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Midterm will be on Thursday, Oct. 25th in class. It will cover material to the end of chapter 9.

Last time we talked about the dG(0) method for $\left\{ \begin{smallmatrix} \dot{u} + au = f \\ u(0) = u_0 \end{smallmatrix} \right.$. We also said that if $|a(t)| \leqslant A$ then

$$|u(t_N) - U_N| \leqslant 3e^{2At_N}|k\dot{u}|_{[0,t_N]}$$

so it is first order w.r.t. the step size k. This is known as the a priori error estimate. if $a(t) \geqslant 0$ then

$$|u(t_N) - U_N| \leqslant 3|k\dot{u}|_{[0,t_N]}$$

We want to estimate $u - U$. We can rewrite this as

$$u - U = u - \pi_k u + \pi_k u - U$$

where $\pi_k u$ is the L^2 projection of u into $W^{(0)}$ and note that $\bar{e} = \pi_k u - U \in W^{(0)}$. We have that

$$|u - \pi_k u| \leqslant \frac{k_n}{2}|\dot{u}|_{I_N}$$

where again $|\varphi|_{I_n} = \max_{x \in I_n} |\varphi(x)|$ and we are just using Taylor expansion. We define the discrete dual problem as finding $\Phi \in W^{(0)}$ s.t.

$$\left\{ \begin{array}{l} \int_{t_{n-1}}^{t_n} (-\dot{\Phi} + a\Phi)v\,dt - [\Phi_n]v_n = 0, \quad \forall v \in W^{(0)} \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad n = N, N-1, \ldots, 1 \\ \Phi_N^+ = \Phi_{N+1} = (\pi_k u - U)_N \end{array} \right. \cdot$$

Take $v = \bar{e} \in W^{(0)}$. Now we get

$$\begin{aligned} 0 &= \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} (-\dot{\Phi} + a\Phi)\bar{e}\,dt - \sum_{n=1}^{N-1} [\Phi_n]\bar{e}_n - \Phi_N^+\bar{e}_N + \Phi_N^-\bar{e}_N \\ 0 &= \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} (-\dot{\Phi} + a\Phi)\bar{e}\,dt - \sum_{n=1}^{N-1} [\Phi_n]\bar{e}_n - \bar{e}_N^2 + \Phi_N^-\bar{e}_N \end{aligned}$$

which gives us

$$|\bar{e}_N|^2 = \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} (-\dot{\Phi} + a\Phi)\bar{e}\,dt - \sum_{n=1}^{N-1} [\Phi_n]\bar{e}_n + \Phi_N^-\bar{e}_N.$$

Let us determine the Galerkin orthogonality for dG(0). The scheme is given by

$$\int_{t_{n-1}}^{t_n} (\dot{U} + aU)v\,dt + [U_{n-1}]v_{n-1}^+ = \int_{t_{n-1}}^{t_n} fv\,dt, \forall v \in W^{(0)}$$

and of course the exact solution gives us

$$\int_{t_{n-1}}^{t_n} (\dot{u} + au)v\,dt = \int_{t_{n-1}}^{t_n} fv\,dt,$$

where $[u_{n-1}] = 0$ since $u \in C^1$ and there are no discontinuities. Moreover, the second equation works for a larger class of v's (that is why we can have two solutions to the scheme - because we are solving on two different spaces!). Now we take the difference and get

$$\int_{t_{n-1}}^{t_n} (\dot{e} + ae)v\,dt + [e_{n-1}]v_{n-1}^+ = 0, \forall v \in W^{(0)}$$

which is known as the error equation. Using integration by parts, we get

$$\begin{aligned} \int_{t_{n-1}}^{t_n} (-e\dot{v} + aev)\,dt + e_n^- v_n^- - e_{n-1}^+ v_{n-1}^+ + e_{n-1}^+ v_{n-1}^+ - e_{n-1}^- v_{n-1}^+ &= 0 \\ \int_{t_{n-1}}^{t_n} (-e\dot{v} + aev)\,dt + e_n^- v_n^- - e_{n-1}^- v_{n-1}^+ &= 0. \end{aligned}$$

Now if we sum these integrals, we get

$$\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}(-\dot{v}+a\,v)e\,dt-\sum_{n=1}^{N}[v_n]e_n^{-}+e_N^{-}v_N^{-} \;=\; 0.$$

where $e_0^{-}=0$ because of the initial condition. Recall that we can choose $v\in W^{(0)}$. So let us choose $v=\Phi\in W^{(0)}$, the solution of the dual problem. Then,

$$\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}(-\dot{\Phi}+a\,\Phi)e\,dt-\sum_{n=1}^{N}[\Phi_n]e_n^{-}+e_N^{-}\Phi_N^{-} \;=\; 0. \tag{1.1}$$

If we take $\bar{e}\;-e=\pi_k u-U-u+U=\pi_k u-u$. Now because of this orthogonality, we can take the equation

$$|\bar{e}_N|^2 \;=\; \sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}(-\dot{\Phi}+a\Phi)\bar{e}\,dt-\sum_{n=1}^{N-1}[\Phi_n]\bar{e}_n+\Phi_N^{-}\bar{e}_N$$

and add this to (2.1) which gives us

$$\begin{aligned}
|\bar{e}_N|^2 \;&=\; \sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}(-\dot{\Phi}+a\Phi)(\bar{e}\;-e)\,dt-\sum_{n=1}^{N-1}[\Phi_n](\bar{e}\;-e)_n+\Phi_N^{-}(\bar{e}\;-e)_N\\
&=\; \sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}(-\dot{\Phi}+a\Phi)(\pi_k u-u)\,dt-\sum_{n=1}^{N-1}[\Phi_n](\pi_k u-u)_n+\Phi_N^{-}(\pi_k u-u)_N.
\end{aligned}$$

We certainly can estimate $\pi_k u-u$ by Taylor expansion. Furthermore,

$$\text{RHS} \;=\; \sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}a\Phi(\pi_k u-u)\,dt-\sum_{n=1}^{N-1}[\Phi_n](\pi_k u-u)_n+\Phi_N^{-}(\pi_k u-u)_N$$

since Φ is piecewise constant so that $\dot{\Phi}=0$.

Lemma 1.1. *(stability of the dual problem) If* $|a(t)|\leqslant A, \forall t\in[0,\,t_N]$ *and* $k_j|a|_{I_j}\leqslant\frac{1}{2}$ *for* $j=1,\dots,N$ *then*

$$\begin{aligned}
|\Phi_n| \;&\leqslant\; e^{2A(t_N-t_{n-1})}|\bar{e}_N|,\\
\sum_{n=1}^{N-1}|[\Phi_n]| \;&\leqslant\; \frac{e}{2}e^{2At_N}|\bar{e}_N|,
\end{aligned}$$

and

$$\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}|a\Phi_n|\,dt \;\leqslant\; \frac{e}{2}e^{2At_N}|\bar{e}_N|.$$

Proof. Recall the scheme for Φ which incorporates finding $\Phi\in W^{(0)}$ s.t.

$$\begin{cases}\int_{t_{n-1}}^{t_n}(-\dot{\Phi}+a\Phi)v\,dt-[\Phi_n]v_n=0, & \forall v\in W^{(0)}\\ & n=N,N-1,\dots,1\;\cdot\\ \Phi_N^{+}=\Phi_{N+1}=(\pi_k u-U)_N & \end{cases}$$

where $\dot{\Phi}=0$, since it is piecewise constant, and $v=1$. This gives us

$$\begin{aligned}
-\Phi_{n+1}+\Phi_n+\Phi_n\int_{t_{n-1}}^{t_n}a(t)\,dt \;&=\; 0\\
\Phi_{N+1} \;&=\; \bar{e}_N.
\end{aligned}$$

Then,

$$\Phi_n \;=\; \frac{1}{1+\int_{t_{n-1}}^{t_n}a(t)\,dt}\Phi_{n+1}$$

and so we can get Φ_n, since we are working backwards. Thus,

$$\Phi_n = \prod_{j=n}^{N} \left(1 + \int_{t_{j-1}}^{t_j} a(t)\,dt\right)^{-1} \Phi_{N+1}.$$

If $|a(t)| \leqslant A$ then

$$\begin{aligned} \frac{1}{1 + \int_{t_{j-1}}^{t_j} a(t)\,dt} &\leqslant \frac{1}{1 - Ak_j} \\ &\leqslant e^{2Ak_j} \end{aligned}$$

because $\frac{1}{1-x} \leqslant e^{2x}$ if $0 \leqslant x \leqslant 1/2$. Hence,

$$\begin{aligned} |\Phi_n| &\leqslant \prod_{j=n}^{N} e^{2Ak_j} |\bar{e}_N| \\ &= e^{2A(t_N - t_{n-1})} |\bar{e}_N|. \end{aligned}$$

As for the jumps,

$$\begin{aligned} |[\Phi_n]| &= \left|\Phi_n \int_{t_{n-1}}^{t_n} a(t)\,dt\right| \\ &\leqslant e^{2A(t_N - t_{n-1})} |\bar{e}_N| A k_n. \end{aligned}$$

Now if we sum these errors, we get

$$\begin{aligned} \sum_{n=1}^{N-1} |[\Phi_n]| &\leqslant A e^{2At_N} |\bar{e}_N| \sum_{n=1}^{N-1} e^{-2At_{n-1}} k_n \\ &\leqslant A e^{2At_N} |\bar{e}_N| e \sum_{n=1}^{N-1} e^{-2At_n} k_n (^{**}) \\ &\leqslant A |\bar{e}_N| e^{2At_N} e \int_0^{t_N} e^{-2At} dt \\ &= \frac{1}{2} |\bar{e}_N| e^{2At_N + 1} (1 - e^{-2At_N}). \end{aligned}$$

(** $e^{2Ak_n} \leqslant e$ since $Ak_n \leqslant \frac{1}{2}$).

Now if $a(t)$ is positive, then

$$\frac{1}{1 + \int_{t_{j-1}}^{t_j} a(t)\,dt} \leqslant 1$$

and

$$\begin{aligned} \Phi_n &= \frac{1}{1 + \int_{t_{n-1}}^{t_n} a(t)\,dt} \Phi_{n+1} \\ &\leqslant \Phi_{n+1} \end{aligned}$$

and so we get that

$$|\Phi_n| \leqslant |\bar{e}_N|$$

and

$$\begin{aligned} \sum_{n=1}^{N-1} |[\Phi_n]| &= \sum_{n=1}^{N-1} |\Phi_{n+1} - \Phi_n| \\ &= \sum_{n=1}^{N-1} \Phi_{n+1} - \Phi_n \\ &= \Phi_N - \Phi_1 \\ &\leqslant \Phi_N \\ &\leqslant |\bar{e}_N|. \end{aligned}$$

□

So now combining the above Lemma with our error $\bar{e}_N$, we get

$$\begin{aligned}
|\bar{e}_N|^2 &= \sum_{n=1}^{N}\int_{t_{n-1}}^{t_n} a\Phi(\pi_k u-u)\,dt-\sum_{n=1}^{N-1}[\Phi_n](\pi_k u-u)_n+\Phi_N^-(\pi_k u-u)_N\\
&\leqslant \sum_{n=1}^{N}\frac{1}{2}k_n|\dot{u}|_{I_n}\int_{t_{n-1}}^{t_n}|a\Phi|\,dt+\sum_{n=1}^{N-1}\frac{1}{2}k_n|\dot{u}|_{I_n}|[\Phi_n]|+|\Phi_N|\frac{1}{2}k_n|\dot{u}|_{I_n}\\
&\leqslant \frac{1}{2}|k_n\dot{u}|_{[0,t_N]}\left(\begin{cases}\frac{e}{2}e^{2At_N}+\frac{e}{2}e^{2At_N}+e^{2\mathrm{At}_N}+1, & \text{if } |a(t)|\leqslant A\\ 1+1+1+1, & \text{if } a(t)\geqslant 0)\end{cases}\right)|\bar{e}_N|.
\end{aligned}$$

Simplifying further, we get

$$|\bar{e}_N| \leqslant \frac{1}{2}|k_n\dot{u}|_{[0,t_N]}\left(\begin{cases}(e+2)e^{2At_N}, & \text{if } |a(t)|\leqslant A\\ 4, & \text{if } a(t)\geqslant 0)\end{cases}\right).$$

Also, remember $|\pi_k u-u|\leqslant\frac{1}{2}|k_n\dot{u}|_{[0,t_N]}$. Hence,

$$|e_N| \leqslant \left(\begin{cases}\frac{e+3}{2}e^{2At_N}, & |a(t)|\leqslant A\\ \frac{5}{2}, & a(t)\geqslant 0\end{cases}\right)|k\dot{u}|_{[0,t_N]}.$$

In summary, we are trying to estimate our solution U within $W^{(0)}$. To do this, we use the triagnle inequality between U and $\pi_k u$ and $\pi_k u$ and u, the true solution. We know $|\pi_k u-u|$ by a simple Taylor expansion since $\pi_k u$ is piecewise constant. To calculate $|U-\pi_k u|$ we evaluated $\bar{e}$. Also, we use the fact that $U-\pi_k u\in W^{(0)}$.

Note that the dual problem is a convenient tool for time dependent problems. It is not necessary, however.

Chapter 10: System of equations:

$$\begin{cases}\dot{u}(t)+A(t)u(t)=f(t) & 0\leqslant t\leqslant T\\ u(0)=u_0\end{cases}$$

where A is a matrix and u, f are vectors. Then $W^{(q)}$, $V^{(q)}$ are defined as before as piecewise polynomials of degree q where W can be discontinuous and V is continuous.

dG(q): find $U\in W^{(q)}$ s.t.

$$\sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}(\dot{U}+AU,v)\,dt+\sum_{n=1}^{N}([U_{n-1}],v_{n-1}^+) = \sum_{n=1}^{N}\int_{t_{n-1}}^{t_n}(f,v)\,dt, \forall v\in W^{(q)}.$$

In particular, dG(0) is just

$$U_n+\int_{t_{n-1}}^{t_n}AU_n dt = U_{n-1}+\int_{t_{n-1}}^{t_n}f dt, U_n=U_n^-, U_0=u_0.$$

The error estimates are the same as before.

Theorem 1.2. *If* $\|A(t)\|\leqslant A, k_n\|A\|_{I_n}\leqslant\frac{1}{2}$ *then dG(0):*

$$|u(t_N)-U_N| \leqslant \frac{e}{2}\big(e^{2At_N}-1\big)\max_{1\leqslant n\leqslant N}k_n|\dot{u}|_{I_n}$$

and the dual problem

$$\begin{cases}-\dot{\varphi}+A^T\varphi=0\\ \varphi(t_N)=e_N\end{cases}.$$

Chapter 13:

Recall the Divergence theorem:

$$\int_\Omega \nabla \cdot v dx = \int_{\partial\Omega} (v \cdot n) d s$$

Green's formula:

$$\int_\Omega \nabla u \cdot \nabla w dx = \int_{\partial\Omega} v \partial_n w ds - \int_\Omega v \Delta w dx.$$

Another variant of Green's formula is

$$\int_\Omega (v\Delta w - w\Delta v) dx = \int_{\partial\Omega} (v\partial_n w - w\partial_n v) ds.$$

Chapter 14:

Consider our traingulation $\mathcal{T}_h$ where $K \in \mathcal{T}_h$ is one element in the traingulation. Call h_k the size of K which is the longest side of K. Also, τ_k is the smallest angle of K and ρ_k is the diameter of the largest inscribed circle in K. We require the following:

$$\min \tau_k \geqslant \tau > 0$$

or

$$\max \frac{h_k}{\rho_k} \leqslant \alpha$$

where a sequence of meshes must satisfy the above properties. If a sequence of meshes do satisfy these two properties, then it is called a *regular mesh.*

Let us definie our finite element space:

$$V_h = \{v : v \text{ continuous in } \Omega, v|_K \in P^1(K), \forall K \in \mathcal{T}_h\}.$$

How do we choose the basis functions?

1. Given the values of v at three nodes, a^1, a^2, and a^3 then there is a unique linear function provided that these three points are not on the same line.
2. If v and w agree on a^1 and a^2 then they agree on the edge a^1a^2. Hence, they are continuous on this edge.

HW #4:
p.209 - 9.9
p.212 - 9.11, 9.12
p.214 - 9.17
p.231 - 9.36, 9.37, 9.39
p.234 - 9.43

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Chapter 14: 2D FEM

Recall that

$$V_h = \{v : v \text{ continuous in } \Omega, v|_K \in P^1(K), \forall K \in \mathcal{T}_h\},$$

with two facts:

1. Given the values of v at three nodes, a^1, a^2, and a^3 then there is a unique linear function provided that these three points are not on the same line.

2. If v and w agree on a^1 and a^2 then they agree on the edge a^1a^2. Hence, they are continuous on this edge.

So this allows us to use the following basis functions:

$$\lambda_i \in P^1(K) \quad i = 1, 2, 3$$

s.t.

$$\lambda_i(a^j) = \delta_{ij}$$

(i.e. zero on one vertice of the triangle and zero on the other two vertices). Consider $\{\varphi_j\}_{j=1}^M$ where $N_1, \ldots, N_m$ are the nodes $N \in \mathcal{N}_h$, where $\mathcal{N}_h$ is the collection of nodes of $\mathcal{T}_h$. Thus,

$$v(x) = \sum_{i=1}^{M} v(N_i)\varphi_i(x)$$

where $v(N_i)$ is the value of the function. What if we have

$$V_h = \{v : v \text{ continuous in } \Omega, v|_K \in P^2(K), \forall K \in \mathcal{T}_h\}.$$

Then we need 6 basis functions: $1, x, y, xy, x^2, y^2$. We can use the vertices and the midpoints of the edges of the triangle as our basis points. Does this give us existence and uniqueness of a quadratic polynomial with the 6 given values. Firstly, for linear problems, existence and uniqueness for 0 are equivalent (e.g. $Ax = b$ has unique solution iff $Ax = 0$ for $x = 0$). If $v(a^i) = 0$, $v(a^{ij}) = 0$ along a^2a^3: v is a one dimensional polynomials of degree $\leqslant 2$ and it vanishes at 3 points a^2, a^{23}, a^3. This implies that v vanishes on a^2a^3. Then, $v(x) = \lambda_1(x)v_1(x)$, which is obtained through polynomial division (e.g. if $x = 2$ is a root to some polynomial then divide the polynomial by $(x - 2)$ to get the remaining polynomial of one degree lower). Furthermore, we can factor out another polynomial to get $v(x) = \lambda_1(x)\lambda_2(x)\lambda_3(x)v_3(x)$ but then $v_3(x)$ must be zero.

Consider the basis given by

$$\begin{aligned}
\psi_1(x) &= \lambda_1(2\lambda_1 - 1)\\
\psi_2(x) &= \lambda_2(2\lambda_2 - 1)\\
\psi_3(x) &= \lambda_3(2\lambda_3 - 1)\\
\psi_4(x) &= 4\lambda_1\lambda_2\\
\psi_5(x) &= 4\lambda_1\lambda_3\\
\psi_6(x) &= 4\lambda_2\lambda_3.
\end{aligned}$$

since $\lambda_1(a^{12}) = \lambda_2(a^{13}) = \frac{1}{2}$.

If one want C^1 functions, we can use $P^5(K)$, piecewise polynomials of degree 5. If two polynomials of degree $\leqslant 5$ agree at three points between first, second and 0 derivatives, then the two polynomials are equal. What about C^1? We need

$$\begin{aligned}
\frac{\partial P_1(x, y)}{\partial x} &= \frac{\partial P_2(x, y)}{\partial x}\\
\frac{\partial P_1(x, y)}{\partial y} &= \frac{\partial P_2(x, y)}{\partial y}
\end{aligned}$$

But this boils down to solving degree $\leqslant 4$ where we know $P_1(0) = P_2(0)$, $P_1'(0) = P_2'(0)$ and $P_1'(\frac{1}{2}) = P_2'(\frac{1}{2})$ and same with $x = 1$.

Approximation results: If $\pi_h v \in P^1(K)$ where $\pi_h v(a^i) = v(a^i), i = 1, 2, 3$ then

Theorem 2.1.

$$\|v - \pi_h v\|_{\infty,K} = \frac{1}{2}h_k^2\|D^2v\|_{\infty,K}$$

and

$$\|\nabla v - \nabla\pi_h v\|_{\infty,K} = \frac{3}{\sin(\tau_k)}\|D^2v\|_{\infty,K}$$

where $\|\cdot\|_{\infty,K}$ *is the* L^∞ *norm over the domain* K*.*

(Recall that τ_k is the smallest angle of K).