Week 6
Partial differential equations

1 Heat equation - boundary conditions

Consider the equation for heat conduction in a rod:

$$u_t = \alpha^2 u_{xx}, \tag{1}$$

on $0 \leqslant x \leqslant L$, $t \geqslant 0$, with initial distribution of temperature $u(x, t = 0) = f(x)$.

Recalling that $u(x,t)$ is the temperature of the rod at position x and time t, discuss in your group what the following boundary conditions correspond to physically:

(1) $u(0,t) = 10$, $u_t(L,t) = 0$.

(2) $u_t(0,t) = -1$, $u_t(L,t) = 1$.

(3) $u(0,t) = 5$, $u_x(0,t) = 0$.

2 Heat equation - separation of variables

In this problem, we will find the temperature $u(x, t)$ in a solid rod of length 50 cm, given that it satisfies the heat equation with thermal diffusivity α^2. Assume that the rod has an initial temperature of $20°$C throughout the rod, and that the rod ends are insulated and maintained at $0°$C for all times.

(1) Write down the heat equation for this rod, as well as the initial and boundary conditions the temperature must satisfy.

(2) Find the temperature $u(x, t)$ using the solution derived in class with separation of variables. This involves solving for the coefficients in a Fourier sine series expansion.

3 Wave equation - nonzero initial velocity

Consider the wave equation problem with nonzero initial velocity:

(2) $$u_{tt} = a^2 u_{xx},$$

on $0 \leqslant x \leqslant L$, $t \geqslant 0$, with boundary conditions

$$u(x=0,t)=0, \qquad u(x=L,t)=0 \qquad \text{for } t \geqslant 0$$

and initial conditions

$$u(x,t=0)=0, \qquad u_t(x,t=0)=g(x) \qquad \text{for } 0 \leqslant x \leqslant L.$$

You will solve this problem in your group using separation of variables in a similar manner as we solved the wave equation problem with nonzero initial displacement in class.

(1) Substitute $u(x,t) = X(x)T(t)$ into the problem above. Make sure to note any changes from the class problem - you can expect a change in the initial condition for T.

(2) Solve the separated x-problem. Are there any changes from the class problem?

(3) Solve the separated t-problem. Are there any changes from the class problem?

(4) Write out the general solution as an infinite series. What do you need to do in order to use the second initial condition given?

(5) Check the answer for the previous question with your instructor, and find the solution of the problem.