11/6/2007

Rayleigh Layer:

So we have that

$$\frac{\partial u_1}{\partial t} = \nu \frac{\partial^2 u_1}{\partial y^2}.$$

We also have that $u_1 = u_0$ at $y = 0$ (for $t \geqslant 0$) and $u_1 \to 0$ as $y \to \infty$. For $t < 0$ there is no flow.

Self Similar Solution:

Let $u_1 = u_0 f(\eta)$ where $\eta = y/\delta(t)$ and $\delta(t)$ is some length scale. Then,

$$\frac{\partial u_1}{\partial y} = u_0 \frac{1}{\delta} f'$$

and then

$$\frac{\partial^2 u_1}{\partial y^2} = u_0 \frac{1}{\delta^2} f''.$$

Also,

$$\begin{aligned} \frac{\partial u_1}{\partial t} &= u_0 \frac{\partial \eta}{\partial t} f' \\ &= u_0 \left(-\frac{y}{\delta^2} \frac{d\delta}{dt} \right) f'. \end{aligned}$$

Hence, since we have $\frac{\partial u_1}{\partial t} = \frac{\partial^2 u_1}{\partial y^2}$ we get the equality

$$\left(-\frac{y}{\delta^2} \frac{d\delta}{dt} \right) f'(\eta) = \frac{\nu u_0}{\delta^2} f''(\eta).$$

Letting $\eta = y/\delta$ the above equation becomes

$$-\left(\frac{1}{\delta} \frac{d\delta}{dt} u_0 \right) \eta f'(\eta) = \nu \frac{u_0}{\delta^2} f''(\eta).$$

To separate out an ODE in η we require that

$$\nu \frac{u_0}{\delta^2} = C \left(\frac{1}{\delta} \frac{d\delta}{dt} u_0 \right).$$

If we can do this, then

$$C f'' + \eta f' = 0.$$

Our condition simplifies to

$$\begin{aligned} \nu &= C \delta \frac{d\delta}{dt} \\ &= C \frac{d}{dt} \left(\frac{1}{2} \delta^2 \right). \end{aligned}$$

So

$$2\nu t \;=\; C(\delta^2 - \delta_0),$$

where δ_0 is our initial scale. For our case, we will let $\delta_0 = 0$. Since we have no flow at all at $t = 0$ then we set $C = 1$ and get $\delta^2 = 2\nu t$.

How so we solve the ODE? Let $g(\eta) = f'(\eta)$ so that we get $g' + \eta g = 0$. Then,

$$g \;=\; g_0 e^{-\frac{1}{2}\eta^2}.$$

Then,

$$f(\eta) \;=\; A + B\int_0^{\eta} e^{-\frac{1}{2}q^2} dq.$$

At $y = 0$ we have that $\eta = 0$ so that $A = 1$. As $y \to \infty$ we also have $\eta \to \infty$ so that

$$\begin{aligned} 0 &= 1 + B\int_0^{\infty} e^{-\frac{1}{2}q^2} dq \\ &= 1 + \frac{1}{2}\sqrt{2\pi}B \end{aligned}$$

and so

$$B \;=\; -\sqrt{\frac{2}{\pi}}.$$

Hence,

$$f(\eta) \;=\; 1 - \sqrt{\frac{2}{\pi}}\int_0^{\eta} e^{-\frac{1}{2}q^2} dq.$$

Note that $\mathrm{er}\,f(x) = \sqrt{\frac{2}{\pi}}\int_0^x e^{-t^2} dx$ where $\mathrm{er}\,f(0) = 0$ and $erf(1) = 1$.

What is the viscous shear stress at the plate ($\eta = 0$)? The force ON the fluid is given by

$$\begin{aligned} -\sigma_{12} &= -\mu\frac{\partial u_1}{\partial x_2} \\ &= -\mu u_0 \frac{1}{\delta} f' \\ &= -\mu\frac{u_0}{\delta} f'|_{\eta=0} \\ &= -\mu\frac{u_0}{\delta} g(0) \\ &= -\mu\frac{u_0}{\delta}\sqrt{\frac{2}{\pi}}. \end{aligned}$$

This is because the normal is $(0, -1, 0)$ then $\sigma_{ij}n_j = -\sigma_{i2}$ and then the tangential component is $-\sigma_{i2}\hat{n_i} = -\sigma_{12}$ where $\hat{\boldsymbol{n}} = (1, 0, 0)$. So the shear stress is $\propto t^{-1/2}$ because $\delta(t) = \sqrt{2\nu t}$. What about the vorticity? $\omega = (0, 0, -\frac{\partial u_1}{\partial x_2})$. Then,

$$\begin{aligned} \omega_3 &= -\frac{u_0}{\delta} f'(\eta) \\ &= -\frac{u_0}{\delta} g(\eta) \\ &= \frac{u_0}{\delta(t)}\sqrt{\frac{2}{\pi}} e^{-\eta^2/2}. \end{aligned}$$

If we graphed the vorticity versus y we have that the vorticity in a layer of thickenss $\delta(t)$ at a wall. Thus, there is a vortex diffusion by viscosity. We have

$$\frac{\partial u_1}{\partial t} = \nu \frac{\partial^2 u_1}{\partial y^2}$$

and

$$\frac{\partial \omega_3}{\partial t} = \nu \frac{\partial^2 \omega_3}{\partial y^2},$$

due to the vorticity equation

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{\omega} = \boldsymbol{\omega} \cdot \nabla \boldsymbol{u} + \nu \nabla^2 \boldsymbol{\omega}.$$

Recall that $u = (u_1(x_2, t), 0, 0)$ so that $\boldsymbol{\omega} \cdot \nabla \boldsymbol{u} = 0$ since $\boldsymbol{\omega} = (0, 0, \omega_3)$ and $\boldsymbol{u} \cdot \nabla \omega_3 = 0$.

So we have vorticity diffusion AND generation of vorticity at a wall, which is determined by the no-slip condition.

11/8/2007

In summary, we have shown that vorticity $\Leftrightarrow$ viscous shear stress at a wall.

Stokes Flow:

This is a viscous dominated flow with $\mathrm{Re} \approx 0$ or $\mathrm{Re} \ll 1$. The equations of motion are usually

$$\rho \frac{D u_i}{D t} = -\frac{\partial p}{\partial x} + \mu \nabla^2 u_i,$$

but in viscous diminated flows we have

$$\begin{aligned} 0 &= -\frac{\partial p}{\partial x} + \mu \nabla^2 u_i \\ &= \frac{\partial \sigma_{ij}}{\partial x_j} \end{aligned}$$

where the stress tensor takes on the pressure and viscous terms. This equation tells us that at all instants, flow is a force equilibrium. That is,

$$0 = \frac{\partial \sigma_{ij}}{\partial x_j} + \rho g_i$$

where ρg_i is usually neglible. Thus, we can say that

$$0 = \frac{\partial \sigma_{ij}}{\partial x_j}.$$

Let us consider a stokes flow past a body (e.g. fixed sphere in a flow). The streamlines are symmetric front and back. The fluid force ON the sphere is

$$\oint_{S_0} \sigma_{ij} n_j \, dS.$$

We could consider the sphere S_1 of radius R (bigger than the body). Consider the closed fluid volume V bounded by S_1 and S_0. Here we know that $\frac{\partial \sigma_{ij}}{\partial x_j} = 0$ because this is a Stokes Flow. The combined surface is $\hat{S} = S_1 - S_0$ where the minus sign is indicating the outward normal to the fluid ($\hat{S}$ bounds V). Now, we can use the divergence theorem for a closed volume, which gives us

$$\begin{aligned}\int_V \frac{\partial \sigma_{ij}}{\partial x_j} dV &= \oint_{\hat{S}} \sigma_{ij} n_j dS \\ &= \int_{S_1} \sigma_{ij} n_j dS - \int_{S_0} \sigma_{ij} n_j dS \\ &= 0\end{aligned}$$

since the divergence is zero. Hence,

$$F_i = \int_{S_1} \sigma_{ij} n_j dS$$

where $F_i = \int_{S_0} \sigma_{ij} n_j dS$, the force on the sphere. Thus, in Stokes flow, the forces are trasnmitted through the fluid.

There are a number of results we have for Stokes flow:

1. This is a linear problem (e.g. in the above example, if we double the fluid velocity, we will double the force on the sphere). Suppose

$$\begin{cases} u^{(1)}, & \sigma_{ij}{}^{(1)} \\ u^{(2)}, & \sigma_{ij}{}^{(2)} \end{cases}$$

are Stokes flows. Then, the sum

$$u^{(1)} + u^{(2)}, \ \ \sigma_{ij}{}^{(1)} + \sigma_{ij}{}^{(2)}$$

is also a Stokes flow. For this to be true, we need to show that

$$\begin{cases} \nabla \cdot u = 0 \\ \frac{\partial \sigma_{ij}}{\partial x_j} = 0 \end{cases}.$$

This is easily verified since all operators, $\nabla \cdot$, $\frac{\partial}{\partial x_j}$, are linear.

2. A Stokes flow is determined by the instantenous boundary conditions (i.e. no memory effect).

Conside the the vorticity $\nabla \times u = \omega$. Consider the equations for a Stokes flow:

$$0 = -\nabla p + \mu \nabla^2 u$$

Taking the curl of both sides, we get

$$\nabla^2 \omega = 0,$$

since the curl of the gradient is zero. So the vorticity satisfies Laplace's equation. We can compare this to steady state heat conduction. The time scale for viscous diffusion of vorticity $\ll$ time scale of the flow development. For example, for a flow past a sphere, if $r = a$ and diffusion length δ

$$\begin{aligned}\delta^2 &\sim \nu t \\ &= \nu \left(\frac{a}{u_0} \right),\end{aligned}$$

which is the time scale for the fluid to move past the sphere (i.e. fluid advection). This diffusion length scale $\gg a$. Hence,

$$\begin{aligned} \left(\nu \frac{a}{u_0}\right)^{1/2} &\gg a \\ \Rightarrow \nu \frac{a}{u_0} &\gg a^2 \\ \Rightarrow 1 &\gg \frac{u_0 a}{\nu}. \end{aligned}$$

The viscous diffusion on a length scale a, takes time $\sim \frac{a^2}{\nu}$. The advection time scale $\frac{a}{u_0}$ and

$$\frac{a}{u_0} \gg \frac{a^2}{\nu}.$$

So we are saying that the timescale on which this is changing, there is so much viscous diffusion, that the vorticity satisfies Laplace equtaion. So when we think of Stokes flow, we think of fully diffused voritcity.

3. Reversability. Suppose we have Stokes flow past a sphere and we are trying to find a force F. We need to solve $\nabla \cdot u = 0$ and $0 = -\nabla p + \mu \nabla^2 u$, with $u = 0$ on $|x| = a$ (i.e. the sphere) and $u \sim U$as $x \to \infty$. The flow u is linear in U. Then, p and σ_{ij} is linear in U. Hence,

$$F_i = A_{ij} U_j,$$

where A_{ij} is determined only by the geoemetry. We see that $\sigma_{ij} = p\delta_{ij} + 2\mu S_{ij}$. We find that

$$u_i(x) = \Phi_{ij}(x) U_j.$$

Then,

$$\frac{\partial p}{\partial x_i} = \mu \nabla^2 u_i$$

so that

$$p \propto \mu, U.$$

So $\sigma_{ij} \propto \mu, U$ and

$$\begin{aligned} F_i &= \oint_S \sigma_{ij} n_j \, dS \\ &= \mu A_{ij} U_j \end{aligned}$$

so that A is only determined by the geometry of the problem. Reversing U reverses F symmetrically. Conside the flow past a sphear again, Since A_{ij} depends on the geometry of the sphere, we have that A_{ij} is isotropic. Hence, $A_{ij} = \alpha \delta_{ij}$. Finally, we need to make sure our dimensions are correct.

Dimensions:

We have a force, F on the LHS and on the RHS we have $\mu \frac{U}{a}$. Thus,

$$\begin{aligned} \mu \frac{U}{a} &= \text{Force / Area} \\ \Rightarrow \mu \frac{U a^2}{a} &= \text{Force.} \end{aligned}$$

Hence,

$$F \text{ or } \mu U a.$$

Hence,

$$F_i = \hat{\alpha} a \mu U_j \delta_{ij}.$$

So the drag on the wphere is given by

$$F_i = \hat{\alpha}\, a \mu U_i.$$

$\alpha = 6\pi$ which is the Stokes drag law for a fixed sphere.

Stokes Flow:

Consider the governing equation for Stokes flow:

$$0 = -\nabla p + \mu \nabla^2 u$$

Taking the curl of both sides, we get that $\nabla^2 \omega = 0$.

2D problem: Use the stream function ψ so that

$$u_1 = \frac{\partial \psi}{\partial x_2}$$
$$u_2 = -\frac{\partial \psi}{\partial x_1}$$

so that

$$u = \nabla \times (0, 0, \psi).$$

For the ω we have that $\omega = \nabla \times (\nabla \times \psi e^{(3)})$ and we get

$$\omega = (0, 0, -\nabla^2 \psi),$$

since ψ is only a function of x and y. So we solve $\nabla^2(\nabla^2 \psi) = 0$ (this is for a 2D stokes flow).

Example: Paint Scraper problem: On the x_1 axis, we have that $u_1 = -V$ and $u_2 = 0$. Hence, $u_2 = 0 = -\frac{\partial \psi}{\partial x_1}$ by the definition of the streamfunction. So $\psi =$ constant w.r.t. x_1 for $x_2 = 0$. Then, let us take $\psi = 0$ on the x axis. Take $\psi = 0$ and we have that $\frac{\partial \psi}{\partial x_2} = -V$. On the $\theta = \alpha$ boundary, we have $u = 0$.

In plane polar coordinates (r, θ) we have $(u_r, u_\theta, 0) = \frac{1}{r}\begin{vmatrix} e^{(r)} & re^{(\theta)} & e^{(z)} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & 0 \\ 0 & 0 & \psi \end{vmatrix}$ (i.e. $u = \nabla \times \psi e^{(3)}$ in polar coordinates) so that $u_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$ and $u_\theta = -\frac{\partial \psi}{\partial r}$. Then, $u_\theta|_{\theta = \alpha} = 0 = -\frac{\partial \psi}{\partial r}$ (the no normal flow bc at the angle α) for all r. Thus, $\psi =$ consant on the boundary. Since we have a fixed wall, then we have no slip and $u_r = 0 = \frac{1}{r}\frac{\partial \psi}{\partial \theta}$. So we need to solve

$$\nabla^2(\nabla^2 \psi) = 0.$$

The way to do the problem is to solve two Laplace's equations.

Flow past a sphere in spherical polar coordinates. We have axissymmetric flow u_r, u_θ nd $u_\phi = 0$, where ϕ is the azimuthal angle (i.e. no ϕ dependence). We have that

$$\begin{aligned}
\nabla \cdot u &= 0 \\
u &= \frac{1}{r^2 \sin\theta} \begin{vmatrix} e^{(r)} & re^{(\theta)} & r\sin\theta e^{(\phi)} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & 0 \\ 0 & 0 & r\sin\theta A_\phi \end{vmatrix}
\end{aligned}$$

so we have $\nabla \times (A_\phi e^{(\phi)})$. We let $\psi = r\sin\theta A_\phi$ so that

$$A = \left(\frac{\psi}{r\sin\theta}\right) e^{(\phi)}.$$

Then,

$$\begin{aligned}
u_r &= \frac{1}{r^2 \sin\theta} \frac{\partial \psi}{\partial \theta} \\
u_\theta &= -\frac{1}{r\sin\theta} \frac{\partial \psi}{\partial r}.
\end{aligned}$$

Furthermore,

$$\begin{aligned}
u \cdot \nabla \psi &= (u_r e^{(r)} + u_\theta e^{(\theta)}) \cdot (\frac{\partial \psi}{\partial r} e^{(r)} + \frac{\partial \psi}{\partial \theta} e^{(\theta)}) \\
&= u_r \frac{\partial \psi}{\partial r} + \frac{1}{r} u_\theta \frac{\partial \psi}{\partial \theta} \\
&= \frac{1}{r^2 \sin\theta} \frac{\partial \psi}{\partial \theta} \frac{\partial \psi}{\partial r} + \frac{1}{r} \frac{1}{r^2 \sin\theta} \frac{\partial \psi}{\partial r} \frac{\partial \psi}{\partial \theta} \\
&= 0.
\end{aligned}$$

Hence, $\psi =$ constant along the streamlines.