1 10/9/2007

Makeup classes:

Tuesdays - 10/9, 10/16, 10/23, 10/30, 11/6, 11/13, 5-5:50 pm B&H 155 (there will be no classes on 11/20, 11/27, and 11/29.

Chapter 9:

Consider

$$\begin{cases} \dot{u}(t)+a(t)u(t)=f(t) & 0<t\leqslant T \\ u(0)=u_0 \end{cases}$$

where the exact soluton is

$$u(t)=e^{-A(t)}u_0+\int_0^t e^{-(A(t)-A(s))}f(s)ds$$

where $A'(t)=a(t), A(0)=0 (A=\int_0^t a(s)ds)$. To find the solution, consider

$$\dot{u}+au \;=\; 0.$$

The solution to this homogeneous ODE is $u_0e^{-A(t)}$. We can use $e^{A(t)}$ as an the integrating factor since

$$\frac{d}{dt}(e^{A(t)}u) \;=\; e^{A(t)}(\dot{u}+au).$$

Thus,

$$\frac{d}{dt}(e^{A(t)}u) \;=\; e^{A(t)}f$$

and integrating

$$\begin{aligned} \int_0^s \frac{d}{dt}(e^{A(t)}u)dt &= \int_0^s e^{A(t)}fdt \\ &\Rightarrow \\ e^{A(s)}u(s) &= \int_0^s e^{A(t)}fdt+u_0. \end{aligned}$$

Finally, switching the s and t, since they are dummy variables, we get

$$u(t) \;=\; u_0e^{-A(t)}+\int_0^t e^{-(A(t)-A(s))}f(s)ds,$$

where $u_0e^{-A(t)}$ was the homogeneous solution.

1. If $a(t)\geqslant\alpha>0$ (parabolic case) then

$$|u(t)| \;\leqslant\; e^{-\alpha t}|u_0|+\frac{1}{\alpha}(1-e^{-\alpha t})\max_{0\leqslant\delta\leqslant t}|f(s)|.$$

2. If $a(t)\geqslant 0$ then

$$|u(t)| \;\leqslant\; |u_0|+\int_0^t |f(s)|ds.$$

Where do these estimates come from? If $a(t)\geqslant a>0$ then using a Taylor expansion we have that $A(t)=A(0)+tA(\xi)\geqslant t\alpha$. Also, if we Taylor expand $g(s)=A(s)-A(t)$ about t then we get $g(s)=g(t)+(s-t)g'(\xi)=(s-t)A'(t)$. Since $s\leqslant t$ then $(s-t)A'(t)\leqslant(s-t)\alpha$ where $A'(t)>0$. Then,

$$\begin{aligned} u_0e^{-A(t)}+\int_0^t e^{A(s)-A(t)}f(s)ds &\leqslant |u_0|e^{-\alpha t}+\int_0^t e^{(s-t)\alpha}f(s)ds \\ &\leqslant |u_0|e^{-\alpha t}+\max_{0\leqslant s\leqslant t}|f(s)|e^{-\alpha t}\int_0^t e^{\alpha s}ds \\ &= |u_0|e^{-\alpha t}+\frac{1}{\alpha}\max_{0\leqslant s\leqslant t}|f(s)|e^{-\alpha t}(e^{\alpha t}-1) \\ &= |u_0|e^{-\alpha t}+\frac{1}{\alpha}(1-e^{-\alpha t})\max_{0\leqslant s\leqslant t}|f(s)|. \end{aligned}$$

Now if we have that $a(t) \geqslant 0$ then $-A(t) \leqslant 0$ and $(s-t)A'(t) \leqslant 0$ which gives us the second inequality.

The Galerkin FEM

cG(q) (continuous P^q, with discontinuous P^{q-1} as test functions). It is perfectly valid to use continuous P^{q-1} test functions, but the amount of work involved is significant as compared with only using the discontinuous kind. Also, we have solved ODE's where the test space is the same space as the trial space. This is merely a convention and using a different test space will make for easier error analysis.

We also have dG(q) (discontinuous P^q, with discontinuous P^q as test functions).

cG(q): To find $U \in P^q(0,T)$ with $U(0) = u_0$ given, s.t.

$$\int_0^T (\dot{U} + aU)v dt = \int_0^T f v dt, \forall v \in P^{q-1}(0,T).$$

(we loose one degree of freedom since we are already given that $U(0) = u_0$ so that testing with only P^{q-1} will still give us a system with N unknowns and N equations). For example, let $q = 1$, then the only test function is $v = 1$. So we are looking for a solution $U = u_0 + \xi t$ to solve the above ODE. If we write U in terms of $U(T)$ we get that $U(t) = u_0 + \frac{U(T) - u_0}{T} t$. Hence, we get

$$U(T) - U(0) + \int_0^T a\Big(U(T)\frac{t}{T} + u_0\frac{T-t}{T}\Big)dt = \int_0^T f dt$$

and we can solve for $U(T)$. Now if we want to solve on the next partition, we let $U(T)$ be the new u_0 in order to generate a continuous function U. In general, for cG(q), $q > 1$, if u is already computed on $(t_{n-2}, t_{n-1}]$ and letting U_{n-1} denote $U(t_{n-1})$, then we compute $U \in P^q(t_{n-1}, t_n)$ that satisfies $U(t_{n-1}) = U_{n-1}$ and

$$\int_{t_{n-1}}^{t_n} (\dot{U} + aU)v dt = \int_{t_{n-1}}^{t_n} f v dt, \forall v \in P^{q-1}(t_{n-1}, t_n).$$

We can write this method globally as : find $U \in V^{(q)}$(space consisting of continuous, piecewise polynomials of deg $\leqslant q$), where $U(0) = u_0$ s.t.

$$\int_0^T (\dot{U} + aU)v dt = \int_0^T f v dt, \forall v \in W^{q-1}$$

where W^{q-1} is a space consisting of discontinuous, piecwise polynomials of degree $\leqslant q-1$. Note that when we move on to the different partitions, (t_n, t_{n+1}) we have to *shift* or *transform* our basis functions so that they form a basis on that particular interval. For cG(1), the test space just consists of $v = 1$ so there is not need to make a change of basis. However, if we were computing cG(2), then the test space would consist of $v = 1, t$. Then $t \rightarrow t - t_n$ for all n.

dG(q): Find $U \in P^q(0,T)$ s.t.

$$\int_0^T (\mathring{U} + aU)v dt + (U(0) - u_0)v(0) = \int_0^T f v dt, \forall v \in P^q(0,T).$$

For the continuous finite element method, the term $(U(0) - u_0)v(0)$ will vanish, but for the discontinuous Galerking approach, the term does not vanish. Where exactly does this term come from? Recall that the variational problem w.r.t to the original ODE is

$$\int_0^T (\dot{U} + aU)v dt = \int_0^T f v dt, \forall v \in V_h,$$

whatever the test space might be. Using integration by parts on the first term of the LHS, we get

$$\begin{aligned}\int_0^T \dot{U} v dt &= -\int_0^T U\dot{v} dt + Uv|_0^T \\ &= -\int_0^T U\dot{v} dt + U(T^-)v(T^-) - U(0^+)v(0^+)\end{aligned} \tag{1.1}$$

where we are careful here not to use $U(T)$ or $U(0)$ since the function is discontinuous at these points. We call the values of U at the boundary, denoted by $\hat{U}$, the single valued flux at a discontinuity point. We have two values at each of the boundary points in the discontinuous Galerking approach. It is reasonably expect that $U(0^+) = U(0^-)$ since we use previous time information to predict the future time. Then,

$$\int_0^T \mathring{U} v dt = -\int_0^T U\mathring{v} dt + U(T^-)v(T)^- - U(0^-)v(0)^+ \tag{1.2}$$

Plugging in (1.1) into (1.2) we get ,

$$\begin{aligned}\int_0^T \mathring{U} v dt &= \left[\int_0^T \mathring{U} v dt - U(T^-)v(T^-) + U(0^+)v(0^+)\right] + U(T^-)v(T^-) - U(0^-)v(0)^+ \\ &= \int_0^T \mathring{U} v dt + (U(0^+) - U(0^-))v(0^+) \\ &= \end{aligned}$$

which of course is true since we say that it reasonable to assumme that $U(0^+) = U(0^-)$. What if we say that $U(0^-) = u_0$ so that

$$\int_0^T \dot{U} v dt \approx \int_0^T \dot{U} v dt + (U(0^+) - u_0)v(0^+)$$

where $(U(0^+) - u_0)v(0^+)$ is the penalty term, written as $(U(0) - u_0)v(0)$ (note that $\approx$ becomes $=$ if U is continuous). So now instead of strictly enforcing continuity at the boundaries, we are letting the variational formula mitigate the boundary conditions in such a way that the difference in boundary points should be small (orthogonal to the test space).

In general, the dG(q) method becomes: for $n = 1, \ldots, N$ we find $U \in P^q(t_{n-1}, t_n)$ s.t.

$$\int_{t_{n-1}}^{t_n} (\mathring{U} + aU) v dt + U_{n-1}^+ v_{n-1}^+ = \int_{t_{n-1}}^{t_n} f v dt + U_{n-1}^- v_{n-1}^+, \forall v \in P^q(t_{n-1}, t_n).$$

or globally as: find $U \in W^{(q)}$ (the set of discontinuous, piecewise polynomials of degree at most q on $[0, T]$ with a given mesh) s.t.

$$\sum_{n=1}^N \int_{t_{n-1}}^{t_n} (\mathring{U} + aU) v dt + \sum_{n=1}^N [U_{n-1}] v_{n-1}^+ = \int_0^{t_N} f v dt, \forall v \in W^{(q)}.$$

Note that $\sum_{n=1}^N \int_{t_{n-1}}^{t_n} (\mathring{U} + aU) v dt \neq \int_0^{t_N} (\mathring{U} + aU) v dt$ since $\mathring{U}$ is discontinuous at the element boundaries where $[U_0] = U_0^+ - U_0^- = U_0^+ - u_0$ and so on for $[U_n]$.

Error Analysis:

cG(1): Let us first consider an a posteriori error estimate. The motivation is the following: Start with

$$\int_0^T (\dot{u} + au) v dt = \int_0^T f v dt \tag{1.3}$$

which is clearly true for the true solution u. Then, integration by parts gives us

$$\int_0^T \dot{u} v = u(T)v(T) - u(0)v(0) - \int_0^T u\dot{v} dt.$$

Plugging this into (1.3) we get

$$u(T)v(T) - u(0)v(0) + \int_0^T u(-\dot{v} + av) dt = \int_0^T f v dt. \tag{1.4}$$

Now we choose v s.t. $-\mathring{v}+av=0$. If we choose such a v, then the above equality simplifies to

$$u(T)v(T) \;=\; u(0)v(0)+\int_0^T fv dt.$$

Finding such a v is called the dual problem. That is, knowing that u satisfies (1.4) for all v and finding a v with some nice properties, gives us information about the final value $u(T)$ in terms of the the initial value and f. More formally, the dual problem as finding a φ s.t.

$$\begin{cases} -\dot{\varphi}+a\varphi=0 & \text{for}\, t_N>t\geqslant 0 \\ \varphi(t_N)=e_N \end{cases},$$

where $e_N=u(t_N)-U_N$, whic is the error we are trying to estimate. Letting $e=u(t)-U(t)$, then

$$\begin{aligned} e_N^2 &= e_N^2+\int_0^{t_N} e(-\dot{\varphi}+a\varphi)dt. \\ &= e_N^2-e\varphi|_0^{t_N}+\int_0^{t_N}\dot{e}\varphi dt+\int_0^{t_N} ea\varphi dt \\ &= e_N^2-e_N\varphi_N+\int_0^{t_N}(\dot{e}+ae)\varphi dt \\ &= e_N^2-e_N^2+\int_0^{t_N}(\dot{e}+ae)\varphi dt \\ &= \int_0^{t_N}(\dot{e}+ae)\varphi dt \end{aligned}$$

Using integration by parts with $e(0)=0$ and the fact that $\varphi_N=e_N$. Then,

$$\begin{aligned} e_N^2 &= \int_0^{t_N}(\dot{e}+ae)\varphi dt \\ &= \int_0^{t_N}(\dot{u}-\dot{U}+au-aU)\varphi dt \\ &= \int_0^{t_N}(\dot{u}+au-\dot{U}-aU)\varphi dt \\ &= \int_0^{t_N}(f-\dot{U}-aU)\varphi dt \\ &= -\int_0^{t_N} r(U)\varphi dt \end{aligned}$$

where $r(U)=\dot{U}+aU-f$. So we have the error in terms of the residual error and the dual problem.

Recall that the Galerking method is formulated so that the residual is orthogonal to any test function in W^{q-1} (this is merely the definition of the Galerking method meaning that this isn't magic. This is how U is defined. U is obtained by setting the residual orthogonal to the test space). That is,

$$-\int_0^{t_N} r(U)\varphi dt=-\int_0^{t_N} r(U)(\varphi-\hat{\varphi})dt, \hat{\varphi}\in W^{q-1}.$$

For example, we can take $\hat{\varphi}=\pi_h\varphi$ where $\pi_h\varphi$ is the L^2 projection of φ into $W^{(0)}$. Then, if we denote $|v|_I=\max_{t\in I}|v(t)|$, we obtain

$$\begin{aligned} -\int_0^{t_N} r(U)(\varphi-\pi_h\varphi)dt &\leqslant \sum_{n=1}^N |r(U)|_{I_n}\int_{I_N}|\varphi-\pi_h\varphi|dt \\ &\leqslant \sum_{n=1}^N |r(U)|_{I_n}k_n\int_{I_n}|\mathring{\varphi}|dt \\ &\leqslant \max_{1\leqslant n\leqslant N}(k_n|r(U)|_{I_n})\int_0^{t_N}|\mathring{\varphi}|dt \\ &\leqslant s(t_N)(|e_N|)(|kr(U)|_{[0,t_n]}) \end{aligned}$$

where

$$s(t_N) \;=\; \frac{\int_0^{t_N} |\dot{\varphi}|\,dt}{|e_N|}$$

is called the *stability factor*. Combining the above equations, we get

$$\begin{aligned} e_N^2 &\leqslant s(t_N)(|e_N|)(|kr(U)|_{[0,t_n]}) \\ \Rightarrow |e_N| &\leqslant s(t_N)(|kr(U)|_{[0,t_n]}) \end{aligned}$$

Lemma 1.1. *(1) If* $|a(t)| \leqslant \alpha$ *for* $0 < t < t_N$ *and* φ *is the solution to the dual problem, then*

$$|\varphi(t)| \;\leqslant\; e^{\alpha t_N}|e_N|$$

and

$$s(t_N) \;\leqslant\; e^{\alpha t_N}.$$

(2) If $a(t) \geqslant 0$ *for all* t*, then* $|\varphi(t)| \leqslant e_N$ *and* $s(t_N) \leqslant 1$*.*

Proof. φ is defined as the solution to

$$\begin{cases} -\dot{\varphi} + a\varphi = 0 & \text{for}\, t_N > t \geqslant 0 \\ \varphi(t_N) = e_N \end{cases}.$$

Letting $A'(t) = a(t)$ and $A(t_N) = 0$ we get that $\varphi(t) = e_N e^{A(t)}$. Taylor expanding about t_N gives us that $A(t) = (t - t_N)a(\eta(\xi)) \leqslant |t - t_N||a(t)| \leqslant \alpha t_N$ which gives us

$$|\varphi(t)| \;\leqslant\; e^{\alpha t_N}|e_N|.$$

If $a(t) \geqslant 0$ then $(t - t_N)a(\eta(\xi)) \leqslant 0$ since $t < t_N$ so that

$$|\varphi(t)| \;\leqslant\; |e_N|.$$

For $|a(t)| \leqslant \alpha$ we have e

$$\begin{aligned} s(t_N) &= \frac{\int_0^{t_N} |\dot{\varphi}|\,dt}{|e_N|} \\ &= \frac{\int_0^{t_N} |a\varphi|\,dt}{|e_N|} \\ &= \frac{\int_0^{t_N} |a(t)|\,e^{A(t)}|e_N|\,dt}{|e_N|} \\ &= \int_0^{t_N} |a(t)|\,e^{(t-t_N)a(\xi(t))}\,dt, \end{aligned}$$

where $t \leqslant \xi(t) \leqslant t_N$ and we used a Taylor expansion of $A(t)$ about t_N. Then,

$$\begin{aligned} \int_0^{t_N} |a(t)|\,e^{(t-t_N)a(\xi(t))}\,dt &\leqslant \int_0^{t_N} \alpha e^{\alpha(t_N - t)}\,dt \\ &= e^{\alpha t_N}\int_0^{t_N} \alpha e^{-\alpha t}\,dt \\ &\leqslant e^{\alpha t_N}. \end{aligned}$$

since $\int_0^{t_N} \alpha e^{-\alpha t}\,dt \leqslant 1$. If $a(t) \geqslant 0$ then this simplifies even further to

$$\begin{aligned} s(t_N) &= \frac{\int_0^{t_N} |a\varphi|\,dt}{|e_N|} \\ &= \frac{\int_0^{t_N} a(t)\,e^{A(t)}|e_N|\,dt}{|e_N|} \\ &= \int_0^{t_N} a(t)\,e^{A(t)}\,dt \\ &= e^{A(t_N)} - e^{A(0)} \\ &= 1 - e^{A(0)} \\ &\leqslant 1. \end{aligned}$$

□

2 10/11/07

Last time we provided a posteriori error estimates for cG(1) applied to

$$\left\{\begin{array}{l}\dot{u}(t)+a(t)u(t)=f(t) \quad 0<t\leqslant T\\ u(0)=u_0\end{array}\right. .$$

Then,

$$|u(t_N)-U_N| \;\leqslant\; s(t_N)|kr(U)|_{[0,t_N]}$$

where

$$s(t_N) \;\leqslant\; \left\{\begin{array}{ll}e^{a t_N} & \text{if } |a(t)|\leqslant a\\ 1 & \text{if } a(t)\geqslant 0\end{array}\right. .$$

Let us do the same thing for dG(0). Let us recall the dG(0) method for the same ODE:

$$\int_0^{t_1} (\dot{U}+aU)v\,dt+(U(0)^+-u_0)v(0)^+ \;=\; \int_0^{t_1} fv\,dt$$

where $U(0)^-=u_0$. Now since this is dG(0) we have that $\dot{U}=0$ on each interval and $v=1$. This gives us

$$\begin{aligned}\int_0^{t_1} aU\,dt+U(0)^+ &= \int_0^{t_1} f\,dt+U(0)^-\\ \Rightarrow (A(t)|_0^{t_1})U(t_1)^-+U(t_1)^- &= \int_0^{t_1} f\,dt+U(0)^-\end{aligned}$$

where U is constant within $t\in(0,t_1)$ so that we can replace U on $(0,t_1)$ with $U(0)^+$ or $U(t_1)^-$. Now we can solve for $U(t_1)^-$. Then, the next step gives us

$$\begin{aligned}\int_{t_1}^{t_2} aU\,dt+U(t_1)^+ &= \int_{t_1}^{t_2} f\,dt+U(t_1)^-\\ \Rightarrow (A(t)|_{t_1}^{t_2})U(t_2)^-+U(t_2)^- &= \int_{t_1}^{t_2} f\,dt+U(t_1)^-.\end{aligned}$$

Now, calling $U_n=U(t_n)^-$ we get that the n^{th} step gives us

$$\begin{aligned}\int_{t_{n-1}}^{t_n} aU\,dt+U(t_{n-1})^+ &= \int_{t_{n-1}}^{t_n} f\,dt+U(t_{n-1})^-\\ \Rightarrow (A(t)|_{t_1}^{t_2})U(t_n)^-+U(t_n)^- &= \int_{t_{n-1}}^{t_n} f\,dt+U(t_{n-1})^-\\ \Rightarrow (A(t)|_{t_1}^{t_2})U_n+U_n &= \int_{t_{n-1}}^{t_n} f\,dt+U_{n-1}.\end{aligned}$$

As before, let us define the dual problem as

$$\left\{\begin{array}{l}-\dot{\varphi}+a\varphi=0\\ \varphi(t_N)=e_N^-\end{array}\right.$$

where $e_n=u_n-U_n$ (we will define $e(t_n)=u(t_n)-U(t_n)^-$ so that $e_n=e_n^-$) and φ is continuous. We have

$$\begin{aligned}e_N^2 &= e_N^2+\int_0^{t_N} e(-\dot{\varphi}+a\varphi)\,dt\\ &= e_N^2+\sum_{n=1}^N \int_{t_{n-1}}^{t_n} e(-\dot{\varphi}+a\varphi)\,dt\\ &= e_N^2+\sum_{n=1}^N \left(-\varphi e|_{t_{n-1}}^{t_n}+\int_{t_{n-1}}^{t_n} \dot{e}\varphi\,dt+\int_{t_{n-1}}^{t_n} ae\varphi\,dt\right)\\ &= e_N^2+\sum_{n=1}^N \int_{t_{n-1}}^{t_n} (\dot{e}+ae)\varphi\,dt+\sum_{n=1}^N (-e_n^-\varphi_n+e_{n-1}^+\varphi_{n-1}).\end{aligned}$$

Note that e is discontinuous since U is discontinuous (i.e. $-\varphi e|_{t_{n-1}}^{t_n} = -\varphi_n e_n^- + \varphi_{n-1} e_{n-1}^+$ and $\varphi_n^+ = \varphi_n^-$ by continuity). Let us write out some of the terms from the last summation.

$$\begin{array}{r} -e_1^- \varphi_1 + e_0^+ \varphi_0 \\ -e_2^- \varphi_2 + e_1^+ \varphi_1 \\ \vdots \\ -e_N^- \varphi_N + e_{N-1}^+ \varphi_{N-1} \end{array}$$

but $-e_N^- \varphi_N = -e_N^2$ so this term cancels with the original e_N^2. Then we are left with the terms $(-e_1^- \varphi_1 + e_1^+ \varphi_1) + \cdots + (-e_{N-1}^- \varphi_{N-1} + e_{N-1}^+ \varphi_{N-1}) + e_0^+ \varphi_0 = \sum_{n=1}^{N-1} [e_n] \varphi_n + e_0^+ \varphi_0$. Thus,

$$\text{RHS} = \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} (\dot{e} + a e) \varphi\, dt + \sum_{n=1}^{N-1} [e_n] \varphi_n + e_0^+ \varphi_0,$$

where $[e_n]$ is the jump at the interface (e.g. $-e_1^- \varphi_1 + e_1^+ \varphi_1 = [e_1] \varphi_1$). We are left with $e_0^+ \varphi_0$ and by definition $e_0^+ = u_0 - U_0^+$. Hence, we get

$$e_N^2 = \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} (\dot{e} + a e) \varphi\, dt + \sum_{n=1}^{N-1} [e_n] \varphi_n + (u_0 - U_0^+) \varphi_0.$$

Since u solves the differential equation and $\dot{U} \equiv 0$ on the intervals (this is for dG(0)) we have that $\dot{e} + a e = \dot{u} - \dot{U} + a u - a U = f - a u - 0 + a u - a U = f - a U$. Also $u_n^+ = u_n^-$ gives us that $[e_n] = e_n^+ - e_n^- = u_n^+ - U_n^+ - u_n^- + U_n^- = -[U_n]$. Then,

$$\begin{aligned} \sum_{n=1}^{N-1} [e_n] \varphi_n + (u_0 - U_0^+) \varphi_0 &= -\sum_{n=1}^{N-1} [U_n] \varphi_n + u_0 \varphi_0 - U_0^+ \varphi_0 \\ &= -\sum_{n=1}^{N-1} [U_n] \varphi_n - (U_0^+ - U_0^-) \varphi_0 \\ &= -\left(\sum_{n=1}^{N-1} [U_n] \varphi_n + [U_0] \varphi_0 \right) \\ &= -\sum_{n=1}^{N} [U_{n-1}] \varphi_{n-1}. \end{aligned}$$

Hence, we get

$$e_n^2 = \sum_{n=1}^{N} \left(\int_{t_{n-1}}^{t_n} (f - a U) \varphi\, dt - [U]_{n-1} \varphi_{n-1} \right). \tag{2.1}$$

Now recall the Galerking orthogonality for dG(0):

$$\sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} (\dot{U} + a U) v\, dt + \sum_{n=1}^{N} [U_{n-1}] v_{n-1}^+ = \int_0^{t_N} f v\, dt, \forall v \in W^{(0)}.$$

Certainly $\pi_k \varphi$ is constant in each interval I_n so that $\pi_k \varphi \in W^{(0)}$. Thus,

$$\sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} (\dot{U} + a U) \pi_k \varphi\, dt + \sum_{n=1}^{N} [U_{n-1}] \pi_k \varphi_{n-1}^+ = \int_0^{t_N} f \pi_k \varphi\, dt.$$

Now $\dot{U} \equiv 0$ and we have that

$$\sum_{n=1}^{N} [U_{n-1}] \pi_k \varphi_{n-1}^+ = \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} (f - a U) \pi_k \varphi\, dt. \tag{2.2}$$

Combining (2.1) and (2.2) we get that

$$e_N^2 = \sum_{n=1}^{N} \left(\int_{t_{n-1}}^{t_n} (f - a U)(\varphi - \pi_h \varphi)\, dt - [U]_{n-1} (\varphi - \pi_h \varphi)_{n-1}^+ \right).$$

(Also, recall that for dG(0) we get

$$\int_{t_{n-1}}^{t_n}(\dot{U}+aU)v\,dt+[U]_{n-1}v_{n-1}^{+} \;=\; \int_{t_{n-1}}^{t_n}fv\,dt$$

which is the Galerkin orthogonality). Then,

$$|e_N^{-}| \;\leqslant\; s(t_N)\,|kR(U)|_{[0,t_N]}$$

where $R(U)=\frac{|U_n-U_{n-1}|}{k_n}+|f-aU|$. The proof is the following:

$$\begin{aligned}\int_{t_{n-1}}^{t_n}(f-aU)(\varphi-\pi_h\varphi)dt &\leqslant |R(U)|_{I_n}\int_{t_{n-1}}^{t_n}|\varphi-\pi_h\varphi|dt\\ &\leqslant |R(U)|_{I_n}k_n\int_{t_{n-1}}^{t_n}|\dot{\varphi}|dt\end{aligned}$$

and

$$\begin{aligned}|[U]_{n-1}(\varphi-\pi_h\varphi)_{n-1}^{+}| &\leqslant |U_{n-1}^{+}-U_{n-1}^{-}|\,|\varphi_{n-1}-\pi_h\varphi_{n-1}^{+}|\\ &\leqslant (?)\;|U_{n-1}^{+}-U_{n-1}^{-}|\int_{I_n}|\dot{\varphi}|dt.\end{aligned}$$

What about **a priori** error estimates for dG(0)? First, let $a(t)=a$. Recall that $U_n=U(t_n^{-})$. Then, the scheme simplifies to

$$U_n-U_{n-1}+k_naU_n \;=\; \int_{I_n}f\,dt \tag{2.3}$$

Furthermore, if we plug in the true solution, u, into the ODE, we get

$$\int_{I_n}\dot{u}+au\,dt+u_{n-1}^{+} \;=\; \int_{I_n}f\,dt+u_{n-1}^{-}, \tag{2.4}$$

where of course $u_{n-1}^{+}=u_{n-1}^{-}$ since the true solution is continuous. Adding and subtractin k_nau_n to both sides of (2.4), we get

$$u_n-u_{n-1}+k_nau_n \;=\; \int_{I_n}f\,dt+k_nau_n-\int_{I_n}au\,dt.$$

Subtracting the dG(0) formula and the above equation with the true solution gives us

$$\begin{aligned}(u_n-U_n)-(u_{n-1}-U_{n-1})+k_na(u_n-U_n) &= \rho_n\\ \Rightarrow\quad e_n-e_{n-1}+k_nae_n &= \rho_n,\end{aligned}$$

where $|\rho_n|=|k_nau_n-\int_{I_n}au\,dt|\leqslant\frac{1}{2}|a|k_n^2|\dot{u}|_{I_n}$which is the rectangular formula. Then,

$$e_n \;=\; (1+k_na)^{-1}(e_{n-1}+\rho_n)$$

which is a recursive error estimate. Let us take the time step s.t. $|k_na|<\frac{1}{2}$ so that the error term is decreasing. Recall that $\frac{1}{1-x}\leqslant e^{2x}$ for $0<x\leqslant\frac{1}{2}$, just by a simple visual argument. We get

$$\begin{aligned}|e_N| &\leqslant (1+|k_na|)^{-1}(e_{n-1}+\rho_n)\\ &\leqslant e^{2|a|k_N}|e_{N-1}|+e^{2|a|k_N|}|\rho_N|\\ &\;\;\vdots\\ &\leqslant \sum_{n=1}^{N}e^{2|a|\sum_{m=n}^{N}k_m}|\rho_n|\\ &\leqslant \frac{1}{2}\sum_{n=1}^{N}e^{2|a|(t_N-t_{n-1})}|a|k_n\max_{1\leqslant n\leqslant N}k_n|\dot{u}|_{I_n}\end{aligned}$$

where we perform the recursion up to $|e_0| = 0$. The maximum term can move out of the sum. The remaining terms looks like a Reimann sum for an approximating integral. Furthermore,

$$\begin{aligned} \text{RHS} &\leqslant \frac{1}{2}\bigg(e \int_0^{t_N} |a| e^{2|a|\tau} d\tau \bigg) \max_{1 \leqslant n \leqslant N} k_n |\dot{u}|_{I_n} \\ &= \frac{e}{4}(e^{2|a|t_N} - 1) \max_{1 \leqslant n \leqslant N} k_n |\dot{u}|_{I_n}. \end{aligned}$$

In summary,

$$|e_N| \leqslant \frac{e}{4}(e^{2|a|t_N} - 1) \max_{1 \leqslant n \leqslant N} k_n |\dot{u}|_{I_n}$$

which is order 1 since we estimating using piecewise constant functions.