10.2.2007

Dynamics:

Recall that we had an equation for mass conservation

$$\frac{D\rho}{Dt} + \rho(\nabla \cdot u) = 0$$

or

$$\frac{\partial\rho}{\partial t} + \nabla \cdot (\rho u) = 0$$

where the density is ρ and the velocity vector is u. We can write these equations in integral form. If we have a steady flow, where $\frac{\partial\rho}{\partial t} = 0$ (and $\frac{\partial u}{\partial t} = 0$) then $\nabla \cdot (\rho u) = 0$ and $\oint_S \rho u \cdot n dS = 0$ (note that $\oint_S \rho u \cdot n dS = 0$ is the mass **outflux**). Then,

$$\int_{S_A} \rho u \cdot n dS + \int_{S_B} \rho u \cdot n dS = 0$$

where the outward normal is n and the S has contributions only from S_A and S_B. Thus, the rate of mass outflow across S_A equal the mass flow rate across S_B (note that $\rho u \cdot n dS$ is in terms of mass/time). We are going to focus on incompressible flows : $\nabla \cdot u = 0$. For example, water or air. This means that the change in density with fluid motion is neglible. For example, the mach number, which is u_0/c where c_{air} is the speed of sound (300 m/s) we look at $\text{Ma} \ll 1$. Usually, this means that ρ is constant. If we look at $\frac{D\rho}{Dt} + \rho(\nabla \cdot u)$ then $\nabla \cdot u = 0$ implies that the total time derivative of ρ is unchanging.

Now we will look at the **momentum equation**. Consider the navier stokes equation:

$$\rho\frac{Du_i}{Dt} = -\frac{\partial p}{\partial x_i} + \mu\nabla^2 u_i + \rho g_i$$

for $\nabla \cdot u_i = 0$. We have that $p(x,t)$ is the pressure, μ is a constant that is the dynamic viscosity, and the acceleration term g which is the acceleration due to gravity. The pressure and viscosity terms represent internal fluid forces. Here, a simple Newtonian fluid, viscosity μ (e.g. water and air are Newtonian fluids). The question arises how important is pressure and viscosity. Pressure, in some cases in related to the gravitational forces (think of moving deeper in water where the pressure increases due to the weight of water).

Also, we have the **Euler equation** - ideal inviscid flow:

$$\rho\frac{Du_i}{Dt} = -\frac{\partial p}{\partial x_i} + \rho g_i$$

where we completely ignore the viscous effects. Typically we use this equation for high speed gas flows, pressure driven flows, and compressible gas dynamics.

The third type are the **Stokes Equations**:

$$0 = -\frac{\partial p}{\partial x_i} + \mu\nabla^2 u_i$$

$$\nabla \cdot u = 0$$

which is a viscous dominated flow (negligible fluid inertia).

When is viscosity important? For now, we are given the above types of equation. We have that

$$\rho\left(\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j}\right) = -\frac{\partial p}{\partial x_i} + \mu\nabla^2 u_i$$

where $\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j} = \frac{D u_i}{D t}$. Let us focus on steady flows where $\frac{\partial u}{\partial t} = 0$. Consider an entry flow in a tube. That is, when the fluid is first entering a tube. We have flow development,

$$\frac{\partial u_1}{\partial x_1} \neq 0.$$

We can estimate $u_j \frac{\partial u_i}{\partial x_j} = (u \cdot \nabla u_i) \sim u_0 \frac{u_0}{H}$ (since we are monitoring the flow close the entry H is a good length scale for the x_1 displacement. Why? Well recall that $\frac{\partial u_i}{\partial x}$ is the rate of change of the velocity in the x direction. For flow development, if we say that the velocity after a certain point is u_0 and it takes about the length of the opening of the tube for the flow to develop, then we are looking at a rate of change of velocity of the order of u_0/H. That is, in H distance, we have changed our velocity by u_0, in terms of the order.In general this is subjective, but empirical data backs up these claims for flows such as pipe flows or flows over a flat plate. So we can say that $\frac{\partial u_i}{\partial x} = \mathcal{O}(\frac{U}{L})$ assuming that the components of $\boldsymbol{u}$ change by the amounts of order U over their distances L. But if we assume this, then it is not suprising that the second derivatives are of order U/L^2. Thus, even if our length scale is wrong, we are still being consistent).

We can also estimate $\mu \nabla^2 u_1 \sim \mu \frac{\partial^2 u_1}{\partial x^2}$ which is the most relevant (since the flow w.r.t. other directions can be negibible). Then, $\mu \nabla^2 u_1 \sim \mu \frac{\partial^2 u_1}{\partial x^2} \sim \mu \frac{u_0}{H^2}$. The ratio : $\rho \frac{u_0^2}{H}$ vs $\mu \frac{u_0}{H^2}$. Then, the ratio is

$$\rho \frac{u_0 H}{\mu} \text{ vs } 1.$$

$\rho \frac{u_0 H}{\mu}$ is called the reynold's number: $\mathrm{Re} = \frac{u_0 H}{v}$ where v is the kinematic viscosity $= \mu/\rho$. If the Reynold's number is large $\mathrm{Re} \gg 1$ then the viscous forces are small - inertia dominated flows - so we can use the Euler equations (if $\mathrm{Re} \gg 1$ then the term multiplied by ρ is a lot larger than that multiplied by μ). If $\mathrm{Re} \ll 1$ then the viscous forces dominate and we can ignore intertial forces so we use Stokes equations.

In water $v = .01\,\mathrm{cm}^2/\mathrm{sec}$, air $v = .15\,\mathrm{cm}^2/\mathrm{sec}$ (in this case the $\rho_{\mathrm{water}} = 10^3 \times \rho_{\mathrm{air}}$).

Suppose we have water flow with $H = 10\mathrm{cm}$, $u_0 = 10\,\mathrm{cm/sec}$. Then, $\mathrm{Re} = 10^4$.

Suppose we have a water + microorganism $L \sim 1 - \mu m$ and $u_0 \sim 100 \mu m/\mathrm{sec}$. Then $\mathrm{Re} = 10^{-3}$. In this case the viscous forces are very important.

Further issues arise, looking to our differential equations: What boundary conditions do we need or have? AU flows - mass conservation $v \cdot n = 0$ at a rigid wall (fixed). That is, fluid does not cross a fixed rigid wall. In viscous flows, we require no-slip so that $u \times n = 0$ at a fixed rigid wall.

In Stokes Flow:

$$\begin{aligned} -\frac{\partial p}{\partial x_i} + \mu \nabla^2 u_i &= 0 \\ \nabla \cdot u &= 0. \end{aligned}$$

The unknowns are u_1, u_2, u_3 and p. Also, we are given the constant ρ and μ. So we have three equations given by $-\frac{\partial p}{\partial x_i} + \mu \nabla^2 u_i = 0$ and $\nabla \cdot u = 0$. Thus, we have 4 equations and 4 unknowns. How do we solve this? First, let us take the divergence of the first equation:

$$\nabla \cdot \left(-\frac{\partial p}{\partial x_i} + \mu \nabla^2 u_i \right) = 0$$

Thus, we get that $\nabla^2 p = 0$ since $\nabla \cdot \mu \nabla^2 u_i = 0$ (divergence of a scalar function is zero). We can also take the curl, to get that

$$\begin{aligned} \nabla \times \left(-\frac{\partial p}{\partial x_i} + \mu \nabla^2 u_i \right) &= 0 \\ \nabla^2 \omega &= 0 \end{aligned}$$

where $\omega = \nabla \times u$. In summary, flow equations are linear in u, p determined by instantaneous boundary conditions.

Inviscid, pressure-driven flows:

$$\begin{aligned} \rho\frac{Du_i}{Dt} &= -\frac{\partial p}{\partial x_i} \\ \nabla \cdot u &= 0. \end{aligned}$$

If we have high pressure on one side and low pressure on the other, the flow moving from high to low, the pressure will decrease so that $\frac{\partial p}{\partial x_i} < 0$ and $-\frac{\partial p}{\partial x_i} > 0$. Thus, u_1 accelerates. So pressure is an agent that can accelerate or slow down the fluid.

Let us focus on Steady Flows:

$$\begin{aligned} \rho(u \cdot \nabla u) &= -\nabla p \\ \nabla \cdot u &= 0. \end{aligned}$$

Mass conservation tells us that $u_1 A_1 = u_2 A_2$ (flow rate on the left is equal to the flow rate on the right with constant ρ). Here we are using the averaged velocities u across the sectioned areas. If $A_1 > A_2$ then we need $u_1 < u_2$ in order to conserve mass. Thus, the flow accelerates from u_1 to u_2. So $p_1 > p_2$.

Bernoulli's Equation:

$$\begin{aligned} u \times \omega &= u \times (\nabla \times u) \\ (u \times \omega)_i &= u_j \frac{\partial u_j}{\partial x_i} - u_j \frac{\partial u_i}{\partial x_j} \\ &= \frac{\partial}{\partial x_i}\left(\frac{1}{2}u_j^2\right) - u_j \frac{\partial u_i}{\partial x_j} \\ &= \nabla(\frac{1}{2}u^2)_i - u \cdot \nabla u_i \end{aligned}$$

Thus,

$$\rho\left\{\nabla(\frac{1}{2}u^2) - u \times \omega\right\} = -\nabla p$$

and constant density gives us

$$\begin{aligned} \nabla(\frac{1}{2}u^2) + \frac{1}{\rho}\nabla p &= u \times \omega \\ \nabla\left\{\frac{1}{2}u^2 + \frac{p}{\rho}\right\} &= u \times \omega \end{aligned}$$

and

$$u \cdot \nabla\left\{\frac{1}{2}u^2 + \frac{p}{\rho}\right\} = 0$$

since $u \cdot (u \times \omega) = 0$. Thus, $u \cdot \nabla f = 0$ tells us that $f =$ constant on a streamline. That is, $u \cdot \nabla f$ is the directitonal derivative of f in the direction of u. So if f does not change in the direction of u then it is constant along u. But what is the function that is always in the direction of u? That is is exactly the streamlines! So since the streamline is everywhere tangent with u, f is constant on the streamline (i.e. f in the direction of u is non changing. But f in the direction of u are the streamlines so f is constant along the streamlines). Thus, following a streamline, $\frac{1}{2}u^2 + \frac{p}{\rho} = c$.

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Dyanmics:

Mass conservation plays an important role:

$$\frac{D\rho}{Dt} + \rho \nabla \cdot u \; = \; 0.$$

Let us consider Force balance:

In the Lagrangian sense, we have some material volume Ω bounded by a surface Σ. We can think of Ω as the finite sum of small volume elements. So as the volume moves, the surface will deform with the flow. The main thing is that we are tracking a fixed mass of fluid. Let us look at the rate of change of the momentum of the fluid (i.e. the acceleration): If $\int_{\Omega} \rho u d\Omega$ is the momentum, then we want

$$\frac{d}{dt}\int_{\Omega} \rho u d\Omega \; = \; \frac{d}{dt}\sum_{n=1}^{N} \rho u \Delta\Omega^{(n)}$$

In Lagrangian coordinates, we will use the material derivative, so that

$$\frac{d}{dt}\sum_{n=1}^{N} \rho u \Delta\Omega^{(n)} \; = \; \sum_{n=1}^{N} \frac{d}{dt}(\rho u \Delta\Omega^{(n)})$$

and note that $\rho\Delta\Omega^{(n)}$ is constant since we have a fixed mass (this is the density multiplied by the small volume).[1] Then,

$$\begin{aligned} \sum_{n=1}^{N} \frac{d}{dt}(\rho u \Delta\Omega^{(n)}) \; &= \; \sum_{n=1}^{N} \rho\Delta\Omega^{(n)} \frac{Du}{Dt} \\ &= \; \int_{\Omega} \rho \frac{Du}{Dt} d\Omega. \end{aligned}$$

Now let us consider the forces on the volume of the fluid:

1. **Body Forces** - action at a distance. For example, gravity produces a force ρg per unit volume, where g is a gravitational acceleration ($9.81 m/\sec^2$). Also, elctromagnetic forces can act upon the fluid, especially if the fluid is an electrical conductor, such as mercury. Here the body force is $J \times B$ where J is the current density (amps/m^2)and B is the magnetic field.

2. **Surface Forces** - acting locally between fluid molecules. These forces are transmitted across Σ. An example is the pressure force (Newtons/m^2) - an effect of molecular collisions from adjacent molecules. Furthermore, the pressure forces acts even if there is no flow. Also, the force of pressure is proportional to the surface area. A general principle is that the surface forces is proportional to the area of contact. That is, the surface force, $\boldsymbol{F}_s$, $\propto$ $\Delta\Sigma$, which depends on x, t and depends on flow. To characterize the surface, we need $\Delta\Sigma$ and n, the unit normal from that surface. This is where we come to the idea of the stress tensor (Cauchy's Stress Principle).

Cauchy's Stress Principle:

1. We have that

$$\frac{d}{dt}u(\rho\Delta\Omega) = \frac{du}{dt}(\rho\Delta\Omega) + u\frac{d}{dt}(\rho\Delta\Omega) = \frac{du}{dt}(\rho\Delta\Omega) + 0 = \frac{Du}{Dt}(\rho\Delta\Omega).$$

Consider a small tetrahedral volume where the sloping face is $n\Delta S$. If we look at the Ox_1x_2 plane, the normal will be $-e^{(3)}$, and the area is $\Delta S(n\cdot e^{(3)}) = \Delta S n_3$ (To obtain the surface area of the bottom triangle in the x_1x_2 plane, we need to determine the height. We already know that the base is the same as the base of the sloping face. The height is exactly the height of the sloping face times $n\cdot e^{(3)}$). On the Ox_2x_3 the normal is $-e^{(1)}$ and the area is $\Delta S(n\cdot e^{(1)})$. So the force balance is given by

$$\begin{aligned}\left(\rho\frac{D\boldsymbol{u}}{Dt}\right)\Delta\Omega &= \boldsymbol{F}_s(\boldsymbol{n})\Delta S+\boldsymbol{F}_s(-e^{(3)})\Delta S n_3+\boldsymbol{F}_s(-e^{(1)})\Delta S n_1+\boldsymbol{F}_s(-e^{(2)})\Delta S n_2\\ (\text{Inertia}-\text{within}) &= (\text{surface forces})\end{aligned}$$

where the intertia is the rate of change of the momentum (Also, this is an instantenuous view). So there is no variation with position, x, and only varies with $\boldsymbol{n}$, the normal vector to the surface. Note that we are looking at a very small element of volume so that we can neglect the forces of gravity. We have a finite ρ, $\frac{D\boldsymbol{u}}{Dt}$, $\boldsymbol{F}_s$. These are not changing with the size of the volume. Then, the intertia time is $\propto \Delta\Omega = (\delta L)^3$. As for the suface area, the RHS $\propto \Delta S = (\delta L)^2$. Now as $\delta L \rightarrow 0$, then the RHS has to be equal to zero. That is, for L small enough, the left hand size is much smaller then the RHS. So

$$0 = \boldsymbol{F}_s(\boldsymbol{n})+\boldsymbol{F}_s(-e^{(3)})n_3+\boldsymbol{F}_s(-e^{(1)})n_1+\boldsymbol{F}_s(-e^{(2)})n_2. \tag{1}$$

and if we look at the respective forces, then the dependence on the normal n is linear. That is, $\boldsymbol{F}_s(n)$ is linearly related to n_1, n_2, n_3. Then,

$$\boldsymbol{F}_s(n)_i = A_{ij}n_j$$

where A is some matrix and the force is a vector function in $\mathbb{R}^3$. We can rewrite (1) as

$$(\boldsymbol{f}_s)_i = (A_{ij}n_j)$$

or

$$(\boldsymbol{f}_s)_i = (\sigma_{ij}n_j),$$

where σ is the stress tensor (what we defined as A before). Since f_s is a vector and $\boldsymbol{n}$ is a vector, then σ_{ij} is a tensor (note that if we had a matrix with only 1 in the diagonal and zero everywhere else, this is not a tensor).

Now, looking at a larger scale, with the effect of gravity added, we get the equilibirum equation

$$\begin{aligned}\frac{d}{dt}\int_\Omega \rho u\, d\Omega &= \int_\Omega \rho g\, d\Omega+\oint_\Sigma f_s d\Sigma\\ &= (\text{Body force})+(\text{Surface force}).\end{aligned}$$

Note again that this is an instantaneous view: this way we can identify Σ with a coincident fixed surface S. Furthermore,

$$\int_\Omega \rho\frac{Du_i}{Dt}d\Omega = \int_\Omega \rho g_i\, d\Omega+\oint_\Sigma \sigma_{ij}n_j\, d\Sigma.$$

In order to get this integral equation into differerential form, we let $\Omega\rightarrow 0$. We can convert the last integral, using the Divergence theorem, into a volume integral: $\int_\Omega \frac{\partial}{\partial x_j}\sigma_{ij}\, d\Omega$. Hence,

$$\rho\frac{Du_i}{Dt} = \rho g_i+\frac{\partial}{\partial x_j}\sigma_{ij}$$

which is the momentum equation. The tricky part is defining the stress tensor σ_{ij}.

Stress Tensor:

The pressure is f_s . **Note** that $\oint_\Sigma \sigma_{ij}\, n_j\, d\Sigma$ is the force on the fluid inside S, from the fluid outside, and $\boldsymbol{n}$ is the unit *outward* normal from S. Then, the pressure force will be $-p\boldsymbol{n}\Delta S$ (acting in the direction opposite to the normal). Then the surface force acting on n_i is given by $\sigma_{ij}n_j = -pn_i$ and so

$$\sigma_{ij} = -p\delta_{ij}\,\text{(pressure term)}.$$

In hydrostatics (fluid statics) we would have

$$\begin{aligned} 0 &= \rho g_i + \frac{\partial \sigma_{ij}}{\partial x_j} \\ &= \rho g_i + \frac{\partial}{\partial x_j}(-p\delta_{ij}) \\ &= \rho g_i - \frac{\partial p}{\partial x_i} \end{aligned}$$

since the acceleration is zero ($\frac{Du_i}{Dt} = 0$). Hence, $0 = \rho \boldsymbol{g} - \nabla p$. A typical example is the swimming pool. At the static surface, the pressure is continuous. Then the fluid (water) p at $x_3 = 0$ is p_{atmos} (pressure does not change any other direction other than x_3). Then, the static equation becomes

$$\begin{aligned} 0 &= -\rho \boldsymbol{g} - \frac{\partial p}{\partial x_3} \\ &\Rightarrow \\ \frac{\partial p}{\partial x_3} = \frac{dp}{dx_3} &= -\rho \boldsymbol{g} \\ &\Rightarrow \\ p &= -\rho \boldsymbol{g} x_3 + p_{\text{atmos}}. \end{aligned}$$

Fluid Motion:

1. Stress tensor is (usually) symmetric.
2. The stress tensor due to flow is based on the rate of strain (the relative motion of fluid elements).

Consider the moment of momentum (ρu is the linear momentum and x is the displacement vector from the origin):

$$\begin{aligned} \frac{d}{dt}\int_\Omega \epsilon_{ijk}\, x_j\, \rho u_k\, d\Omega &\overset{\text{product rule}}{=} \int_\Omega \epsilon_{ijk}\left[\frac{dx_j}{dt}\,\rho u_k d\Omega + x_j \frac{Du_k}{Dt}\rho d\Omega\right] \\ \frac{d}{dt}\left(\int x \times \rho u\, d\Omega\right) &= \int_\Omega \epsilon_{ijk}\left[u_j\, \rho u_k d\Omega + x_j \frac{Du_k}{Dt}\rho d\Omega\right] \\ &= \int_\Omega \epsilon_{ijk} x_j \frac{Du_k}{Dt}\rho d\Omega \\ &= \int_\Omega \epsilon_{ijk} x_j \left[\rho g_k + \frac{\partial}{\partial x_m}\sigma_{km}\right] d\Omega \end{aligned}$$

since $\epsilon_{ijk} u_j\, \rho u_k = u \times u = 0$. $\int_\Omega \epsilon_{ijk} x_j \frac{Du_k}{Dt}\rho d\Omega$ for a small volume $\Delta\Omega \propto (\delta L)^4$ if we assume $\|\frac{Du_k}{Dt}\|_\infty < +\infty$. This means that

$$\int_\Omega \epsilon_{ijk} x_j \frac{Du_k}{Dt}\rho d\Omega \sim (\delta L)^4.$$

As for $\int_\Omega \epsilon_{ijk}x_j\Big[\rho g_k + \frac{\partial}{\partial x_m}\sigma_{km}\Big]d\Omega$, the first term in the sum the body force term, is $\propto (\delta L)^4$ becuase x_j gives us one order and the $d\Omega$ gives us a third order term. For the second term, the divergence theorem says that

$$\int_\Omega \frac{\partial}{\partial x_m}\{\epsilon_{ijk}x_j\sigma_{km}\}d\Omega \;=\; \oint_\Sigma \epsilon_{ijk}\,x_j\sigma_{km}n_m d\Sigma$$

or more typically as

$$\int_\Omega \frac{\partial}{\partial x_m}F_m\,d\Omega \;=\; \oint_\Sigma F_m n_m d\Sigma$$

where $F_m = \epsilon_{ijk}x_j\sigma_{km}$. But, by the product rule, $\int_\Omega \frac{\partial}{\partial x_m}\{\epsilon_{ijk}x_j\sigma_{km}\}d\Omega$ becomes

$$\int_\Omega \epsilon_{ijk}x_j\Big(\frac{\partial}{\partial x_m}\sigma_{km}\Big)d\Omega + \int_\Omega \epsilon_{ijk}\,\frac{\partial x_j}{\partial x_m}\sigma_{km} \;=\; \oint_\Sigma \epsilon_{ijk}\,x_j\sigma_{km}n_m d\Sigma.$$

Thus,

$$\begin{aligned}\int_\Omega \epsilon_{ijk}x_j\Big(\frac{\partial}{\partial x_m}\sigma_{km}\Big)d\Omega &= \oint_\Sigma \epsilon_{ijk}x_j\sigma_{km}n_m d\Sigma - \int_\Omega \epsilon_{ijk}\,\frac{\partial x_j}{\partial x_m}\sigma_{km}d\Omega\\ &= \oint_\Sigma \epsilon_{ijk}x_j\sigma_{km}n_m d\Sigma - \int_\Omega \epsilon_{ijk}\,\delta_{jm}\sigma_{km}d\Omega.\end{aligned}$$

We can assume that we have a finite divergence of the stress tensor $\Big(\nabla\cdot\sigma_k = \frac{\partial}{\partial x_m}\sigma_{km}\Big)$. Recall that we have

$$\begin{aligned}\frac{d}{dt}\int_\Omega \epsilon_{ijk}\,x_j\,\rho u_k\,d\Omega &= \int_\Omega \epsilon_{ijk}x_j\frac{Du_k}{Dt}\rho d\Omega\\ &= \int_\Omega \epsilon_{ijk}x_j\Big[\rho g_k + \frac{\partial}{\partial x_m}\sigma_{km}\Big]d\Omega,\end{aligned}$$

where $\int_\Omega \epsilon_{ijk}x_j\frac{Du_k}{Dt}\rho d\Omega \sim \mathcal{O}(\delta L)^4$ and $\int_\Omega \epsilon_{ijk}x_j\rho g_k\,d\Omega \sim \mathcal{O}\,(\delta L)^4$. Also, we have

$$\Big(\int_\Omega \frac{\partial}{\partial x_m}\{\epsilon_{ijk}x_j\sigma_{km}\}d\Omega \;=\Big)\;\begin{aligned}\int_\Omega \epsilon_{ijk}x_j\frac{\partial\sigma_{km}}{\partial x_m}d\Omega &\sim \mathcal{O}(\delta L)^4\\ \oint_\Sigma \epsilon_{ijk}x_j\sigma_{km}n_m d\Sigma &\sim \mathcal{O}(\delta L)^{3<x<4}\\ \int_\Omega \epsilon_{ijk}\,\delta_{jm}\,\sigma_{km}d\Omega &\sim \mathcal{O}(\delta L)^3.\end{aligned}$$

which all go to zero as $\delta L \to 0$. Hence, $\frac{1}{\delta L^3}\int_\Omega \epsilon_{ijk}\,\delta_{jm}\,\sigma_{km}d\Omega \approx 0$ and so $\epsilon_{ijk}\sigma_{kj} = 0 \Rightarrow \sigma_{kj} = \sigma_{jk}$. We neglected to talk about torques and angular momentum.

(The torque balance is given by

$$\oint_S x\times f_s\,dS$$

where $x\times f_s$ is the torque on the fluid by surface forces.

Now we consider

$$\sigma_{ij} \;=\; -\,p\delta_{ij} + \sigma'_{ij}$$

where σ'_{ij} is due to the flow. In a simple newtonian fluid, we suppose that σ'_{ij} depends linearly on S_{ij}, the strain rate. Since they are both tensors of rank 2, then

$$\sigma'_{ij} = A_{ijkm} S_{km}$$

which has 81 terms. Here we have an isotropic relationship and so there is no preferred orientation. The general form of an isotropic tensor of rank 4 is

$$A_{ijkm} = \alpha \delta_{ij} \delta_{km} + \beta \delta_{ik} \delta_{jm} + y \delta_{im} \delta_{jk}.$$

To show this if suffices to show the relationship between the terms with $\pi/2$ and π rotations.)