1 10/4/2007

(make up class 5-5:50 on Tuesdays)

Recall that we have $\left\{ \begin{smallmatrix} -(au')' = f \\ u(0)=0, u(1)=0 \end{smallmatrix} \right.$ with FEM V_h. We want to find $U \in V_h$ s.t.

$$\int aU'v'dx = \int fvdx, \forall v \in V_h.$$

Furthermore, in our energy norm,

$$\|u-U\|_E \leqslant \|u-v\|_E, \forall v \in V_h$$

and

$$\|u-U\|_E \leqslant c\|hu''\|_a$$

where $\|v\|_E^2 = \int_0^1 a(v')^2 dx$, $\|v\|_a^2 = \int_0^1 av^2 dx$. This is called the a priori estimate. That is, we know that U gives us the best estimate to the true solution u. As for the a posteriori estimate, we don't want u in the error estimate. This type of estimate is much more practical because in many cases we do not know the true solution. Recall that after some work, we get

$$\|u-U\|_E^2 \leqslant c\|hR(U)\|_{a^{-1}}$$

where $R(U) = f + (aU')'$. Note that if U is piecewise linear, then the derivative, U', is piecewise constant and $(aU')' = 0$ only if a is also a constant. Thus, this type of estimate is completely computable whether or not we have the true solution u.

Now we consider an adaptive error control. That is, given a tolerance ε, we start from a coarse mesh and compute U. Then we check if $c_i\|hR(U)\|_{a^{-1}} \leqslant \varepsilon$ is satisfied or not (c must be determined to perform this adaptive error control). If this is not satisfied, then we refine our mesh and compute another U to get within the given tolerance. But how do we refine our mesh? Uniform refinement is too expensive in certain models, so we might want to know in which interval is the error giving us problems.

What if we have data and modeling error. That is, what if a and f change a little. How sensitive is our numerical scheme to perturbations in a and f. Given $\hat{a}$, $\hat{f}$ our Galerkin orthogonality turns into

$$\int_0^1 \hat{a}\hat{U}'v'dx = \int_0^1 \hat{f}vdx,$$

where $\hat{U} \in V_h$. We refer to the modeling error as the error attributed from approximating a and the data error as the error attributed from approximating f.

If $e = u - \hat{U}$ then

$$\begin{aligned}
\|(u-\hat{U})'\|_a^2 &= \int_0^1 a(e')e'dx \\
&= \int_0^1 a(u'-\hat{U}')e'dx \\
&= \int_0^1 (aue' - a\hat{U}'e')dx \\
&= \int_0^1 (fe - a\hat{U}e')dx \left(\int_0^1 auv' = \int_0^1 fvdx, \forall v \in V \right).
\end{aligned}$$

Now, by Galerking orthogonality we know that $\int_0^1 \hat{a}\hat{U}'(\pi_h e)'dx - \int_0^1 \hat{f}\pi_h e dx = 0$. Thus,

$$\int_0^1 (fe - a\hat{U}e')dx = \int_0^1 (fe - a\hat{U}e')dx + \left(\int_0^1 \hat{a}\hat{U}'(\pi_h e)'dx - \int_0^1 \hat{f}\pi_h e dx \right).$$

Now if we add some more zero terms such as $\int_0^1 \hat{f} e\,dx - \int_0^1 \hat{f} e$ and $\int_0^1 \hat{a}\hat{U}e - \int_0^1 \hat{a}\hat{U}e$ we get

$$\begin{aligned}
\int_0^1 (fe - a\hat{U}e')\,dx &= \int_0^1 (fe - a\hat{U}e')\,dx + \left(\int_0^1 \hat{a}\hat{U}'(\pi_h e)'\,dx - \int_0^1 \hat{f}\pi_h e\,dx\right) \\
&\quad + \left(\int_0^1 \hat{f}e\,dx - \int_0^1 \hat{f}e\right) + \left(\int_0^1 \hat{a}\hat{U}e' - \int_0^1 \hat{a}\hat{U}e'\right) \\
&= \int_0^1 \hat{f}e\,dx - \int_0^1 \hat{f}\pi_h e\,dx - \int_0^1 \hat{a}\hat{U}e' + \int_0^1 \hat{a}\hat{U}'(\pi_h e)'\,dx \\
&\quad + \int_0^1 fe\,dx - \int_0^1 \hat{f}e - \int_0^1 a\hat{U}e'dx + \int_0^1 \hat{a}\hat{U}e \\
&= \int_0^1 \hat{f}(e - \pi_h e)dx - \int_0^1 \hat{a}\hat{U}(e - \pi_h e)'\,dx \\
&\quad + \int_0^1 (f - \hat{f})e\,dx - \int_0^1 (a - \hat{a})\hat{U}e'\,dx,
\end{aligned}$$

where all we did was regroup the terms together. Thus,

$$\begin{aligned}
\|(u - \hat{U})'\|_a^2 &= \int_0^1 \hat{f}(e - \pi_h e)dx - \sum_{j=1}^{M+1} \int_{I_j} \hat{a}\hat{U}(e - \pi_h e)'\,dx \\
&\quad + \int_0^1 (f - \hat{f})e\,dx - \int_0^1 (a - \hat{a})\hat{U}e'\,dx \\
&= I + II - III
\end{aligned}$$

where we have broken the integral into sums in order to perform integration by parts (since e' is discontinuous it is important that we break the integrals into intervals in which the functions are continuous). We will label the first two terms as I and the last two as II and $-III$, respectively. By the Cauchy inequality we proved in the last lecture, we have that

$$|III| \leqslant \|(a - \hat{a})\hat{U}\|_{a^{-1}}\|e'\|_a.$$

For II, using integration by parts, we get that

$$\int_0^1 (f - \hat{f})e\,dx = (F - \hat{F})e|_0^1 - \int_0^1 (F - \hat{F})e'\,dx,$$

where $F' = f$ and $\hat{F}' = \hat{f}$, just for notation, and we will let $F(0) = 0$ just for simplicity. Now since $e(1) = e(0) = 0$ we have that

$$\begin{aligned}
\int_0^1 (f - \hat{f})e\,dx &= -\int_0^1 (F - \hat{F})e'\,dx \\
&= \int_0^1 (\hat{F} - F)e'\,dx \\
&\leqslant \|\hat{F} - F\|_{a^{-1}}\|e'\|_a.
\end{aligned}$$

Finally, we have already shown, using integration by parts, that

$$\int_0^1 \hat{f}(e - \pi_h e)dx - \sum_{j=1}^{M+1} \int_{I_j} \hat{a}\hat{U}(e - \pi_h e)'\,dx = \int_0^1 \hat{R}(\hat{U})(e - \pi_h e)\,dx$$

where $\hat{R}(\hat{U}) = \hat{f} + (\hat{a}\hat{U}')'$. Thus, again by the Cauchy inequality, we have

$$\text{LHS} \leqslant C\|hR(U)\|_{a^{-1}}\|h^{-1}(e - \pi_h e)\|_a$$

where $\|h^{-1}(e - \pi_h e)\|_a \leqslant C\|e'\|_a$. Thus, we have that

$$\begin{aligned}
\|(u - \hat{U})'\|_a^2 &\leqslant C\|hR(U)\|_{a^{-1}}\|e'\|_a + \|\hat{F} - F\|_{a^{-1}}\|e'\|_a + \|(a - \hat{a})\hat{U}\|_{a^{-1}}\|e'\|_a \\
&= (C\|hR(U)\|_{a^{-1}} + \|\hat{F} - F\|_{a^{-1}} + \|(a - \hat{a})\hat{U}\|_{a^{-1}})\|e'\|_a
\end{aligned}$$

and dividing both sides by $\|e'\|_a$ gives us

$$\|(u-U)'\| \leqslant c\|h\hat{R}(\hat{U})\|_{a^{-1}} + \|\hat{F}-F\|_{a^{-1}} + \|(a-\hat{a})\hat{U}'\|_{a^{-1}}.$$

Higher Order Finite Elements:

What if we want $cG(2)$, using piecewise polynomials of degree 2. Since we want a quadratic interpolant on each interval. There are many ways to create a basis. A nice way is to still make the basis functions 1 on the boundaries of the interval. Since we want our basis functions to be continuous, the support of our basis functions will be over two cells. If we write out this basis function, we get that

$$\varphi_i(x) = \begin{cases} \frac{2(x-x_{i+1/2})(x-x_{i+1})}{h_{i+1}^2} & , \; x \in I_{i+1} \\ \frac{2(x-x_{i-1/2})(x-x_{i-1})}{h_i^2} & , \; x \in I_i \\ 0 & , \; \text{otherwise} \end{cases}$$

for $i=1,2,\ldots,M$. But these basis functions are not enough. We have $M+1$ intervals and $3(M+1)-(M+2)=2(M+1)$, degrees of freedom. So we need a basis of $2(M+1)$ functions. Thus, we have an additional set of basis function

$$\varphi_{i-1/2}(x) = \begin{cases} \frac{4(x_i-x)(x-x_{i-1})}{h_i^2} & , \; x \in I_i \\ 0 & , \; \text{otherwise} \end{cases}$$

for $i=1,2,\ldots,M+1$. This gives us a total of $2(M+1)$ basis functions corresponding to $2(M+1)$ degrees of freedom. Given these basis functions, for any $v \in V_h$ we write

$$v(x) = \sum_{i=1}^{M+1} v(x_{i-1/2})\varphi_{i-1/2}(x) + \sum_{i=1}^{M} v(x_i)\varphi_i(x).$$

Then, if we write $U(x)=\xi_{1/2}\varphi_{1/2}(x)+\cdots\xi_1\varphi_1(x)+\cdots+\xi_{M+1/2}\varphi_{M+1/2}(x)$ we get $A\xi=b$ where A is banded (with width 5) and symmetric positive definite.

Higher Order ODEs:

Consider

$$\begin{cases} u^{(4)}(x)=f \\ u(0)=u'(0)=u(1)=u'(1)=0 \end{cases}$$

(V):

$$\int_0^1 u''v''\,dx = \int_0^1 fv\,dx, \forall v \in W$$

which is obtained by integration by parts twice. To go backwards we assume that $u \in C^4$. Also,

$$W = \left\{ v \colon \int_0^1 v^2+(v')^2+(v'')^2\,dx < \infty, v(0)=v'(0)=v(1)=v'(1)=0 \right\}.$$

Now consider $W_h \subset W$ (conforming FE). We want that W_h: piecewise polynomials, C^1. This is very difficult in general. In order to construct a numerical scheme, we need our basis functions to be C^1. Before, we used piecewise linear polynomials, which were NOT C^1. Consider the following basis functions.

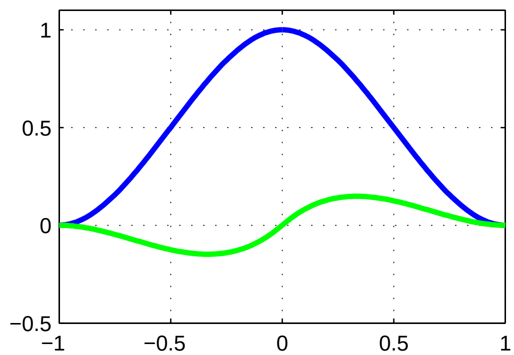

Figure 1.1. $\hat{\varphi}$ is in green, φ is in blue.

Notice here that

$$\begin{aligned} \varphi(x) &= \begin{cases} 1 & x=0 \\ 0 & x=\pm 1 \end{cases} & \varphi'(x) &= \{ 0 \quad x=0,\pm 1 \\ \hat{\varphi}(x) &= \{ 0 \quad x=0,\pm 1 & \hat{\varphi}'(x) &= \begin{cases} 1 & x=0 \\ 0 & x=\pm 1 \end{cases} \end{aligned}.$$

So $\varphi(x)$ resolves the issue of continuity in C^0 (i.e. $\varphi(x)$ acts as a global basis function so that the interpolating coefficients between adjacent elements matches up so as to make the function continuous) and $\hat{\varphi}(x)$ resolved the issue of the continuity of the first derivative. In other words, if we seek a solution of the form

$$u(x) = \sum_{k=1}^{M} \xi_k \varphi_k(x) + \hat{\xi}_k \hat{\varphi}_k(x)$$

then the solution will be C^1. But is this basis enough to determine our interpolating polynomial uniqely? And what is the interpolating polynomial. The basis functions in teh above figure consitute a cubic spline. Moreover, given these basis functions, knowing the value of the first derivatives and the function values at the nodes will **uniquely** determine the cubic spline!

HW: (due 10/16)

p.183 8.7, 8.8; p.184 8.9, 8.10; p.186 8.11; p.188 8.12,8.13; p.189 8.15; p.191 8.18

p. 199 8.30; p.201 8.35 (except the last sentence; computational: $h = \frac{1}{10}, \frac{1}{20}, \frac{1}{40}, \frac{1}{80}$, and print out the L^2 and energy norm and orders of accuracy).

How to computer the L^2 error: If $e(h) = ch^r$ then $\frac{e(2h)}{e(h)} = \frac{c(2h)^r}{ch^r} = 2^r \Rightarrow r = \log\frac{e(2h)}{e(h)}/\log 2$.