10/16/2007

For the videos we saw last week, you can go to

www.efluids.com

Last time, we talked about the Cauchy stress principle, which has to do with the stress on a fluid given by

$$\sigma_{ij} n_j \Delta S.$$

This principle states that the change in momentum w.r.t. to the surface area is zero. This leads to conclude that the forces on the surface is governed by a linear relationship:So

$$0 = \boldsymbol{F}_s(\boldsymbol{n})\Delta S + \boldsymbol{F}_s(-e^{(3)})\Delta S n_3 + \boldsymbol{F}_s(-e^{(1)})\Delta S n_1 + \boldsymbol{F}_s(-e^{(2)})\Delta S n_2.$$

Which tells us that $\boldsymbol{F}_s(n)$ is linearly related to n_1, n_2, n_3. So in general, since n_3 can have components in other directions if we are not dealing with a simple tetrahedron, we have that

$$(f_s)_i = \sigma_{ij} n_j.$$

Also, last time we showed that the equation of motion becomes

$$\rho \frac{D u_i}{Dt} = \frac{\partial}{\partial x_j}\sigma_{ij} + \rho g_i.$$

Note that the stress tensor is symmetric. Why? Consider a cube of side length h. Right now, assume a constant stress tensor (ignore spacial variations). The angular momentum of the fluid in the box/ cube is given by

$$\int_{\text{Box}} (x \times \rho u)\rho V.$$

This will change in response to a net torque. Consider

$$\frac{d}{dt}\int_{\text{Box}} (x \times \rho u)\rho V.$$

This implies that the rate of change $\sim h^4$ ($V \sim h^3$,$x \times \rho u \sim h$ for $\|u\|_\infty < +\infty$) . What is the torque on the fluid?

1. Body torque density (microscopic scale, torque/unit volume) ($\neq x \times \rho g$, etc.). Then, the net torque will be $\propto h^3$. Consider a ferrofluid (e.g. a fluid that contains minute particles of iron). When a magnetic nears the fluid, the tiny particles change direction to align their poles, thus resulting in a torque per small volume. Since the volume is directly proportional to the number of these iron particles, then the net torque $\propto h^3$.

2. Torque from the surface stress.

As $h \rightarrow 0$, we have that

$$0 = (\text{Body torque})_3 h^3 e^{(3)} + (\sigma_{21} h^3 - \sigma_{12} h^3) e^{(3)}.$$

Most fluids have no microscopic body torque density. That is,

$$\sigma_{12} = \sigma_{21}, \text{etc.}$$

Thus, in general, the stress tensor is symmetric.

Angular momentum budget:

The momentum equation is given by

$$\rho\frac{Du_i}{Dt} = \frac{\partial}{\partial x_j}\sigma_{ij}+f_i$$

and

$$\begin{aligned}
\frac{d}{dt}\int_{\text{Box}}(x\times\rho u)\rho V &= \text{rate of change of angular momentum in a material vol, }\Omega\\
&= \int_\Omega x\times\rho\left(\frac{Du}{Dt}\right)d\Omega\\
&= \int_\Omega \epsilon_{ijk}x_j\rho\frac{Du_k}{Dt}d\Omega\\
&= \int_\Omega \epsilon_{ijk}x_j\rho\left(\frac{\partial}{\partial x_p}\sigma_{kp}+f_k\right)d\Omega\\
&= \int_\Omega \epsilon_{ijk}x_j\rho\left(\frac{\partial}{\partial x_p}\sigma_{kp}\right)d\Omega+\int_\Omega \epsilon_{ijk}x_j\rho f_k d\Omega
\end{aligned}$$

since $\rho d\Omega$ is fixed so the rate of change w.r.t. time is not effected. Then, $\int_\Omega \epsilon_{ijk}x_j\rho f_k d\Omega$ is the torque exerted from the body force and

$$\begin{aligned}
\int_\Omega \epsilon_{ijk}x_j\rho\left(\frac{\partial}{\partial x_p}\sigma_{kp}\right)d\Omega &= \int_\Omega \epsilon_{ijk}\rho\left(\frac{\partial}{\partial x_p}x_j\sigma_{kp}-\frac{\partial x_j}{\partial x_p}\sigma_{kp}\right)d\Omega\\
&= \int_\Omega \epsilon_{ijk}\rho\left(\frac{\partial}{\partial x_p}x_j\sigma_{kp}\right)d\Omega
\end{aligned}$$

where $\frac{\partial x_j}{\partial x_p}\sigma_{kp}=0$ since $\epsilon_{ijk}\delta_{jp}\sigma_{kp}=\epsilon_{ijk}\sigma_{kj}=0$ due to symmetry of σ and the antisymmetry of ϵ_{ijk}. Then we use the divergence theorem on this remaining integral to get

$$\int_\Omega \epsilon_{ijk}\rho\left(\frac{\partial}{\partial x_p}x_j\sigma_{kp}\right)d\Omega = \oint_S \epsilon_{ijk}x_j\sigma_{kp}n_p dS,$$

which is the torque by the surface stresses. So the rate of change of the angular momentum is equal to the body torque and the torque exerted by the surface stresses:

$$\frac{d}{dt}\int_{\text{Box}}(x\times\rho u)_i\rho V = \oint_S (x\times f_s)_i dS+\int_\Omega \rho(x\times f)_i d\Omega,$$

where f_s is the body surface force and f is external body force.

Newtonian Viscous Fluid:

We will relate the stress tensor σ_{ij} to the fluid motion.

1. Recall that $(f_s)_i=(\sigma_{ij}n_j)\Delta S$ where f_s is the surface force. This internal surface force in the i direction should be equal to the pressure force externally given by $-p\boldsymbol{n}\Delta S$. Setting the two forces equal gives us $\sigma_{ij}n_j=-pn_i$ and so

$$\sigma_{ij} = -p\delta_{ij}$$

where p is the thermodynamic pressure. Here the $\sigma_{ij}n_j = -p\delta_{ij}n_j = -pn_i$ meaning that the pressure force is normal to the surface.

Fluid motion or statics: If no other effects, then

$$\begin{aligned} \frac{\partial}{\partial x_j}\sigma_{ij} &= \frac{\partial}{\partial x_j}(-p\delta_{ij}) \\ &= -\frac{\partial p}{\partial x_i} \end{aligned}$$

or $-\nabla p$. Now using the momentum equation ($\rho\frac{Du_i}{Dt} = \frac{\partial}{\partial x_j}\sigma_{ij} + f_i$) for an ideal, inviscid (frictionless) fluid, we have

$$\rho\frac{Du}{Dt} = -\nabla p + \rho g$$

which is the Euler equation.

2. Viscous terms: we have that

$$\sigma_{ij} = -p\delta_{ij} + \sigma'_{ij}$$

where σ'_{ij} is the Deviatoric stress tensor. This depends on the relative motion of fluid elements. That is, it depends on the rate of strain S_{ij}, but not the vorticity ω. In a Newtonian fluid, σ'_{ij} depends isotropically, linearly, and on instantaneous, local S_{ij}. So we have that

$$\sigma'_{ij} = A_{ijkm}S_{km}.$$

So we need a tensor of rank 4 that is isotropic. Thus,

$$A_{ijkm} = \alpha\delta_{ij}\delta_{km} + \beta\delta_{ik}\delta_{jm} + \gamma\delta_{im}\delta_{jk}.$$

We can verify that each term is isotropic by taking simple reflections and rotations. So we know that σ_{ij}, S_{km} are symmetric. This means that we can group β, γ together (i.e. $\beta = \beta + \gamma$). Thus,

$$A_{ijkm} = \alpha\delta_{ij}\delta_{km} + \beta(\delta_{ik}\delta_{jm} + \delta_{im}\delta_{jk})$$

where the RHS is symmetric in i, j. Why?

$$\begin{aligned} \sigma'_{ij} &= (\alpha\delta_{ij}\delta_{km} + \beta\delta_{ik}\delta_{jm} + \gamma\delta_{im}\delta_{jk})S_{km} \\ &= \alpha\delta_{ij}\delta_{km}S_{km} + \beta\delta_{ik}\delta_{jm}S_{km} + \gamma\delta_{im}\delta_{jk}S_{km} \\ &= \alpha\delta_{ij}S_{kk} + \beta S_{ij} + \gamma S_{ji}. \end{aligned}$$

Recall that S_{ij} is symmetric so that $S_{ij} = S_{ji}$ so that

$$\sigma'_{ij} = \alpha\delta_{ij}S_{kk} + (\beta + \gamma)S_{ij}$$

and if we let $\beta + \gamma = \beta'$ then

$$\sigma'_{ij} = \alpha\delta_{ij}S_{kk} + 2\beta'S_{ij}$$

where we are using the symmetry of S_{ij}. This tells us that σ'_{ij} is symmetric.

Recall that β' usually written as μ, which is the dynamic viscosity. Also, $\mu = \rho\nu$ where ν is the kinematic viscosity. Next, $S_{kk} = \frac{\partial u_k}{\partial x_k}$ which is exactly $\nabla \cdot u$. For incompressible flow, the $\alpha\delta_{ij}S_{kk}$ term would vanish. Thus, for incompressible flow we have

$$\sigma'_{ij} = 2\mu S_{ij.}$$

Suppose the flow is **compressible**. Then,

$$\sigma_{ij} = -p\delta_{ij} + \alpha\delta_{ij}(\nabla \cdot u) + 2\mu S_{ij}.$$

Let us define the *mechanical pressure* as $p_M = -\frac{1}{3}\sigma_{kk}$. Then,

$$\begin{aligned} p_M &= -\frac{1}{3}(-p\delta_{kk} + \alpha\delta_{kk}(\nabla \cdot u) + 2\mu S_{kk}) \\ &= -\frac{1}{3}(-3p + \alpha 3(\nabla \cdot u) + 2\mu(\nabla \cdot u)) \\ &= p - \alpha(\nabla \cdot u) - \frac{2}{3}\mu(\nabla \cdot u) \\ &= p - (\nabla \cdot u)\left(\alpha + \frac{2}{3}\mu\right). \end{aligned}$$

Summarizing, we get

$$p_M = p - (\nabla \cdot u)\left(\frac{2}{3}\mu + \alpha\right).$$

Stokes' assumption states that

$$p_M = p \;\Rightarrow\; \alpha = -\frac{2}{3}\mu,$$

of course for noncompressible flows. Moreover, this is deriveable from the kinetic theory of ideal gases. We can also define the coefficient of *bulk viscosity* as

$$\kappa = \left(\alpha + \frac{2}{3}\mu\right)$$

so that by Stokes' assumption $\kappa = 0$.

Let us look at an example. Suppose

$$u = \alpha(x_1, x_2, x_3),$$

which is a pure expansion flow. That is, if we draw the vector field, the streamlines moe radially outward. Then, for this compressible flow,

$$\begin{aligned} \sigma'_{ij} &= \alpha\delta_{ij}S_{kk} + 2\beta S_{ij} \\ &= \alpha\delta_{ij}(\nabla \cdot u) + 2\mu S_{ij} \\ &= \alpha\delta_{ij}3\alpha + 2\mu S_{ij}. \end{aligned}$$

Note that $S_{ij} = a$ only on the diagonals and zero everywhere else. Similarly for δ_{ij}. Thus, for $i = 1, 2, 3$ we have

$$\begin{aligned} \sigma'_{ii} &= \alpha(3\alpha) + 2\mu\alpha \\ &= 3\alpha\left(\alpha + \frac{2}{3}\mu\right). \end{aligned}$$

Or we can write this as

$$\sigma'_{ij} \;=\; 3S_{ij}\left(\alpha+\frac{2}{3}\mu\right)$$

where $S_{ij}=\left\{\begin{smallmatrix}\alpha & i=j\\ 0 & i\neq j\end{smallmatrix}\right.$.

10/18/2007

Take home midterm on Friday October 26th, posted at 12 noon and due 1pm october 29th. The material will cover material through Oct. 18th.

Recall that we have our equations of motion:

Mass conservation:

$$\frac{D\rho}{Dt}+\rho(\nabla\cdot u)=0.$$

(Special Case (incompressible flow): $\nabla\cdot u=0$).

Last time we talked about the momentum equation:

$$\rho\frac{Du_i}{Dt} \;=\; \frac{\partial}{\partial x_j}\sigma_{ij}+\rho g_i,$$

where σ_{ij} is the stress tensor. We showed that for a general fluid (not necessarily incompressible) we have that $\sigma_{ij}=-p\delta_{ij}+2\mu S_{ij}+\alpha\delta_{ij}(\nabla\cdot u)$. Stokes' model tells us that the bulk viscosity term, is zero so that $\kappa=\alpha+\frac{2}{3}\mu=0\Rightarrow\alpha=-\frac{2}{3}\mu$.

From the momentum equation, let us take the divergence of σ_{ij} given by $\frac{\partial}{\partial x_j}\sigma_{ij}$:

$$\begin{aligned}
\frac{\partial}{\partial x_j}\sigma_{ij} &= \frac{\partial}{\partial x_j}(-p\delta_{ij}+2\mu S_{ij}+\alpha\delta_{ij}(\nabla\cdot u))\\
&= -\frac{\partial}{\partial x_j}p\delta_{ij}+2\mu\frac{\partial}{\partial x_j}\left(\frac{1}{2}\left[\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right]\right)+\alpha\frac{\partial}{\partial x_j}\delta_{ij}(\nabla\cdot u)\\
&= -\frac{\partial p}{\partial x_i}+\mu\frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_j}\right)+\mu\frac{\partial}{\partial x_j}\left(\frac{\partial u_j}{\partial x_i}\right)+\alpha\frac{\partial}{\partial x_i}(\nabla\cdot u)\\
&= -\frac{\partial p}{\partial x_i}+\mu\nabla^2u_i+\mu\frac{\partial}{\partial x_i}\left(\frac{\partial u_j}{\partial x_j}\right)+\alpha\frac{\partial}{\partial x_i}(\nabla\cdot u)\\
&= -\frac{\partial p}{\partial x_i}+\mu\nabla^2u_i+\mu\frac{\partial}{\partial x_i}(\nabla\cdot u)+\alpha\frac{\partial}{\partial x_i}(\nabla\cdot u)\\
&= -\frac{\partial p}{\partial x_i}+\mu\nabla^2u_i+(\alpha+\mu)\frac{\partial}{\partial x_i}(\nabla\cdot u)\\
&= -\frac{\partial p}{\partial x_i}+\mu\nabla^2u_i+\frac{1}{3}\mu\frac{\partial}{\partial x_i}(\nabla\cdot u)
\end{aligned}$$

since $\alpha+\mu=(-\frac{2}{3}\mu+\mu)=\frac{1}{3}\mu$ by Stokes' assumption. These are the viscosity terms. Hence, the momentum equation becomes

Newtonian viscous fluids: $\rho\frac{Du_i}{Dt}=-\frac{\partial p}{\partial x_i}+\mu\nabla^2u_i+\frac{1}{3}\mu\frac{\partial}{\partial x_i}(\nabla\cdot u)+\rho g_i$ where $\mu\nabla^2u_i+\frac{1}{3}\mu\frac{\partial}{\partial x_i}(\nabla\cdot u)$ on the RHS are the viscous terms (Stokes').

Taking the divergence of σ_{ij} we get

$$\rho\frac{Du_i}{Dt} \;=\; -\frac{\partial p}{\partial x_i}+\mu\nabla^2u_i+\frac{1}{3}\mu\frac{\partial}{\partial x_i}(\nabla\cdot u)+\rho g_i.$$

Boundary conditions:

1. Kinematic/ Mass conservation.

   a) Fluid + Wall. For a fixed wall $u\cdot n=0$ which means that the fluid does not cross the wall (i.e. no normal flow).

   b) Moving interface between immiscible (does not mix) fluids. The condition here is that $(u^{(1)}-v)\cdot n=(u^{(2)}-v)\cdot n=0$ (no material exchange across boundary).

2. Dynamic conditions.

$$\rho\frac{Du_i}{Dt} = \frac{\partial\sigma_{ij}}{\partial x_j}+\rho g_i.$$

Pill box volume is area of face ΔA with thickness δh with $\Delta A\gg(\delta h)^2$. We can consider the instantaneous volume integral:

$$\int_V\rho\frac{Du_i}{Dt}\,dV = \oint_S\sigma_{ij}n_j\,dS+\int_V\rho g_i\,dV.$$

All terms in the integrand are finite. Thus we have that $\int_V\rho\frac{Du_i}{Dt}\,dV\leqslant c|V|$ so that

$$\begin{aligned}\mathcal{O}(\Delta A\cdot\delta h) &= (\text{surface integral on pill box})+\mathcal{O}(\Delta A\cdot\delta h)\\ &= ({\sigma_{ij}}^{(2)}\hat{n}_j-{\sigma_{ij}}^{(1)}\hat{n}_j)\Delta A+\text{side wall}+\mathcal{O}(\Delta A\cdot\delta h)\\ &= ({\sigma_{ij}}^{(2)}\hat{n}_j-{\sigma_{ij}}^{(1)}\hat{n}_j)\Delta A+(\propto(\Delta A)^{1/2}\delta h+\mathcal{O}(\Delta A\cdot\delta h)\end{aligned}$$

So as $\frac{\delta h}{(\Delta A)^{1/2}}\to0$ we are left with $({\sigma_{ij}}^{(2)}\hat{n}_j-{\sigma_{ij}}^{(1)}\hat{n}_j)\Delta A$. Now let $\Delta A\cdot\delta h\to0$ so the balance is

$$\begin{aligned}0 &= [\sigma_{ij}\hat{n}_j]_1^2\\ \Rightarrow\sigma_{ij}^{(2)}\hat{n}_j &= \sigma_{ij}^{(1)}\hat{n}_j.\end{aligned}$$

Thus, the **stress vector** is continuous (not necessarily the stress tensor).

Stress Vector:

Let $\sigma_{ij}\hat{n}_j=\mathcal{S}_i$ the stress vector (for example, suppose we have a flow over a flat plate and the normal is $\boldsymbol{n}=(0,\ 1,\ 0)$. Then $\sigma_{ij}\hat{n}_j$ represents the stress in the direction of this normal. It has components in all three directions from the other forces. Now, the tangential direction to $\boldsymbol{n}$ is (1,0,0). Now, we project onto the tangent to find the force in the tangent). This has two components: the normal stress $\mathcal{S}_in_i$ and the tangential stress $\mathcal{S}_i-(\mathcal{S}\cdot\hat{n})\hat{n}$. The normal stress, is given by $\sigma_{ij}\hat{n}_j\hat{n}_i$. Then,

$$\begin{aligned}\sigma_{ij} &= -p\delta_{ij}+2\mu S_{ij}\,(+\nabla\cdot u\ \text{ terms dropped})\\ \Rightarrow &= -p\delta_{ij}\hat{n}_j\hat{n}_i+2\mu S_{ij}\hat{n}_j\hat{n}_i\\ &= -p+2\mu S_{ij}\hat{n}_j\hat{n}_i.\end{aligned}$$

So with an inviscid flow ($2\mu S_{ij}\hat{n}_j\hat{n}_i=0$), the normal stress is continuous w.r.t. p is continuous ($p^{(1)}=p^{(2)}$). With a viscous flow,

$$(-p+2\mu S_{ij}\hat{n}_j\hat{n}_i)^{(1)} = (-p+2\mu S_{ij}\hat{n}_j\hat{n}_i)^{(2)}.$$

Warning: The surface tension is a concentrated surface force. Can contribute to normal stress balance and can give a change a pressure. For instance, consider a gas bubble. The surface tension keeps the bubble spherical. Excess pressure in the bubble is given by $\frac{2T}{R}$ where R is the radius of the bubble, T is the coefficient of the surface tension ($\sim 70\,\text{dyne}/\text{cm}$).

The tangential stress in continuous - viscous term only (shear stress). Consider again a rigid wall. What condition applies? We have the kinematic condition $u \cdot n = 0$. As for the tangential motion, s.t. $u \times n = 0$ (where $u \times n$ is in the tangential direction). This is a no slip condition. Thus, the viscous stresses create a finite strain rate $\frac{1}{2}(\nabla u + \nabla u^T)$ but the velocity(tangential) is equal to the tangential velocity of the wall $= 0$.

Kinetic Theory of gases:

If we have a large number of gas molecule-wall collisions then we take the average diffuse reflection. Then on average, we have there is no slip at the wall. At low density, we can have partial slip. Recall the Knudsen number $=$ mean free path/ flow scale. So when the Knudsen number is $\geqslant 1$ then we get a partial slip. For example, an aerosol particle (less than one micron) we have partial slip. This is nown as cunningham's slip correction.

Boundary conditional with fixed walls: $u \cdot n = 0$ (normal flow $= 0$). and in a viscous fluid $u \times n = 0$ (no slip).

Consider a 2 fluid system with simple shear flow. We have that

$$\begin{aligned} u_1 &= u_0 + \beta^{(2)} y \\ u_1 &= u_0 + \beta^{(1)} y, \end{aligned}$$

where $\beta^{(i)}$ are constants. We have a purely tangential flow with no normal flow. Moreover, the velocity is continuous at $y = 0$ (otherwise we would get enormous viscous shear stresses). The normal stress, where n_j is the normal, is given by

$$-p + 2\mu S_{ij} n_i n_j.$$

If $n = (0, 1, 0)$ and we have $S_{ij} = \begin{pmatrix} 0 & \beta & 0 \\ \beta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, where $S_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$, so that $2S_{ij} n_j = \beta e^{(1)}$ and $S_{ij} n_i n_j = 0$. So we only have shear viscous stress and no viscous normal stress. Thus, $p^{(1)} = p^{(2)}$.

What about the tangential (shear) stress? The viscous stress vector

$$\begin{aligned} 2\mu S_{ij} n_j &= \mu \begin{pmatrix} 0 & \beta & 0 \\ \beta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \\ &= \begin{pmatrix} \mu\beta & 0 & 0 \end{pmatrix} \end{aligned}$$

so that $\mu^{(1)} \beta^{(1)} = \mu^{(2)} \beta^{(2)}$. So we have that the velocity is continuous and the velocity gradient changes.