Group Session 2

Thursday, July 2nd

Problem 1

In this problem, we will solve example 3 from class using the concept of matrix exponentials.

1. First, calculate the eigenvalues and eigenvectors of matrix $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$ (or find them from your class notes).

2. Is matrix A diagonal/diagonalizable/not diagonalizable? Why?

3. Find the transformation matrix T for A.

4. Calculate $T^{-1}AT$. Call this matrix D and have a member of your group explain why the result makes sense given your answer to part 2.

5. Calculate e^{Dt}. Recall that this is a fundamental matrix for $\mathbf{y}' = D\mathbf{y}$ where $\mathbf{x} = T\mathbf{y}$.

6. Use this matrix to calculate the fundamental matrix for the system of differential equations in $\mathbf{x}$: $\mathbf{x}' = A\mathbf{x}$.

7. Have another member in your group summarize the steps you took in this problem and check with your instructors.

Problem 2

In this problem, we consider example 3 in section 7.6 in the book, but solve it in a different way. Let

$$m_1 \frac{d^2 x_1}{dt^2} = -(k_1 + k_2)x_1 + k_2 x_2\,, \tag{1}$$

$$m_2 \frac{d^2 x_2}{dt^2} = k_2 x_1 - (k_2 + k_3)x_2\,. \tag{2}$$

These equations describe the motion of 2 masses and 3 springs.

1. Re-write the system in matrix form: $\mathbf{x}'' = A\mathbf{x}$, where $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$.

2. Let $m_1 = 2$, $m_2 = \frac{9}{4}$, $k_1 = 1$, $k_2 = 3$, $k_3 = \frac{15}{4}$. Write down the matrix A in this case, and make sure it is correct before you proceed.

3. Let $\mathbf{x} = \mathbf{v}e^{rt}$ and plug it into the $\mathbf{x}'' = A\mathbf{x}$ equation. Show that $(A - r^2 I)\mathbf{v} = \mathbf{0}$.
   Note that now r^2 (as opposed to r) are eigenvalues of matrix A.

4. Compute the eigenvalues of matrix A, and then find r and $\mathbf{v}$ from part 3.

5. Find x_1 and x_2 by recalling the solution guess in part 3. Note that there should be 4 arbitrary constants in these expressions.

6. Have another member in your group summarize the steps you took in this problem and check with your instructors.

Problem 3

In class, we only looked at solutions of equation $\mathbf{x}' = A\mathbf{x}$ for A a real-valued matrix. In this problem, we examine what happens in the case of a specific type of complex matrix.

A square matrix A is called **Hermitian** if $A = \bar{A}^T$ (in other words, if it equals the conjugate of its transpose). This is usually denoted as $A = A^*$. Note that $(AB)^* = B^*A^*$, and $(A\mathbf{x})^* = \mathbf{x}^*A^*$.

Note: If there is not enough time, go straight to part 4 of this question.

1. What properties does a Hermitian matrix with real entries have?

2. If a Hermitian matrix has complex entries, what can you say about its diagonal elements?

3. In this part, we will prove that all eigenvalues of a Hermitian matrix are real.

   (a) Start with the definition of an eigenvalue-eigenvector pair:

   $$A\mathbf{v} = \lambda\mathbf{v}\,. \tag{3}$$

   Apply * to both sides of this equation, then multiply the right of both sides by $\mathbf{v}$.

   (b) Use the fact that A is Hermitian: $A = A^*$ and equation (3) to simplify your result from part (b).

   (c) Conclude that $\lambda = \lambda^*$ and discuss with this group what this suggests about the eigenvalues of a Hermitian matrix.

It can also be shown that a Hermitian matrix A must have a full set of linearly independent eigenvectors.

4. Let
$$\mathsf{H} = \begin{bmatrix} 1 & 2+i \\ 2-i & 5 \end{bmatrix}.$$

(a) Verify that matrix H is Hermitian.

(b) Find the eigenvalues and eigenvectors of matrix H.

(c) Discuss any observations with your group.

(d) Write down the general solution of $\mathbf{x}' = \mathsf{H}\mathbf{x}$.

Have someone in your group explain what this problem suggests about solving $\mathbf{x}' = A\mathbf{x}$ where A a Hermitian matrix.

Problem 4

In the last part of the problem session, we will show you an example of a numerical tool that can create the phase portraits for a given planar (2-equation) system of differential equations. We will be using the tool later in the course as well.