9/25/07

(TA help section Wednesdays at 5pm)

Let us conclude our discussion of Cartesian Tensors:

Theorem 1. *(tensor detection theorem) We have T_{ijk} is a rank 3 system. Is this a tensor? We do know that $T_{ijk}b_k = S_{ij}$. If b_k is any vector and S_{ij} is a rank 2 tensor then T_{ijk} is a rank 3 tensor.*

That is, if the unknown contracted with an arbitrary tensor gives a tensor, then the unknown is a tensor. To prove this, we consider $T'_{ijk}b'_k = S'_{ij} = Q_{ip}Q_{jq}S_{pq}$ since S_{ij} is a tensor. Also $b'_k = Q_{kr}b_r$. Pluggin this in we get $T'_{ijk}Q_{kr}b_r = Q_{ip}Q_{jq}S_{pq}$. But $S_{pq} = T_{pqn}b_n$ and now we have that $T'_{ijk}Q_{kr}b_r = Q_{ip}Q_{jq}T_{pqn}b_n$. We can replace the dummy index $n \to r$ so that $T'_{ijk}Q_{kr}b_r - Q_{ip}Q_{jq}T_{pqr}b_r = 0, \forall b$. This is only true if $(\,\cdot\,) = 0$. Thus, $T'_{ijk}Q_{kr} = Q_{ip}Q_{jq}T_{pqr}$. Now multiply through by Q_{nr} and we get

$$\begin{aligned} \delta_{nk}T'_{ijk} &= Q_{ip}Q_{jq}Q_{nr}T_{pqr} \\ \Rightarrow T'_{ijk} &= Q_{ip}Q_{jq}Q_{kr}T_{pqr} \end{aligned}$$

and we are done.

Product Rule:

$\epsilon_{ijk}\epsilon_{pqr} = \delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp}$. We have that ε_{ijk} is a pseudo tensor so that when we multiply the two we get a proper tensor. We have i, j, p, q as free indices (total of 81 terms) (i.e. once we know i, j, p, q we know that there can only be one non zero value of k and r. For example, if $i, j = 1, 2$ then we are only interested in $k = 3$). So right away we reduce our possibilities by 3^2 corresponding to the varieties of k and r. Now we have 3^4 possibilities. Notice that we have antisymmetry in i, j on the LHS and RHS. That is, we just think of ij as the free indices, we have nine terms, but because of antisymmetry, we reduce the terms by a factor of 3. For example, if we know $i = 1, j = 2$ we know $i = 2, j = 1$. Of course, we know what happens if $i = j$ so that we only have to worry about three cases of i and j so that we reduce our problem by a third (we already know what happens if $i = j$). Now we have 3^3 terms to determine. Note that if A_{ij} is antisymmetric matrix then we have a total of 9 terms such that

$$\begin{pmatrix} 0 & a_{12} & a_{13} \\ -a_{12} & 0 & a_{23} \\ -a_{13} & -a_{23} & 0 \end{pmatrix}.$$

Similarly, we have antisymmetry of p, q on LHS and RHS ($9 \to 3$). So we have only to show for 3×3 terms. Select $i = 1$ and $j = 2$. Then we have $\epsilon_{12k}\epsilon_{pqk} \neq 0$ only if $k = 3$. On the right hand side we have $\delta_{1p}\delta_{2q} - \delta_{1q}\delta_{2p}$. On LHS we have that $\epsilon_{123}\epsilon_{pq3} = (+1)\epsilon_{pqk} = 0$ unles $p, q = 1, 2$ or 2,1. On the RHS we have that if $p, q = 3$ then RHS=0. Then we take $p, q = 1, 2$ or 2,1. On the LHS we have that $\epsilon_{123}\epsilon_{123} = 1$ and $\epsilon_{123}\epsilon_{213} = -1$. Similarly, on the RHS we have that $\delta_{11}\delta_{22} - \delta_{11}\delta_{22} = 1$ and $\delta_{12}\delta_{21} - \delta_{11}\delta_{22} = -1$. The rest follows because of the cyclic symmetry of i, j.

Now if we have $\epsilon_{kij}\epsilon_{kpq}$ is the same since we have a cyclic permutation. Let us look at $a \times (b \times c)$. Then we have

$$\begin{aligned} (a \times (b \times c))_i &= f_i \\ &= \epsilon_{ijk}a_i(b \times c)_k \\ &= \epsilon_{ijk}a_i\epsilon_{rpq}b_pc_q \\ &= \epsilon_{ijk}\epsilon_{pqk}a_ib_pc_q \\ &= (\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp})a_ib_pc_q \\ &= a_jb_ic_j - a_jb_jc_i. \end{aligned}$$

(we have that $(a\times b)_i = \epsilon_{ijk} a_j b_k$).

Example:

$$\begin{aligned}\nabla\cdot(A\times B) &= \frac{\partial}{\partial x_i}(\epsilon_{ijk}A_jB_k)\\ &= \epsilon_{ijk}\Big[B_k\frac{\partial}{\partial x_i}A_j + A_k\frac{\partial}{\partial x_i}B_k\Big]\end{aligned}$$

and now we use $(\nabla\times A)_p = \epsilon_{pqr}\frac{\partial A_r}{\partial x_q}$. Then, $\epsilon_{ijk}\Big[B_k\frac{\partial}{\partial x_i}A_j + A_k\frac{\partial}{\partial x_i}B_k\Big] = \epsilon_{kij}\Big[B_k\frac{\partial}{\partial x_i}A_j + A_k\frac{\partial}{\partial x_i}B_k\Big]$ by cylic permutation. Then,

$$\begin{aligned}\nabla\cdot(A\times B) &= \Big(\epsilon_{kij}\frac{\partial}{\partial x_i}A_j\Big)B_k - \Big(\epsilon_{jik}\frac{\partial}{\partial x_i}B_k\Big)A_k\\ &= (\nabla\times A)\cdot B - (\nabla\times B)\cdot A\end{aligned}$$

Example:

$$u\times(\nabla\times u)$$

where the i^{th} component is $=$

$$\begin{aligned}\epsilon_{ijk}u_j(\nabla\times u)_k &= \epsilon_{ijk}u_j\epsilon_{kpq}\frac{\partial u_q}{\partial x_p}\\ &= \epsilon_{ijk}\epsilon_{pqk}u_j\frac{\partial u_q}{\partial x_p}\\ &= (\delta_{ip}\delta_{jq}-\delta_{iq}\delta_{jp})u_j\frac{\partial u_q}{\partial x_p}\\ &= u_j\frac{\partial u_j}{\partial x_i} - u_j\frac{\partial u_i}{\partial x_j}\\ &= \frac{\partial}{\partial x_i}\Big(\frac{1}{2}u_j^2\Big) - (u\cdot\nabla)u_i\\ &= \nabla(\frac{1}{2}u^2) - u\cdot\nabla u\end{aligned}$$

(Note that $\frac{1}{2}u_j^2$ is summed over j since $\frac{1}{2}u_j^2 = \frac{1}{2}u_ju_j$).

Gauss' Theorem:

If we have a vector field $A(\vec{x})$ then

$$\int_V \nabla\cdot A\,dV = \oint_S A\cdot n\,dS.$$

This result is generalizeable because we can write it as

$$\int_V \frac{\partial A_i}{\partial x_i}dV = \oint_S n_iA_i\,dS.$$

Now consider $A_i = T_{ij\ldots k}$ for a given $j\ldots k$. We can imply this result for each choice of $j\ldots k$. Now

$$\int_V\Big(\frac{\partial}{\partial x_i}T_{ij\ldots k}\Big)dV = \oint_S n_iT_{ij\ldots k}\,dS.$$

So we fix $j\ldots k$ and use Gauss' theorem as usual.

Example:

Let $T_{ij} = \epsilon_{ijk} A_k$. Now we apply Gauss' Divergence Theorem to get

$$\begin{aligned}
\int_V \frac{\partial}{\partial x_i}(\epsilon_{ijk} A_k)\, dV &= \oint_S n_i \epsilon_{ijk} A_k\, dS \\
-\int_V \epsilon_{jik} \frac{\partial}{\partial x_i} A_k\, dV &= \oint_S \epsilon_{kij} A_k n_i\, dS \\
-\int_V (\nabla \times A)_j\, dV &= \oint_S (A \times n)_j\, dS \\
\int_V (\nabla \times A)_j\, dV &= \oint_S (n \times A)_j\, dS.
\end{aligned}$$

Now we will study Kinematics (how to describe fluid motion, Kundu Ch.3):

(We already covered sections 1-4)

Local fluid motion:

Conisder some simple flows:

1. Couette flow (Shear):

   $u_1 = \beta x_2$

2. Pure straining flow:

   $u = (\alpha x_1, -\alpha x_2, 0)$

3. Rigid Body Rotation:

   $u = \Omega \times x$

   (Note that rigid body rotations have circular stream lines but stream lines of circular motion do not necessarily imply rigid body rotations!)

Relative velocity of fluids:

1st element at x and the 2nd element at $x + \xi$, where $x, \xi \in \mathbb{R}^3$. We have that

$$u_i(x + \xi) = u_i(x) + \xi_j \frac{\partial u_i}{\partial x_j} + O(\xi^2)$$

so that the relative velocity is approximately $\xi_j \frac{\partial u_i}{\partial x_j}$ (using Taylor expansion in $\mathbb{R}^3$ only about x, y, and z) for $\xi \ll 1$. Consider the motion of a material line element $\xi(t)$. We have

$$\begin{aligned}
\frac{D}{Dt}\xi(t) &= u_i(x + \xi) - u_i(x) \\
&= \xi_j \frac{\partial u_i}{\partial x_j} + \mathcal{O}(\xi^2) \\
&= \text{relative velocity},
\end{aligned}$$

where $u_i(x + \xi(t)) - u_i(x)$ is the material line element. So we have

$$\begin{aligned}
\frac{D}{Dt}\xi_i &= \xi_j \frac{\partial u_i}{\partial x_j} \\
&= (\xi \cdot \nabla) u
\end{aligned}$$

where ξ is determined by end-points. Notice that $\frac{\partial u_i}{\partial x_j}$ is a rank 2 tensor where

$$\frac{\partial u_i}{\partial x_j} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$$

where the first () is the rate of strain tensor (symmetric) and the other is the rotation tensor (antisymetric).

9/27/07

Will meet regularly on Thurs Oct. 9, but not class on Thurs. Oct 11.

Let us look at local fluid motion. We have a line element $H(t)$ which moves with the fluid. At time t we have that $x = X_A(t)$ and $x + h = X_B(t) = X_A(t) + H(t)$ (so we have fixed a t and looked at two fluid particles at x and $x + h$. In other words, this is a snapshot at time t. When we let t move forwards, these two particles will move). We can say that

$$\frac{dH(t)}{dt} = u(X_B, t) - u(X_A, t).$$

So at time t we have

$$\frac{dH}{dt} = u(x + h, t) - u(x, t) = u(x, t) + H \cdot \nabla u(x, t) + \mathcal{O}(h^2) - u(x, t)$$

(the relative velocity) or

$$\frac{dH_i}{dt} = u_i(x, t) + \frac{\partial u_i(x, t)}{\partial x_j} H_j + O(h^2) - u_i(x, t) = \frac{\partial u_i(x, t)}{\partial x_j} h_j + O(h^2)$$

and for short we have that

$$\frac{dH_i}{dt} = \frac{\partial u_i(x, t)}{\partial x_j} H_j \text{ at time } t,$$

for $|h|$ small. We will see later, in proving Kelvin's theorem, that we indeed use this approximation since we let $|h| \to 0$.

Generally,

$$\frac{dH_i}{dt} = \frac{\partial u_i}{\partial x_j} h_j \,|_{X_A(t)}$$

We mentioned last time that we can write

$$\begin{aligned} \frac{\partial u_i}{\partial x_j} &= \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) \\ &= \frac{1}{2}(S_{ij} + R_{ij}) \end{aligned}$$

where S is the rate of strain and R is the rotation, both are tensors of rank 2. We also see this as

$$\frac{1}{2}(\nabla u + \nabla u^T) = S \quad , \quad \frac{1}{2}(\nabla u - \nabla u^T) = R$$

which is called the dyadic form, where S and R are second order tensors. Note that $\frac{dH}{dt} = \frac{DH}{Dt}$ because we are looking at the lagrangian motion of the particles. The length of H is given by $H^2 = H \cdot H$ and $\frac{d}{dt} H^2 = 2H \frac{dH}{dt} = 2H_i(\frac{\partial u_i}{\partial x_j})H_j$ which looks familiar as $H^T u H$ and we get that

$$2H_i(\frac{\partial u_i}{\partial x_j})H_j = H_i H_j S_{ij}$$

which means the change in relative distance of adjacent fluid elements depends only on the rate of strain, S_{ij}. What happened to the rotation tensor term? For the diagonal terms, $R_{ij} = 0$. For $i < j$, for the terms above the diagonal, the values for $H_i R_{ij} H_j$ are idential with opposite sign for the terms $i > j$, under the diagonal. Hence, $H_i R_{ij} H_j$ vanishes. This means that the rate of strain is the important flow field to be tracking. Note that R is antisymmetric with zeros on the diagonal so that $v^T R v = 0$ where $R_{n\times n}$ and $v \in \mathbb{R}^n$.

Let us look at the rotation R_{ij}, which is anti-symmetric.

Definition 2. *Vorticity* $\omega = \nabla \times u$.

We will show that R is very closely related to the vorticity.

We have that $\omega_1 = \frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}$, $\omega_2 = \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}$, $\omega_3 = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}$ after taking the curl of the velocity field. We also defined R_{ij} to be

$$\begin{aligned}
2R_{12} &= -\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2} \\
&= -\omega_3 \\
2R_{13} &= \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1} \\
&= \omega_2 \\
2R_{23} &= -\frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3} \\
&= -\omega_1,
\end{aligned}$$

where we only need to define three terms since R is anti-symmetric. Hence,

$$2R = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}.$$

Let us try to find a relationship between R_{ij} and ω_{ij}. Generally, for anti-symmetric tensors A_{ij} we have a dual vector a where

$$A_{ij} = \epsilon_{ijk} a_k.$$

For instance, $A_{12} = \epsilon_{12k} a_k = a_3$, $A_{13} = \epsilon_{13k} a_k = -a_2$, $A_{23} = \epsilon_{23k} a_k = a_1$. So, for $A_{ij} = R_{ij}$ we have that if $R_{ij} = \epsilon_{ijk} a_k$ then $R_{12} = -\frac{1}{2}\omega_3 = a_3$, $R_{13} = \frac{1}{2}\omega_2 = -a_2$, $R_{23} = -\frac{1}{2}\omega_1 = a_1$. Hence, $a = -\frac{1}{2}\omega$ and we get

$$R_{ij} = -\frac{1}{2}\epsilon_{ijk}\omega_k.$$

In order to find a_p in terms of A_{ij} we can do the following.

$$\begin{aligned}
a_p &= \alpha \epsilon_{pmn} A_{mn} \\
&= \alpha \epsilon_{pmn} \epsilon_{mnk} a_k \\
&= -\alpha \epsilon_{mpn} \epsilon_{mnk} a_k \\
&= -\alpha (\delta_{pn}\delta_{nk} - \delta_{pk}\delta_{nn}) a_k \\
&= -\alpha (\delta_{pk} - 3\delta_{pk}) a_k \\
&= 2\alpha \delta_{pk} a_k.
\end{aligned}$$

So we have that

$$\begin{aligned} a_p &= 2\alpha\delta_{pk}a_k \\ &= 2\alpha a_p. \end{aligned}$$

Now, it must be true that $\alpha = 1/2$ so that $a_p = \frac{1}{2}\epsilon_{pmn}A_{mn}$ (since we are summing over m now ($\epsilon_{mpn}\epsilon_{mnk} = \epsilon_{pnm}\epsilon_{nkm}$)). ($\delta_{1n}\delta_{n2} - \delta_{12}\delta_{nn} = \Sigma_{n=1}^{3}\delta_{1n}\delta_{n2} - \delta_{12}\delta_{nn} = -\delta_{12} - \delta_{12} = -2\delta_{12}$). Then, for R_{ij}, we have $a = -\frac{1}{2}\omega$ so that

$$\begin{aligned} -\frac{1}{2}\omega_i &= \frac{1}{2}\epsilon_{ijk}R_{jk} \\ \Rightarrow \omega_i &= -\epsilon_{ijk}R_{jk}. \end{aligned}$$

In summary, we get that

$$\begin{aligned} R_{ij} &= -\frac{1}{2}\epsilon_{ijk}\omega_k, \\ \omega_i &= -\epsilon_{ijk}R_{jk} \end{aligned}$$

Simple Flows:

1. $u_1 = \alpha x_1, u_2 = -\alpha x_2, u_3 = 0$. If we look at $\nabla u = \begin{pmatrix} \alpha & 0 & 0 \\ 0 & -\alpha & 0 \\ 0 & 0 & 0 \end{pmatrix}$. So $\omega = 0$ ($R_{ij} = 0$) and $S_{ij} = \nabla u$. If we look at the divergence $\nabla \cdot u = \frac{\partial u_i}{\partial x_i} =$ sum of the diagonal terms of ∇u, which is zero in our case. This is equal to the trace(∇u). Since the divergence is a rank two tensor contracted to a rank 0 tensor the $\nabla \cdot u$ is invariant of coordinates (since it is always a scalar). This type of flow is called a **pure straining flow** (stagnation point flow)

2. $u_1 = -\Omega x_2, u_2 = \Omega x_1, u_3 = 0$. This gives us circular stream lines where $\vec{\Omega} = (0, 0, \Omega)$. Here we have that $\omega_1 = \omega_2 = 0$ and $\omega_3 = 2\Omega$. Then $\omega = 2\vec{\Omega}$. For $S_{ij} = 0$.

In flow 1 we have compression along x_2 and extension along x_1. That is

$$\begin{aligned} H_i\frac{\partial u_i}{\partial x_j}H_j &= \frac{1}{2}\frac{dH^2}{dt} \\ &= H_1^2\alpha - H_2^2\alpha \end{aligned}$$

Using

$$\frac{dH_i}{dt} = \frac{\partial u_i(x,t)}{\partial x_j}H_j \text{ at time } t,$$

it is more useful to think of

$$\begin{aligned} \frac{dH_1}{dt} &= (H \cdot \nabla)u_1 \\ &= \alpha H_1 \\ \frac{dH_2}{dt} &= -\alpha H_2 \end{aligned}$$

so that $H_1 = H_1(0)e^{\alpha t}, H_2 = H_2(0)e^{-\alpha t}$.

For flow 2 (rigid body flow) we have zero rate of strain. That is the line element just rotates.

3. $u_1 = \beta x_2, u_2 = u_3 = 0$ (shear flow). What happens with the line elements over time?

$$\frac{dH}{dt} = (H \cdot \nabla)u.$$

How quickly does the length of the line element grow? We have

$$\begin{aligned}\frac{dH_1}{dt} &= (H\cdot\nabla)u_1\\ &= \beta H_2,\\ \frac{dH_2}{dt},\frac{dH_3}{dt} &= 0.\end{aligned}$$

Hence, H_2 is a constant w.r.t. time. So H_1 grows linearly w.r.t. t (i.e. $H_1 = \beta H_2(0)t + H_2(0)$).

(Recall that the vorticity is $\omega = \nabla \times u$. If we are given angular momentum, we have that the velocity is $u = \Omega \times x$ and so $\omega = 2\Omega$. Thus, the vorticity is twice the local angular velocity. Hence, the vorticity is very closely related to the angular momentum in the flow).

Vorticity and Circulation

Stokes Theorem:

$$\int_S (\nabla\times A)\cdot n\,dS = \oint_C A\cdot dx$$

What if we inflated the surface of A. Would this change the integral? No! Take the inflated surface and consider the closed volume created by the inflated surface S' and the original surface S. The surface integral of the inflated side S' is

$$\oint_{S'} (\nabla\times A)\cdot n\,dS.$$

For the original surface S, the outward normal now points the opposite direction. Hence, the surface integral over S becomes

$$-\oint_S (\nabla\times A)\cdot n\,dS.$$

The total surface, S^* is then

$$\oint_{S^*} (\nabla\times A)\cdot n\,dS = \oint_{S'} (\nabla\times A)\cdot n\,dS - \oint_S (\nabla\times A)\cdot n\,dS.$$

But, now, S^* is a closed surface, and by the divergence theorem we have

$$\begin{aligned}\oint_{S^*} (\nabla\times A)\cdot n\,dS &= \oint_{V^*} \nabla\cdot(\nabla\times A)dV\\ &= 0.\end{aligned}$$

Hence,

$$\oint_{S'} (\nabla\times A)\cdot n\,dS = \oint_S (\nabla\times A)\cdot n\,dS,$$

so that the inflation does not change integral in Stokes' Theorem.

Now, by Stokes' Theorem, we have

$$\int_S \omega\cdot n\,dS = \oint_C u\cdot dx$$

where the RHS is the circulation on the curve C. If $\omega \equiv 0$ then $\nabla\times u = 0$ and we can find $\phi(x)$, where $u = \nabla\phi$. Moreover, $\omega \equiv 0$ then $\oint_C u\cdot dx = 0$. If we define $\phi(\boldsymbol{x}) = \int_0^{\boldsymbol{x}} u\cdot d\boldsymbol{x}$ +constant then certainly $u = \nabla\phi$ (see Stewart's calculus for explanation).

Consider $u_\theta = \Gamma/2\pi r$, $u_r = 0$, $u_z = 0$ or that $u_1 = -\frac{x_2}{r}\frac{\Gamma}{2\pi r} = -\sin\theta\frac{\Gamma}{2\pi r}$, $u_2 = \frac{x_1}{r}\frac{\Gamma}{2\pi r} = \cos\theta\frac{\Gamma}{2\pi r}$, since $e^{(\theta)} = (-\sin\theta, \cos\theta, 0)$. This is a circular flow where we have that $u_1 = u_r\cos\theta - u_\theta\sin\theta$ and $u_2 = u_r\sin\theta + u_\theta\cos\theta$. Recall that $r^2 = x_1^2 + x_2^2$ and that

$$\nabla \times \boldsymbol{F} = \frac{1}{r}\begin{vmatrix} e^{(r)} & r\,e^{(\theta)} & e^{(z)} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} & \frac{\partial}{\partial z} \\ F_r & r\,F_\theta & F_z \end{vmatrix},$$

in cylindrical polar coordinates. Then,

$$(\nabla \times \boldsymbol{u}) = \frac{1}{r}\begin{vmatrix} e^{(r)} & r\,e^{(\theta)} & e^{(z)} \\ \frac{\partial}{\partial r} & 0 & 0 \\ 0 & \frac{\Gamma}{2\pi} & 0 \end{vmatrix}$$

which gives us

$$(\nabla \times \boldsymbol{u}) = 0,$$

so that the flow is irrotational even though we have circulation!

$$\begin{aligned} \omega_3 &= \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \\ &= \frac{\partial}{\partial x_1}\left(\frac{\Gamma}{2\pi r^2}\,x_1\right) + \frac{\partial}{\partial x_2}\left(\frac{\Gamma}{2\pi r^2}\,x_2\right) \\ &= \frac{\Gamma}{2\pi}\left\{\frac{\partial}{\partial x_1}\left(\frac{1}{r^2}\,x_1\right) + \frac{\partial}{\partial x_2}\left(\frac{1}{r^2}\,x_2\right)\right\} \\ &= \frac{\Gamma}{2\pi}\left\{\frac{1}{r^2} + x_1\frac{\partial}{\partial x_1}\left(\frac{1}{r^2}\right) + \frac{1}{r^2} + x_2\frac{\partial}{\partial x_2}\left(\frac{1}{r^2}\right)\right\} \\ &= 0. \end{aligned}$$