree of χ_i for each i in the McKay quiver considered as a digraph D. Thus, the extended Cartan matrix $\tilde{C}$ is the same as the Laplacian matrix $\tilde{L}$ for D. In fact, D can be considered as the *Cayley digraph* for $\widehat{G}$ in which each potential generator χ_j is given multiplicity m_{0j} if $\chi_\gamma = \sum_{j=0}^{\ell} m_{0j}\chi_j$. One can check that the set $\{\chi_j : m_{0j} \geq 1\}$ actually *does* generate $\widehat{G}$ if and only if γ is faithful. In particular, all vertices in D look the same, up to translation by elements of $\widehat{G}$, so that the choice of the row/column in $\tilde{C}$ to strike out in forming C is immaterial.

Example 6.18. Examples 5.24 and 5.25 both had $G = \langle g : g^m = e \rangle \cong \mathbb{Z}/m\mathbb{Z}$, and therefore $\tilde{C} = \tilde{L}$ for a circulant digraph D on vertex set $\{0, 1, \dots, m-1\}$.

- In the classical McKay case of type A_{m-1} in Example 5.24, where $g \mapsto \begin{bmatrix} \omega_m & 0 \\ 0 & \omega_m^{-1} \end{bmatrix}$, one has $\chi_\gamma = \chi_1 + \chi_{m-1}$, and hence the digraph D has both arrows $\chi_i \to \chi_{i+1}$ and $\chi_{i+1} \to \chi_i$ for each $i = 0, 1, 2, \dots, m-1$ (subscripts modulo m).
- In Example 5.25, where $g \mapsto \omega_m I$, one has $\chi_\gamma = n\chi_1$, and hence the digraph D has n copies of the same arrow $\chi_i \to \chi_{i+1}$ for each $i = 0, 1, 2, \dots, m-1$ (subscripts modulo m).

Interestingly, in the case where G is abelian, one can resolve Question 6.11(i). The authors are grateful to S. Koplewitz for providing a proof of the following proposition.

Proposition 6.19. *Let G be a finite abelian group, and $\gamma : G \hookrightarrow \mathsf{SL}_n(\mathbb{C})$ be a faithful representation with no copies of $\mathbf{1}_G$. Then $\mathsf{K}(\gamma) \cong \widehat{G}$ if and only if $n = 2$, that is, if and only if $G \cong \mathbb{Z}/m\mathbb{Z}$ with $G \subset \mathsf{SL}_2(\mathbb{C})$ as in McKay's type A_{m-1}.*

Sketch proof. Recall that Theorem 1.3 provides a surjection $\mathsf{K}(\gamma) \twoheadrightarrow \widehat{G}$. Since for digraphs D, one has an interpretation (see Remark 2.18) for $|K(D)|$ as the number of arborescences in D directed toward the vertex χ_0, this proposition can be rephrased as follows: Given a

- finite abelian group $(A, +)$ (corresponding to $A = \widehat{G}$),
- a multiset $a_1, \ldots, a_r$ of $A \backslash \{0\}$ (corresponding to $\chi_\gamma = \sum_{i=1}^r \chi_{a_i}$ not containing the character $\chi_0 = \mathbf{1}_G$),
- with $\{a_i\}_{i=1}^r$ generating A (corresponding to χ_γ being faithful),
- and $\sum_{i=1}^r a_i = 0$ (corresponding to $\gamma(G) \subset \mathsf{SL}_n(\mathbb{C})$),

then the associated Cayley digraph for $(A, (a_i)_{i=1}^r)$ has the number of arborescences directed toward 0 strictly greater than $|A|$ whenever $r \geq 3$.

To see this, use the fact (see, e.g., [18, Thm. 3.3]) that any Cayley digraph D for an abelian group A has a (directed) Hamilton path $(b_{|A|} \to \cdots \to b_2 \to b_1 \to b_0 := 0)$, and any element a_j of the generating multiset $\{a_i\}_{i=1}^r$ can be chosen as the label $a_j = b_1 - b_0$ on the last step $b_1 \to b_0$. The idea is then to construct a map f from A to the set of arborescences that takes b_i to an arborescence containing the subpath $(b_i \to \cdots \to b_2 \to b_1 \to b_0 := 0)$ but not containing the arc $b_{i+1} \to b_i$. Roughly speaking, there is flexibility to do this, since the condition $\sum_{i=1}^r a_i = 0$ implies that the generator $a_j := b_{i+1} - b_i$ is redundant for generating A, and hence partial arborescences can be completed to full arborescences avoiding arcs (such as $b_{i+1} \to b_i$) labeled by a_j. This gives an injective map f, which one can show is *not* surjective for $r \geq 3$. $\qquad\square$

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