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Let us focus on incompressible flows.

Euler equation, inviscid flow:

$$\rho \frac{Du}{Dt} = -\nabla p + \rho g,$$

where u, g are vector valued functions. Typically we have non normal flows for boundary conditions.

Newtonian viscous flow:

$$\rho \frac{Du}{Dt} = -\nabla p + \mu \nabla^2 u + \rho g.$$

For the boundary condition, we typically have no normal flow and no slip at the rigid walls.

Recall that in Eulerian form, we have that $\frac{Du}{Dt} = \frac{\partial \rho}{\partial t} + (u \cdot \nabla) u$ and $\mathrm{Re} = u_0 L / v$.

We've already covered the topics up to chapter 4. Bernoulli equation for stead inviscid flow:

$$u \cdot \nabla (p/\rho + \frac{1}{2} u^2) = 0$$

so that $p/\rho + \frac{1}{2} u^2$ is constant along the streamlines.

Vorticity:

$$\omega = \nabla \times u.$$

The local angular velocity in a fluid motion is $\frac{1}{2}\omega$ (Recall that the fluid flow due to angular momentum is given y $\Omega \times \boldsymbol{x}$, where $\Omega = (0, 0, \Omega)$ Thus, $\boldsymbol{u} = \Omega(-y, x, 0)$ and $\omega = (0, 0, 2\Omega)$). We can formulate equations by the vorticity using Euler's equation and incompressibility. If $\omega \equiv 0$, then $\nabla \times u = 0$ everywhere. This means that we can find a scalar $\phi(x, t)$ so that $u = \nabla \phi$. Why? Stokes' theorem tells us that

$$\begin{aligned} \int_S (\nabla \times u) \cdot n \, dS &= \int_C u \cdot dx \\ &= 0. \end{aligned}$$

So the integral around a closed path is zero so that the integral is path independent. Let $\phi(x) = \phi(0) + \int_0^x u \cdot dx$. So suppose $\nabla \times u \equiv 0$ and that $u = \nabla \phi$ (potential flow). Is this dynamically feasible? Incompressibility tells us that

$$\begin{aligned} \nabla \cdot (\nabla \phi) &= 0 \\ \Rightarrow \nabla^2 \phi &= 0. \end{aligned}$$

Thus, we have to solve Laplace's equation and the solutions are determined by the b.c.'s.

In invisicid flows, we have

$$\rho\left(\frac{\partial u}{\partial t}+u\cdot\nabla u\right) = -\nabla p+\rho g.$$

Now recall that

$$\begin{aligned} u\times\omega &= u\times(\nabla\times u)\\ &= \nabla(\frac{1}{2}u^2)-(u\cdot\nabla)u. \end{aligned}$$

If $\omega=0$ then $u\times\omega=0$ and $\nabla(\frac{1}{2}u^2)=(u\cdot\nabla)u$. Also, $\rho g=\nabla(\rho(g\cdot x))$ provided $\rho=$constant. So we have

$$\begin{aligned} \rho\left(\frac{\partial}{\partial t}(\nabla\phi)+\nabla(\frac{1}{2}u^2)\right) &= -\nabla p+\nabla(\rho(g\cdot x))\\ \Rightarrow\quad \nabla\left(\frac{\partial\phi}{\partial t}+\frac{1}{2}u^2+\frac{p}{\rho}-g\cdot x\right) &= 0. \end{aligned}$$

Hence, $\frac{\partial\phi}{\partial t}+\frac{1}{2}u^2+\frac{p}{\rho}-g\cdot x=f(t)$ (i.e. no dependence on x and thus it is spatially uniform). So the pressure p is found.

In summary, if we are dealing with potential flows, then $u=\nabla\phi$ $(\nabla\times u=0)$, $\nabla^2\phi=0$, and we have no normal flow (invisicid) b.c.'s. Thus, instantenous b.c.'s imply instantenuous flow and we can find p from $\frac{\partial\phi}{\partial t}+\frac{1}{2}u^2+\frac{p}{\rho}-g\cdot x=f(t)$.

Examples of Potential flows:

1. Water waves.
2. Airfoils and lift forces.
3. Dynamics of gas bubbles in liquid (mostly potential).

Remark: If we have a point source, such as when flow is coming out radially from one particular point, where $Q=$ volume outflow from the point soure, then the flow is spherically symmetric. Hence, $u=u_re^{(r)}=\nabla\phi$ where $\phi=\phi(r)$. Furthermore, $\nabla\cdot u=0$ and $\oint_S v\cdot n\,dS=0$ as long as the we do not include the source point,. The solume outflow from teh point source is a constant Q. Thus, the volume outflow from a smaller radius is the same as the outflow of a surface from a bigger radius. That is,

$$\begin{aligned} 4\pi a^2u_r(a) &= 4\pi R^2u_r(R)\\ &= Q. \end{aligned}$$

Hence, $u_r=\frac{Q}{4\pi r^2}=\frac{\partial\phi}{\partial r}$ so that $\phi=-\frac{Q}{4\pi r}$. We will cover 2D and 3 potential flows.

(Consdier Line vortex in 2D. If we have constant circulation Γ then $2\pi r u_\theta=\Gamma$ so that $u_\theta=\frac{\Gamma}{2\pi r}$ where $r\neq 0$).

Vorticity:

$\omega=\nabla\times u$. Can we find a dynamic equation for the vorticity, ω? Consider the momentum equation:

$$\begin{aligned} \rho\left(\frac{\partial u}{\partial t}+u\cdot\nabla u\right) &= -\nabla p+\mu\nabla^2u+\rho g\\ \nabla\cdot u &= 0. \end{aligned}$$

What if we take the curl of this equation? Note that $\nabla \times \frac{\partial u}{\partial t} = \frac{\partial}{\partial t}(\nabla \times u) = \frac{\partial \omega}{\partial t}$ so this seems like the right idea. If we divide the momentum equation by ρ we get

$$\frac{\partial u}{\partial t} + u \cdot \nabla u = -\frac{1}{\rho}\nabla p + \nu\mu\nabla^2 u + g,$$

where $\nu = \mu/\rho$. Now using again $u \times \omega = \nabla(\frac{1}{2}u^2) - (u \cdot \nabla)u$ we get

$$\frac{\partial u}{\partial t} + \nabla(\frac{1}{2}u^2) - u \times \omega = -\nabla p + \mu\nabla^2 u + \rho g.$$

Then the curl of the this is

$$\frac{\partial \omega}{\partial t} + \nabla \times (u \times \omega) = \nabla \times \left(-\frac{1}{\rho}\nabla p \right) + \nu\nabla^2\omega.$$

Now if we have constant density, then $\nabla \times \left(-\frac{1}{\rho}\nabla p \right) = 0$ (it is certainly true that $\nabla \times \nabla p = 0$, but the ρ term may not have zero curl). If there are density variations, possible $-\frac{1}{\rho^2}\nabla\rho \times \nabla p$ then we call this a Baroclinic torque. So for our sake, assume that $\nabla \times \left(-\frac{1}{\rho}\nabla p \right) = 0$. Hence,

$$\frac{\partial \omega}{\partial t} + \nabla \times (u \times \omega) = \nu\nabla^2\omega.$$

Now we have that $\nabla \times (u \times \omega)$ is

$$\begin{aligned}
\epsilon_{ijk}\frac{\partial}{\partial x_j}\{\epsilon_{kpq}u_p\omega_q\} &= \epsilon_{ijk}\epsilon_{pqk}\frac{\partial}{\partial x_j}\{u_p\omega_q\} \\
&= (\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp})\frac{\partial}{\partial x_j}\{u_p\omega_q\} \\
&= \delta_{ip}\delta_{jq}\frac{\partial}{\partial x_j}\{u_p\omega_q\} - \delta_{iq}\delta_{jp}\frac{\partial}{\partial x_j}\{u_p\omega_q\} \\
&= \frac{\partial}{\partial x_j}\{u_i\omega_j\} - \frac{\partial}{\partial x_j}\{u_j\omega_i\} \\
&= \left\{\omega_j\frac{\partial u_i}{\partial x_j} + u_i\frac{\partial \omega_j}{\partial x_j}\right\} - \left\{\omega_i\frac{\partial u_j}{\partial x_j} + u_j\frac{\partial \omega_i}{\partial x_j}\right\} \\
&= \omega_j\frac{\partial u_i}{\partial x_j} - u_j\frac{\partial \omega_i}{\partial x_j}
\end{aligned}$$

since $\frac{\partial \omega_j}{\partial x_j} = \nabla \cdot \omega = \nabla \cdot (\nabla \times u) \equiv 0$ and $\frac{\partial u_j}{\partial x_j} = \nabla \cdot u = 0$. Hence,

$$\frac{\partial \omega_i}{\partial t} - \left(\omega_j\frac{\partial u_i}{\partial x_j} - u_j\frac{\partial \omega_i}{\partial x_j}\right) = \nu\nabla^2\omega_i.$$

which is the same as

$$\begin{aligned}
\frac{\partial \omega}{\partial t} - ((\omega \cdot \nabla)u - u \cdot \nabla\omega) &= \nu\nabla^2\omega \\
\Rightarrow \frac{\partial \omega}{\partial t} + u \cdot \nabla\omega &= (\omega \cdot \nabla)u + \nu\nabla^2\omega.
\end{aligned}$$

Also, recall that $\frac{d}{dt}h = \nabla u \cdot h$ where h is the material line element. $\omega \cdot \nabla u$ is similar to the motion of the line element. Remember that line elements rotate, stretch and compress. Consider a 2D flow: $u = (u(x, y), v(x, y), 0)$. Then $\omega = \nabla \times u = (0, 0, \partial_x v - \partial_y v) = (0, 0, \omega_3)$. Then,

$$\begin{aligned}
\omega \cdot \nabla u &= \omega_3\partial_z u \\
&= 0
\end{aligned}$$

since there is no z dependence. So in a 2D flow, we get

$$\begin{aligned}\frac{\partial\omega}{\partial t} + u\cdot\nabla\omega &= \nu\nabla^2\omega \\ \frac{D\omega}{Dt} &= \nu\nabla^2\omega.\end{aligned}$$

Still in 2D, if the flow is inviscid, then $\nu = 0$ and $\frac{D\omega}{Dt} = 0$. This means that the vorticity of a fluid element in constant. Everything is determined by the initial voriticity. That is, we specifiy the initial vorticity which determines $\omega(x,t)$. So this is not a potential flow (because a potential flow is irrotational - i.e. $\omega = \nabla\times u \equiv 0$ so that we can write $u = \nabla\phi$). If $\omega \equiv 0$ at $t = 0$ then $\omega(x, t) \equiv 0$ always. Hence, a potential flow remains potential (also remember that a potential flow is only used only for invisid, irrotational flows, The above case is only sometimes irrotational).

With a viscous flows,

$$\frac{\partial\omega}{\partial t} + u\cdot\nabla\omega = \nu\nabla^2\omega.$$

If we look at $\frac{\partial\omega}{\partial t} = \nu\nabla^2\omega$ this is the diffusion equation. So the viscous term represents the diffusion of vorticity. Thus, initially localized vorticity will diffuse. If we have viscous dominated flows (Re $\ll 1$) then $0 = \nu\nabla^2\omega$ and the vorticity is fully diffused (this is what characterizes Stokes' flow - steady flow in which the viscous term dominates).

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Recall our discussion of vorticity, $\omega = \nabla\times u$. Getting the vorticity dynamics of an incompressible flow is crucial.

For incompressible flows, constant density ρ: We have

$$\begin{aligned}\nabla\cdot u &= 0 \\ \rho\frac{Du}{Dt} &= -\nabla p + \mu\nabla^2 u + \rho g.\end{aligned}$$

Then,

$$\begin{aligned}\frac{\partial\omega}{\partial t} &= -\nabla\times(u\times\omega) \\ &= v\nabla^2\omega\end{aligned}$$

which gives us

$$\frac{\partial\omega}{\partial t} + u\cdot\nabla\omega = \omega\cdot\nabla u + v\nabla^2\omega.$$

1. If $\omega(x,0) \equiv 0$ then $\omega(x,t) \equiv 0$ for $t > 0$ for inviscid flows. This is irrotational flow. This gives us a potential flow (i.e. $u = \nabla\phi$).

2. 2D planar flows, the vorticity is always in the third direction x_3. e.g. $u = (u_1, u_2, u_3)$ then $\frac{\partial}{\partial x_3}u_i = 0$. Now, $\omega = (0, 0, \omega_3)$ and $\omega\cdot\nabla u = 0$ then

$$\begin{aligned}\frac{\partial\omega}{\partial t} + u\cdot\nabla\omega &= v\nabla^2\omega \\ &= 0 \text{ (if inviscid).}\end{aligned}$$

Thus, in inviscid flow, the vorticity of a fluid element is constant. That is, the pressure forces exert no torques on a spherical fluid element.

3. Viscosity leads to vortex diffusion. This is the key principle when looking at laminar flows. In viscous dominated flows (Stokes flow, Re = 0) we have that $0 = \nabla^2 \omega$.

Now let us discuss a general 3D flow (i.e. we do not have the case where $\omega \cdot \nabla u \neq 0$).

For an invisicid flow ($\nu = 0$):

$$\frac{D\omega}{Dt} = \omega \cdot \nabla u.$$

We will take a local snapshot of the flow (i.e. follow a fluid element and look at its vorticity). Locally we have $\omega, \nabla u$. Then,

$$\frac{d\omega_i}{dt} = \omega_j \frac{\partial u_i}{\partial x_j},$$

where $\frac{D}{Dt}$ is replaced by $\frac{d}{dt}$. Consider

$$\begin{aligned} \frac{d}{dt}\left(\frac{\omega_i^2}{2}\right) &= \omega_i \frac{d\omega_i}{dt} \\ &= \omega_i\left(\omega_j \frac{\partial u_i}{\partial x_j}\right), \end{aligned}$$

from the inviscid vorticity equation. We have defined

$$\frac{\partial u_i}{\partial x_j} = S_{ij} + R_{ij}.$$

Thus, we get

$$\begin{aligned} \frac{d}{dt}\left(\frac{\omega_i^2}{2}\right) &= \omega_i (S_{ij} + R_{ij}) \omega_j \\ &= \omega_i S_{ij} \omega_j, \end{aligned}$$

since S is symmetric and R is antisymmetric. The symmetry of S_{ij} allows us to diagonlize the matrix with **orthogonal** matrices Q. So, let us diagonalize S_{ij} so that $S = Q \Lambda Q^T$ where Λ are the eignenvalues of S and Q are orthogonal meaning that the column vectors are orthogonal. Then,

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\omega_i^2 &= \hat{\omega}_i \begin{pmatrix} \alpha & & \\ & \beta & \\ & & -\alpha - \beta \end{pmatrix} \hat{\omega}_j \\ &= \alpha \hat{\omega}_1^2 + \beta \hat{\omega}_2^2 - (\alpha + \beta) \hat{\omega}_3^2, \end{aligned}$$

(the trace of Λ must add to zero since the flow is incompressible)

Example: Let us try $\alpha > 0, \beta = 0$. Then

$$\frac{1}{2}\frac{d}{dt}\omega_i^2 = \alpha \hat{\omega}_1^2 - \alpha \hat{\omega}_2^2$$

where the flow is

$$(\alpha x_1, 0, -\alpha x_3).$$

Recall that the fluid element is given by

$$\frac{dh}{dt} = h \cdot \nabla u.$$

So there is an analogy of h to ω.

$$\begin{array}{rl} \omega_1, \alpha > 0 & \text{streething, increasing vorticity} \\ \omega_3 & \text{compression, decreasing vorticity.} \end{array}$$

If we look at the rotation R_{ij} on ω we have a change in direction. We have $\frac{d\omega}{dt} = \omega \cdot \nabla u$ and compare it to $\frac{dh}{dt} = h \cdot \nabla u$.

Example: We have an initial flow $u = (-\alpha x_1, -\alpha x_2, 2\alpha x_3)$, $\alpha > 0$ plus a vorticity $\omega = (0, 0, \omega_3)$ added, for example, by a Ranking vortex. So the only contribution from vorticity comes from the added ω. Let us use cylindrical polar coordinates (r, θ, z). So we have a swirling flow. Let us say that v_θ depends on r so that is axissymmetric: $\frac{\partial}{\partial\theta} = 0$.

e.g. Let us add the Rankine vortex: $v_\theta = \begin{cases} \Omega r & r \leqslant a \\ \Omega a^2 / r & r \geqslant a \end{cases}$ so in the core it is rigid body rotation and outside the core irrotational flow. The vorticity is 2Ω inside the core and zero outside.

We are still dealing with an inviscid flow so that $\frac{D\omega}{Dt} = \omega \cdot \nabla u$. Note that we have the flow from the Rankin vortex plus the flow from $u = (-\alpha x_1, -\alpha x_2, 2\alpha x_3)$. Then,

$$\begin{aligned} (v_\theta e^{(\theta)} \cdot \nabla) &= v_\theta \frac{\partial}{\partial\theta} \\ &= 0. \end{aligned}$$

Now, note that $\omega = (0, 0, \omega_3)$. Then, the vorticity equation

$$\frac{D\omega_i}{Dt} = \omega \cdot \nabla u_i.$$

Then, the RHS is

$$(0, 0, \omega_3) \cdot \nabla u_i = \omega_3 \frac{\partial u_i}{\partial z}.$$

The flow due to the Rankine vortex is only in the r and θ direction so there is no flow in the z direction. The straining flow is the only flow that has dependence on z. Hence, we get

$$\begin{aligned} \frac{D\omega_3}{Dt} &= \omega_3 \frac{\partial u_3}{\partial z} \\ &= \omega_3 \, 2\alpha. \end{aligned}$$

So the stretching flow increases vorticity. That is, the vorticity increases w.r.t. time in our straining flow. For a cylinder of vorticity, the pressure forces act normally so there is no torque (in z direction).

The above equation tells us that the vorticity increases w.r.t. t. However, we expect angular momentum to be conserved. So it must be that the radius of the vortex "tube" decreases and the cylinder is elongated. If we look at the flow, it seems as though it would elongate and narrow the tube of the vortex. The volume of the cylinder is the same since we have incompressible flow. The angular momentum is $\propto r^2\Omega$. So $r\downarrow$ so that $\Omega\uparrow$ to conserve angular momentum. We can think of our cylinder with length L and radius a. Then $\pi a^2 L$ is constant. So as $a(t)$, the radius, decreases, $L(t)$ increases. Thus, $\Omega \propto L$ over time. This is specifically a feature of 3D flows. This does not apply to 2D!

Helmoltz Laws (Inviscid flows):

1. Vortex lines move with the fluid.

2. Strength of a vortex tube is constant along its length.

3. A vortex tube cannot end within the fluid (vortex tube is closed or attached to a surface).

4. Circulation of a vortex tube remains constant (Kelvin's circulation theorem).

This equation says all four statements roled into one:

$$\frac{D\omega}{Dt} \;=\; \omega \cdot \nabla u.$$

#3 is a consequence of the fact that $\nabla \cdot \omega = 0$ (compare to $\nabla \cdot u = 0$ and that the streamlines cannot end in a fluid). The fluid cannot just end unless it hits a stagnation point (e.g. rigid body wall). Furthermore, the tube is bounded by streamlines.

(The vortex tube is a bundle of vortex lines parallel to ω.)

Confined vorticity:

1. Vortex line. Tangent to ω. The tube is a bundle of lines.

Recall that the circulation is $\oint_C u \cdot dx$ (Stokes' Theorem: $\int_S \omega \cdot n \, dS = \oint_C u \cdot dx$). So the strength of a confined vortex tube is specified by its circulation, Γ.

Kelvin's Theorem:

For a material curve C, $\Gamma =$ constant in inviscid flows.

Proof. Consider $\frac{d}{dt}\{\oint_C u \cdot dx\}$. The circulation is defined as

$$\oint_C u \cdot dx \;=\; \sum_{n=1}^{N} u \cdot \Delta X$$

where ΔX is a sequence of material line elements ΔX. Think of partitioning our curve C into elements of $X_n(t)$, changing with the fluid. Then $u = (x = X(t), t)$ and $\Delta X = X_{n+1}(t) - X_n(t)$. Thus,

$$\oint_C u \cdot dx \;=\; \sum_{n=1}^{N} u(X_n(t), t) \cdot (X_{n+1} - X_n).$$

Then,

$$\begin{aligned}\frac{d\Gamma}{dt} \;&=\; \sum_{n=1}^{N} \frac{Du}{Dt} \cdot \Delta X + u \cdot \frac{D}{Dt}(\Delta X) \\ &=\; \sum_{n=1}^{N} \left(-\frac{1}{\rho} \nabla p \right) \cdot \Delta X + u \cdot (\Delta X \cdot \nabla u),\end{aligned}$$

since $\rho\frac{Du}{Dt} = -\nabla p$ (for inviscid flows) and $\frac{D}{Dt}(\Delta X) = \Delta X \cdot \nabla u$ (also for invisicd flows. This is what we derived from the rate of change of the line element $H(t)$). Furthermore,

$$\begin{aligned}
\frac{d\Gamma}{dt} &= \sum_{n=1}^{N} \frac{Du}{Dt} \cdot \Delta X + u \cdot \frac{D}{Dt}(\Delta X) \\
&= \sum_{n=1}^{N} \left(-\frac{1}{\rho}\nabla p \right) \cdot \Delta X + u \cdot (\Delta X \cdot \nabla u) \\
&= \sum_{n=1}^{N} \left(-\frac{1}{\rho}\nabla p \right) \cdot \Delta X + (u \cdot \nabla u) \cdot \Delta X \\
&\overset{\lim_{\Delta X \to 0}}{=} -\oint_C \nabla(p/\rho) \cdot dx + \oint_C (u \cdot \nabla u) \cdot dx \\
&\overset{\lim_{\Delta X \to 0}}{=} -\oint_C \nabla(p/\rho) \cdot dx + \oint_C u_i \frac{\partial u_i}{\partial x_j} dx_j \\
&= 0 + \oint_C \frac{\partial}{\partial x_j} \left(\frac{1}{2} u^2 \right) dx_j \\
&= 0 + \oint_C \nabla(\frac{1}{2}u^2) \cdot dx \\
&= 0,
\end{aligned}$$

where $\oint_C \nabla(p/\rho) \cdot dx = 0$, $\oint_C \nabla(\frac{1}{2}u^2) \cdot dx = 0$ by the fundamental theorem of calculus for line integrals. □