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Consider again the conservation law,

$$\begin{cases} u_t + f(u)_x = 0 \\ u(x,0) = u^0(x) \end{cases}.$$

Then, we want to find $u_h \in V_h$ s.t.

$$\int_{I_j} (u_h)_t v_h dx - \int_{I_j} f(u_h)(v_h)_x dx + \hat{f}_{j+1/2}(v_h)^-_{j+1/2} - \hat{f}_{j-1/2}(v_h)^+_{j-1/2} = 0, \tag{1.1}$$

for all $v_h \in V_h$ and for $\hat{f}_{j+1/2} = \hat{f}((u_h)^-_{j+1/2}, (u_h)^+_{j+1/2})$.

We introduced the cell entropy inequality:

Theorem 1.1. *The DG scheme satisfies the following cell entropy inequality:*

$$\int_{I_j} U(u_h)_t dx + \hat{F}_{j+1/2} - \hat{F}_{j-1/2} \leqslant 0$$

for $U(u) = \frac{u^2}{2}$, and a consistent entropy flux $\hat{F}_{j+1/2} = \hat{F}((u_h)^-_{j+1/2}, (u_h)^+_{j+1/2})$ in the sense that

$$\begin{aligned} \hat{F}(u,u) &= F(u) \\ &= \int^u U'(u) f'(u) du \\ &= \int^u u f'(u) du. \end{aligned}$$

Proof. First,

$$\begin{aligned} \frac{d}{dt}\int_{I_j} U(u_h) dx &= \int_{I_j} \frac{d}{dt}\left(\frac{1}{2}u_h^2\right) dx \\ &= \int_{I_j} u_h (u_h)_t dx. \end{aligned}$$

For (3.1), let us take $v_h = u_h$ since $u_h \in V_h$ (note that $u_h^2 \notin V_h$). We get

$$\int_{I_j} (u_h)_t u_h dx - \int_{I_j} f(u_h)(u_h)_x dx + \hat{f}_{j+1/2}(u_h)^-_{j+1/2} - \hat{f}_{j-1/2}(u_h)^+_{j-1/2} = 0. \tag{1.2}$$

Let us look a the integrand of the second integral on the LHS. Let $g(u) = \int^u f(u) du$ so that $g'(u) = f(u)$. Then,

$$\int_{I_j} f(u_h)(u_h)_x dx = \int_{I_j} g(u_h)_x dx.$$

Thus, the second integral becomes

$$g((u_h)^-_{j+1/2}) - g((u_h)^+_{j-1/2}).$$

Hence, we get

$$\int_{I_j} (u_h)_t u_h dx - \left[g((u_h)^-_{j+1/2}) - g((u_h)^+_{j-1/2})\right] + \hat{f}_{j+1/2}(u_h)^-_{j+1/2} - \hat{f}_{j-1/2}(u_h)^+_{j-1/2} = 0$$

and let $\hat{F}_{j+1/2} \mathrel{\hat{=}} -g((u_h)^-_{j+1/2}) + \hat{f}_{j+1/2}(u_h)^-_{j+1/2}$. Then,

$$\int_{I_j} (u_h)_t u_h dx + \hat{F}_{j+1/2} - \hat{F}_{j-1/2} + H_{j-1/2} = 0.$$

Is the flux consistent?

$$\begin{aligned} \hat{F}(u,u) &= -g(u) + \hat{f}(u,u)u \\ &= -g(u) + f(u)u. \end{aligned}$$

Now,

$$\begin{aligned} F(u) &\mathrel{\mathring{=}} \int^u u f'(u)\, du \\ &= u f(u) - \int^u f(u)\, du \\ &= u f(u) - g(u). \end{aligned}$$

Thus,

$$\begin{aligned} \hat{F}(u,u) &= \int^u u f'(u)\, du \\ &= F(u), \end{aligned}$$

so it is consistent by our definition. Also, we need

$$-\hat{F}_{j-1/2} + H_{j-1/2} = g((u_h)^-_{j-1/2}) - \hat{f}_{j+1/2}(u_h)^-_{j-1/2}$$

so that

$$H = g(u^+) - \hat{f}u^+ + \Big[-g(u^-) + \hat{f}u^- \Big]$$

where everything is evaluated at $j - 1/2$. If we prove that H is positive, then the remaining quantity has to be negative. We have

$$\begin{aligned} H &= g(u^+) - g(u^-) - \hat{f}(u^+ - u^-) \\ &\overset{\text{MVT}}{=} g'(\xi)(u^+ - u^-) - \hat{f}(u^+ - u^-) \\ &\mathrel{\mathring{=}} f(\xi)(u^+ - u^-) - \hat{f}(u^+ - u^-) \\ &= (u^+ - u^-)(f(\xi) - \hat{f}(u^+, u^-)) \\ &= (u^+ - u^-)(\hat{f}(\xi,\xi) - \hat{f}(u^+, u^-)) \\ &= (u^+ - u^-)(\hat{f}(\xi,\xi) - \hat{f}(\xi,u^+) + \hat{f}(\xi,u^+) - \hat{f}(u^+, u^-)) \\ &= (u^+ - u^-)\Big\{\hat{f}(\xi,\xi) - \hat{f}(\xi,u^+)\Big\} + (u^+ - u^-)\Big\{\hat{f}(\xi,u^+) - \hat{f}(u^+, u^-)\Big\}. \end{aligned}$$

by consistency of $\hat{f}$ and ξ is between u^- and u^+. Recall that $\hat{f}(\uparrow, \downarrow)$. Then, if $u^- < u^+$ we have $u^- \leqslant \xi \leqslant u^+$ so that $\hat{f}(\xi,\xi) - \hat{f}(\xi, u^+) \geqslant 0$. If $u^- > u^+$ then $u^- \geqslant \xi \geqslant u^+$. Then, the product $(u^+ - u^-)\Big\{\hat{f}(\xi,\xi) - \hat{f}(\xi,u^+)\Big\}$ is still positive since both terms are negative. At the end of the day

$$H \geqslant 0.$$

We are only using the monotonicity of $\hat{f}$. □

Corollary 1.2. *If the boundary condition is periodic, then*

$$\frac{d}{dt}\int_a^b (u_h)^2 dx \leqslant 0.$$

That is, the L_2 is nonincreasing w.r.t. time.

Proof. We just sum $\int_{I_j} U(u_h)_t dx + \hat{F}_{j+1/2} - \hat{F}_{j-1/2}$ over all j. Then the sum is telescoping and we get the desired result. □

Theorem 1.3. *Consider the PDE*

$$u_t + u_x = 0,$$

where u is smooth. Let $e = u - u_h$. Then

$$e \sim \mathcal{O}(h^{k+1}).$$

Proof. Let us use the following projection $\mathcal{P} \in P^k(I_j)$:

$$\int_{I_j} (\mathcal{P}u - u)\, v\, dx = 0, \forall v \in P^{k-1}(I_j)$$

and

$$(\mathcal{P}u - u)^-_{j+1/2} = 0.$$

We are asking for k conditions in th orthogonality statement and another 1 condition in this boundary condition above. Thus, we can solve for our projection by knowing $k+1$ conditions.

As we did before, we estimate e by using the triangle inequality. Hence,

$$e = u - \mathcal{P}_h u + \mathcal{P}_h u - u_h.$$

Note that

$$\|u - \mathcal{P}_h u\|_{L^2} \leqslant Ch^{k+1}.$$

For interpolation, we can show this (see proof by brenna and scott, c allay). Actually, any polynomial projection which remains polynomials of degree k untouched, is of this order. So we can control $u - \mathcal{P}_h u$. What about $\mathcal{P}_h u - u_h$? Let us use the stability condition to bound this term.

Let us denote

$$B_j(u_h, v_h) \mathrel{\hat{=}} \int_{I_j} (u_h)_t u_h dx - \int_{I_j} f(u_h)(u_h)_x\, dx + (u_h)^-_{j+1/2}(v_h)^-_{j+1/2} - u_h)^-_{j-1/2}(v_h)^+_{j-1/2},$$

where $\hat{f}_{j+1/2} = \hat{f}((u_h)^-_{j+1/2}, (u_h)^+_{j+1/2})$. But since $f(u) = u$ then $\hat{f}_{j+1/2} = (u_h)^-_{j+1/2}$ the upwinding scheme (from left to right). So the scheme is $B_j(u_h, v_h) = 0, \forall v_h \in V_h$. Then, certainly $B_j(u, v_h) = 0, \forall v_h \in V_h$ since $\int_{I_j} (u_t + u_x) v_h = 0$. Hence,

$$B(e, v_h) = 0, \forall v_h \in V_h.$$

Note that we can take the difference and get this orthogonality because our PDE is linear ($f(u) = u$). Recall that

$$\begin{aligned} 0 &= B_j(u_h, v_h) \\ &= \frac{d}{dt}\int_{I_j} U(u_h)\, dx + \hat{F}_{j+1/2} - \hat{F}_{j-1/2} + H_{j-1/2}. \end{aligned}$$

Note that if we were to let $v_h = e$ (which is not correct) then

$$\begin{aligned} B_j(e, e) &= \frac{d}{dt}\int e^2 dx \\ &\leqslant 0, \end{aligned}$$

which is too good to be true. Let us try $v = \mathcal{P}_h u - u_h = e_h \in V_h$. Then,

$$B_j(e, e_h) = 0,$$

by the orthogonality relationship. Furthermore, letting $e = \varepsilon_h + e_h$,

$$B_j(\varepsilon_h + e_h, e_h) = 0$$

and so

$$B_j(e_h, e_h) = B_j(-\varepsilon_h, e_h).$$

We have

$$\text{LHS} = \frac{d}{dt}\int_{I_j} \frac{1}{2}(e_h)^2\, dx + \hat{F}_{j+1/2} - \hat{F}_{j-1/2} + H_{j-1/2},$$

where $H_{j-1/2} \geqslant 0$. For the other side,

$$\begin{aligned} \text{RHS} &= B_j(-\varepsilon_h, e_h) \\ &= \int_{I_j} (-\varepsilon_h)_t e_h dx - \int_{I_j} (-\varepsilon_h)(e_h)_x\, dx + (-\varepsilon_h)^-_{j+1/2}(e_h)^-_{j+1/2} - (-\varepsilon_h)^-_{j-1/2}(e_h)^+_{j-1/2} \\ &= \int_{I_j} (-\varepsilon_h)_t e_h dx - \int_{I_j} (-\varepsilon_h)(e_h)_x\, dx, \end{aligned}$$

since $(\varepsilon_h)^-_{j\pm 1/2} = 0$ by construction. Also, $\int_{I_j} (-\varepsilon_h)(e_h)_x\, dx = 0$ because $(e_h)_x \in \mathcal{P}^{k-1}$ and this vanishes also by construction. Hence,

$$\begin{aligned} \text{RHS} &= \int_{I_j} (-\varepsilon_h)_t\, e_h dx \\ &\leqslant \frac{1}{2}\int_{I_j} ((\varepsilon_h)_t)^2 dx + \frac{1}{2}\int_{I_j} (e_h)^2, \end{aligned}$$

which comes from the fact that $(a-b)^2 \geqslant 0 \Rightarrow \frac{1}{2}(a^2+b^2) \geqslant 2ab$. Thus,

$$\frac{d}{dt}\int_{I_j} \frac{1}{2}(e_h)^2\, dx + \hat{F}_{j+1/2} - \hat{F}_{j-1/2} + H_{j-1/2} \leqslant \frac{1}{2}\int_{I_j} ((\varepsilon_h)_t)^2 dx + \frac{1}{2}\int_{I_j} (e_h)^2.$$

If we sum over j, then

$$\begin{aligned} \frac{d}{dt}\int_a^b (e_h)^2 &\leqslant \int_a^b ((\varepsilon_h)_t)^2 dx \\ &\leqslant Ch^{2k+2} + \int_a^b (e_h)^2, \end{aligned}$$

because $\sqrt{\int_a^b ((\varepsilon_h)_t)^2 dx} \leqslant Ch^{k+1}$. Now we have an inequality

$$\frac{d}{dt}E \leqslant a + bE,$$

for $E > 0$, which is solved by letting $y = a + bE$ so that $\frac{d}{dt}y = b\frac{d}{dt}E$ which gives us $\frac{1}{b}\frac{d}{dt}y \leqslant y$. Then,

$$\begin{aligned} \frac{dy}{y} &\leqslant b \\ \Rightarrow y &\leqslant y(0)e^{bt}, \end{aligned}$$

which can be re-written as

$$(a+bE)(t) \leqslant (a+bE)(0)e^{bt}.$$

This is Gronwall's inequality. Then,

$$Ch^{2k+2} + \int_a^b (e_h(x,t))^2\, dx \leqslant e^t\left(Ch^{2k+2} + \int_a^b (e_h(x,t))^2\right)$$

and so

$$\begin{aligned} \|e_h(\cdot,T)\|^2_{L^2} &\leqslant Ch^{2k+2} + \hat{C}h^{2k+2} \\ &= ch^{2k+2}. \end{aligned}$$

Then,

$$\begin{aligned} \|e(\cdot,T)\|_{L^2} &\leqslant \|\varepsilon_h(\cdot,T)\|_{L^2} + \|e_h(\cdot,T)\|_{L^2} \\ &\leqslant ch^{k+1}, \end{aligned}$$

where c is dependent on T, but since we are dealing with finite time, this is alright.

□

HW6, due 11/13/2007:

Solve $u_t + u_x = 0$ with $u(x,0) = \sin x$ and $0 \leqslant x < 2\pi$, periodic boundary conditions to $T = 6$ using $\mathcal{P}^1$ DG method with second order TVD(SSP) Runge-Kutta time discretization:

$$\begin{aligned} u^{(1)} &= u^n + \Delta t L(u^n) \text{ (Euler Forward)} \\ u^{n+1} &= \frac{1}{2}\left[u^n + u^{(1)} + \Delta t L(u^{(1)})\right]. \end{aligned}$$

For the CFL, we take $\frac{\Delta t}{\Delta x} = .3$. Show the L^2 errors and numerical order of accuracy. Take the sequence of meshes to be $\frac{2\pi}{10}, \frac{2\pi}{20}, \ldots, \frac{2\pi}{320}$.

It is important to note that our results do not apply to say L^∞. If we have a non smooth initial condition, then the numerical solution is not necessarily smooth.

Now let us get back to nonlinear PDEs... Consider

$$u_t + f(u)_x \;=\; 0.$$

Unfortunately, this is the weakness of DG. The method for this type of PDE is to find $u_h \in V_h$ s.t.

$$\int_{I_j} (u_h)_t v_h dx - \int_{I_j} f(u_h)(v_h)_x\, dx + \hat{f}_{j+1/2}(v_h)^-_{j+1/2} - \hat{f}_{j-1/2}(v_h)^+_{j-1/2} \;=\; 0.$$

Discretizing $(u_h)_t$ with an Euler forward method gives us

$$\begin{aligned} & \int_{I_j} u_h^{n+1} v_h dx - \int_{I_j} u_h^n v_h dx \\ -\Delta t\Bigg[\int_{I_j} f(u_h)(v_h)_x\, dx - \hat{f}_{j+1/2}(v_h)^-_{j+1/2} + \hat{f}_{j-1/2}(v_h)^+_{j-1/2}\Bigg] \;&=\; 0. \end{aligned}$$

This scheme will blow up if we do not modify this.

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Recall the DG method for $u_t + f(u)_x = 0$. Then, we want to find $u_h^{n+1} \in V_h$ s.t.

$$\begin{aligned} \int_{I_j} (u_h^{n+1}) v_h dx \;&=\; \int_{I_j} (u_h^n) v_h dx \\ &\quad + \Delta t\Bigg\{\int_{I_j} f(u_h)(v_h)_x\, dx - \hat{f}_{j+1/2}(v_h)^-_{j+1/2} + \hat{f}_{j-1/2}(v_h)^+_{j-1/2}\Bigg\}, \forall v_h \in V_h. \end{aligned}$$

We showed that this scheme is not always stable. So we will try to find $u_h^{n+1,\text{tmp}} \in V_h$. Then, $u_h^{n+1} = \Pi u_h^{n+1,\text{tmp}}$, where Π is called the limiter. We will pay special attention to the following two quantities:

$$\begin{aligned} \tilde{u}_j \;&\doteq\; (u_h)^-_{j+1/2} - (\overline{u}_h)_j \\ \tilde{\tilde{u}}_j \;&\doteq\; (\overline{u}_h)_j - (u_h)^+_{j-1/2} \end{aligned}$$

where

$$\overline{u}_h \;\doteq\; \frac{1}{h_j}\int_{I_j} u_h(x,t)\, dx.$$

Now consider

$$\tilde{u}_j^{\text{mod}} \;=\; m(\tilde{u}_j, (\overline{u}_h)_{j+1} - (\overline{u}_h)_j, (\overline{u}_h)_j - (\overline{u}_h)_{j-1}),$$

where m is the so called min-mod function s.t.

$$m(a_1, \ldots, a_l) \;=\; \begin{cases} S \min(|a_1|, \ldots, |a_l|) & \text{if } \operatorname{sign}(a_1) = \cdots = \operatorname{sign}(a_l) = S \\ 0 & \text{otherwise} \end{cases}.$$

Similarly, we define

$$\tilde{\tilde{u}}_j^{\text{mod}} \;=\; m(\tilde{\tilde{u}}_j, (\overline{u}_h)_{j+1} - (\overline{u}_h)_j, (\overline{u}_h)_j - (\overline{u}_h)_{j-1}).$$

Then,

$$\begin{aligned}(u_h^{(\mathrm{mod})})_{j+1/2}^- &= (\overline{u}_h)_j + \tilde{u}_j^{(\mathrm{mod})}\\ (u_h^{(\mathrm{mod})})_{j-1/2}^+ &= (\overline{u}_h)_j - \tilde{\tilde{u}}_j^{(\mathrm{mod})}\end{aligned}$$

where the cell averages do not change. We have enough information for a $\mathcal{P}^2$ polynomial. Simialrly, if we have a polynomial of degree one we get a polynomial of degree one. But now if we have a polynomial of degree $\geqslant 3$ then we only have three pieces of information. Hence, there is not unique way to determine a limiter.

Theorem 2.1. *The total variation of the means of u_h does not increase (TVDM). That is,*

$$\mathrm{TV}(\overline{u}_h)^{n+1} \leqslant \mathrm{TV}(\overline{u}_h^n),$$

where

$$\mathrm{TV}(\overline{u}_h) \doteqdot \sum_j |(\overline{u}_h)_{j+1} - (\overline{u}_h)_j|,$$

which is in fact only a semi − norm.

Proof. If we take our test function to be

$$v_h = \begin{cases} 1, & x \in I_j \\ 0, & \text{otherwise} \end{cases} \in V_h,$$

then,

$$h_j(\overline{u}_h)_j^{n+1} = h_j(\overline{u}_h)_j^n + \Delta t\Big\{-\hat{f}_{j+1/2}^n + \hat{f}_{j-1/2}^n\Big\}.$$

This becomes

$$(\overline{u}_h)_j^{n+1} = (\overline{u}_h)_j^n + \frac{\Delta t}{h_j}\Big\{-\hat{f}((u_h^n)_{j+1/2}^-, (u_h^n)_{j+1/2}^+) + \hat{f}((u_h^n)_{j-1/2}^-, (u_h^n)_{j-1/2}^+)\Big\}.$$

We will try to write the second term on the RHS is

$$C_{j+1/2}[(\overline{u}_h)_{j+1} - (\overline{u}_h)_j] - D_{j-1/2}[(\overline{u}_h)_j - (\overline{u}_h)_{j-1}],$$

where $C_{j+1/2}, D_{j+1/2} \geqslant 0$ and $C_{j+1/2} + D_{j+1/2} \leqslant 1$. To show this, let us add and subtract terms to get

$$\begin{aligned}\frac{\Delta t}{h_j}\Big\{&-\hat{f}((u_h^n)_{j+1/2}^-, (u_h^n)_{j+1/2}^+) + \hat{f}((u_h^n)_{j+1/2}^-, (u_h^n)_{j-1/2}^+) - \hat{f}((u_h^n)_{j+1/2}^-, (u_h^n)_{j-1/2}^+)\\ &+ \hat{f}((u_h^n)_{j-1/2}^-, (u_h^n)_{j-1/2}^+)\Big\},\end{aligned}$$

where we added $\hat{f}((u_h^n)_{j+1/2}^-, (u_h^n)_{j-1/2}^+) - \hat{f}((u_h^n)_{j+1/2}^-, (u_h^n)_{j-1/2}^+) = 0$. Now we call,

$$\begin{aligned}C_{j+1/2} &= \frac{\Delta t}{h_j}\Big\{-\hat{f}((u_h^n)_{j+1/2}^-, (u_h^n)_{j+1/2}^+) + \hat{f}((u_h^n)_{j+1/2}^-, (u_h^n)_{j-1/2}^+)\Big\}\\ D_{j+1/2} &= \frac{\Delta t}{h_j}\Big\{\hat{f}((u_h^n)_{j+1/2}^-, (u_h^n)_{j-1/2}^+) - \hat{f}((u_h^n)_{j-1/2}^-, (u_h^n)_{j-1/2}^+)\Big\}.\end{aligned}$$

For $C_{j+1/2}$ let us assume that $\hat{f}$ is differentiable w.r.t. the second arguments

$$C_{j+1/2} = -\frac{\Delta t}{h_j}\frac{\hat{f}_2(\cdot,\xi)\left[(\overline{u}_h)_{j+1/2}^+ - (\overline{u}_h)_{j-1/2}^+\right]}{(\overline{u}_h)_{j+1}^n - (\overline{u}_h)_j^n}$$

where we used the MVT of the difference and $\hat{f}_2$ denotes the derivative. Now $-\frac{\Delta t}{h_j}\frac{\hat{f}_2(\cdot,\xi)}{1} \geqslant 0$ since $\hat{f}(\uparrow,\downarrow)$. What about the remaining fraction term?

$$\begin{aligned}\frac{\left[(\overline{u}_h)_{j+1/2}^+ - (\overline{u}_h)_{j-1/2}^+\right]}{(\overline{u}_h)_{j+1}^n - (\overline{u}_h)_j^n} &= \frac{(\overline{u}_h^n)_{j+1} - \tilde{\tilde{u}}_{j+1}^{(\mathrm{mod})} - (\overline{u}_h^n)_j + \tilde{\tilde{u}}_j^{(\mathrm{mod})}}{(\overline{u}_h)_{j+1}^n - (\overline{u}_h)_j^n}\\ &= 1 - \frac{\tilde{\tilde{u}}_{j+1}^{(\mathrm{mod})}}{(\overline{u}_h)_{j+1}^n - (\overline{u}_h)_j^n} + \frac{\tilde{\tilde{u}}_j^{(\mathrm{mod})}}{(\overline{u}_h)_{j+1}^n - (\overline{u}_h)_j^n}.\end{aligned}$$

We need this quantity to be positive. But $\frac{\tilde{u}_j^{(\mathrm{mod})}}{(\overline{u}_h)_{j+1}^n - (\overline{u}_h)_j^n} \geqslant 0$ by the properties of the minmod function. $\frac{\tilde{\tilde{u}}_{j+1}^{(\mathrm{mod})}}{(\overline{u}_h)_{j+1}^n - (\overline{u}_h)_j^n}$ is also positive AND it is at most 1 again by the properties of the minmod function. Moreover, $\frac{\left[(\overline{u}_h)_{j+1/2}^+ - (\overline{u}_h)_{j-1/2}^+\right]}{(\overline{u}_h)_{j+1}^n - (\overline{u}_h)_j^n} \in [0, 2]$. This works similarly for the $D_{j-1/2}$ and so if we call $\lambda = \Delta t / h_j$ then

$$\lambda(-\hat{f}_2 + \hat{f}_1) \leqslant \frac{1}{2}.$$

For example, if we have $f(u) = u$ and we choose a simple upwinding flux (i.e. Lax Friedricks flux) then $(-\hat{f}_2 + \hat{f}_1) = 1$ ($\hat{f}_1 = 1$ and $\hat{f}_2 = 0$ or vice versa if we did a downwind scheme) and we just need that $\lambda \leqslant \frac{1}{2}$.

So now we have that

$$(\overline{u}_h)_j^{n+1} = (\overline{u}_h)_j^n + \frac{\Delta t}{h_j}\Big\{-\hat{f}((u_h^n)_{j+1/2}^-, (u_h^n)_{j+1/2}^+) + \hat{f}((u_h^n)_{j-1/2}^-, (u_h^n)_{j-1/2}^+)\Big\}.$$

can be written as

$$(\overline{u}_h)_j^{n+1} = (\overline{u}_h)_j^n + C_{j+1/2}[(\overline{u}_h)_{j+1} - (\overline{u}_h)_j] - D_{j-1/2}[(\overline{u}_h)_j - (\overline{u}_h)_{j-1}]$$

where $C_{j+1/2}, D_{j+1/2} \geqslant 0$ and $C_{j+1/2} + D_{j+1/2} \leqslant 1$, by using the appropriate limiter $\tilde{u}$, $\tilde{\tilde{u}}$. We can rewrite this as

$$(\overline{u}_h)_j^{n+1} = (\overline{u}_h)_j^n + C_{j+1/2}\Delta_+(\overline{u}_h)_j - D_{j-1/2}\Delta_+(\overline{u}_h)_{j-1}$$

and then at $j+1$ we get the same equality and take the difference to get

$$\begin{aligned}
\Delta_+(\overline{u}_h)_j^{n+1} &= \Delta_+(\overline{u}_h)_j^n + C_{j+3/2}\Delta_+(\overline{u}_h)_{j+1} + C_{j+1/2}\Delta_+(\overline{u}_h)_j \\
&\quad - D_{j+1/2}\Delta_+(\overline{u}_h)_j + D_{j-1/2}\Delta_+(\overline{u}_h)_{j-1} \\
&= (1 - C_{j+1/2} - D_{j+1/2})\Delta_+(\overline{u}_h)_j + C_{j+3/2}\Delta_+(\overline{u}_h)_{j+1} + D_{j-1/2}\Delta_+(\overline{u}_h)_{j-1} \\
&\leqslant (1 - C_{j+1/2} - D_{j+1/2})|\Delta_+(\overline{u}_h)_j| + C_{j+3/2}|\Delta_+(\overline{u}_h)_{j+1}| + D_{j-1/2}|\Delta_+(\overline{u}_h)_{j-1}|,
\end{aligned}$$

since the coefficients are all positive. Now we sum over all j to get

$$\begin{aligned}
\mathrm{TV}(\overline{u}_h^{n+1}) &\leqslant \sum_j |\Delta_+(\overline{u}_h)_j| \\
&= \mathrm{TV}(\overline{u}_h^n).
\end{aligned}$$

(This latter part due to Harten's Lemma).

□

Now, recall that we have

$$\tilde{u}_j^{\mathrm{mod}} = m(\tilde{u}_j, (\overline{u}_h)_{j+1} - (\overline{u}_h)_j, (\overline{u}_h)_j - (\overline{u}_h)_{j-1}).$$

We need to show that $\tilde{u}_j^{\mathrm{mod}} = \tilde{u}_j$ when u is smooth. That is, we want to use a limiter to gain accuracy, but we want to use it in a limited manner. Using Taylor expansion, we get

$$\begin{aligned}
\tilde{u}_j &= u_{j+1/2} - \overline{u}_j \\
&= u_j + (u_x)_j\frac{h_j}{2} + \mathcal{O}(h_j^2) - (u_j + \mathcal{O}(h_j^2)) \\
&= (u_x)_j\frac{h_j}{2} + \mathcal{O}(h_j^2).
\end{aligned}$$

where $\overline{u}_j = \frac{1}{h_j}\int_{I_j} u(x)\,dx = \frac{1}{h_j}\int_{I_j} (u_j + (u_x)_j(x - x_j) + \mathcal{O}(h_j^2) = u_j + \mathcal{O}(h_j^2)$ where u_j is in the center of the interval. Similarly, we get

$$\begin{aligned}
(\overline{u}_h)_{j+1} - (\overline{u}_h)_j &= (u_x)_j h_j + \mathcal{O}(h_j^2) \\
(\overline{u}_h)_j - (\overline{u}_h)_{j-1} &= (u_x)_j h_j + \mathcal{O}(h_j^2)
\end{aligned}$$

and so

$$\tilde{u}_j^{\mathrm{mod}} = m(\frac{1}{2}(u_x)_j h_j + \mathcal{O}(h_j^2), (u_x)_j h_j + \mathcal{O}(h_j^2), (u_x)_j h_j + \mathcal{O}(h_j^2).$$

So this is true if u is monotone away from extrema.